On q-SZASZ- Mirakyan Operators of functions of Two Variables

Ali J. Mohammad and Rafah F.Kadum
Department of Mathematics, College of Education for pure science, University of Basrah, Basrah,Iraq.

Abstract

In this paper, we define two operators of summation and summation-integral of q-type in two dimensional spaces. Firstly, we study the convergence of these operators and then we prove Voronovskaya- type asymptotic formulas for these operators.

Keywords: q-SZASZ- Mirakyan Operators, convergence, Voronovskaya- type asymptotic.

حول المؤثرات q-SZASZ- MIRAKYAN لدوال بمتغيرين

علي جاسم محمد الأسدي و رفاه فؤاد كاظم
قسم الرياضيات ، كلية التربية للعلوم الصرفة ،جامعة البصرة، البصرة، العراق

الخلاصة

في هذا البحث, قمنا بتعريف مؤثرين لمجموع ومجموع تكامل من نوع q– في فضاء البعدين . أولا, ندرس تقارب المؤثرين ومن ثم نثبت الصيغة المشابهة لـ Voronvskaya لهذين المؤثرين.

Introduction.

We start our note to introduce some works of the other researchers. In 1912, Bernstein defined a sequence of linear positive operators called the Bernstein polynomials as: [1].

$$\beta_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1]. \qquad .. \quad (1.1)$$

In 1932, Voronovskaya showed that the convergence of $\beta_n(f;x)$ to $f(x)$ as $n \to \infty$ is slow but sure. [2].

In 1950, Szasz generalized the Bernstein operators to infinite interval $[0,\infty)$, which is called Szasz-Mirakyan operators defined as: [3].

$$S_n(f(t);x) = \sum_{k=0}^{\infty} z_{n,k}(x) \; f\left(\frac{k}{n}\right); \qquad (1.2)$$

$$where \quad z_{n,k}(x) = \frac{e^{-nx}(nx)^k}{k!} .$$

In 1987, Lupas generalized the Bernstein polynomials involving q-integers which defined as: [4].

$\beta_n(f,q;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)_q^{n-k} f\left(\frac{[k]}{[n]}\right)$, $x \in [0,1]$, $q \in (0,1)$. (1.3)

In 1998, Lucyna and Mariola proposed two sequences of Szasz-Mirakyan operators of two variables defined as:[5]

$$B_{n,m}(f(t,s);x,y) = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} z_{n,k}(x)\, z_{m,p}(y) f\left(\frac{k}{n},\frac{p}{m}\right) ; \quad (1.4)$$

$$C_n(f(t,s);x,y) = nm\sum_{k=0}^{\infty}\sum_{p=0}^{\infty} z_{n,k}(x)\, z_{m,p}(y) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{\frac{p}{m}}^{\frac{p+1}{m}} f(t,s)\,ds\,dt . \quad (1.5)$$

In 2006, Aral and Gupta developed the approximation property of the Szasz –Mirakyan operators in q- type defined as: [6].

$$\mu_n^q(f,x) = \sum_{k=0}^{\infty} f\left(\frac{[k]_q b_n}{[n]_q}\right) \frac{([n]_q x)^k}{[k]!(b_n)^k} E_q\left(-[n]_q \frac{x}{b_n}\right) , \quad q \in (0,1). \quad (1.6)$$

In 2010, Mahamudov introduced another type of q-Szasz-Mirakyan operators define as: [7].

$$L_n(f,q;x) = \sum_{k=0}^{\infty} z_{n,k}(q;x) f\left(\frac{[k]_q}{[n]_q}\right), \quad (1.7)$$

where

$$z_{n,k}(q;x) = \frac{1}{q^{k(k-1)/2}} \frac{[n]_q{}^k x^k}{[k]_q!} e_q\left(-[n]_q \frac{x}{q^k}\right),$$

where $q > 1$.

In 2012, Ghadhban introduced a new modification of q-Szasz-Mirakyan operators define as: [8].

$$A_n(f,q_n;x) = \sum_{k=0}^{\infty} z_{n,k}(q_n;x) f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\right), \quad (1.8)$$

$$B_n(f(t),q_n;x) = [n]_{q_n} \sum_{k=0}^{\infty} z_{n,k}(q_n;x) \int_{[k]_{q_n}/[n]_{q_n}}^{[k+1]_{q_n}/[n]_{q_n}} f(t)\, d_{q_n}t , \quad (1.9)$$

$q_n \in (0,1)$ and $q_n \to 1$ as $n \to \infty$.

In this paper, we define two types of q-Szasz operators as follows:

For $h > 0$, the space of all continuous real-valued functions on the are $[0,\infty) \times [0,\infty)$ such that $|f(t,s)| \le A e_q^{h(t+s)}$ for some constant $A > 0$.

Suppose that $q_n, \breve{q}_n \in (0,1)$; $q_n, \breve{q}_n \to 1$ as $n \to \infty$, and $f \in C_h([0,\infty) \times [0,\infty))$ $x, y \in [0,\infty)$, we define:

$$G_n(f(t,s), q_n, \breve{q}_n; (x,y)) = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} z_{n,k}(q_n;x)\, z_{n,p}(\breve{q}_n;y) f\left(\frac{[k]_{q_n}}{[n]_{q_n}}, \frac{[p]_{\breve{q}_n}}{[n]_{\breve{q}_n}}\right) ; \quad (1.10)$$

and for $(q_n - 1) = o([n]^{-2})$ and $(\breve{q}_n - 1) = o([n]^{-2})$ we define:

$$R_n(f(t,s), q_n, \breve{q}_n; (x,y)) = [n]^2 \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} z_{n,k}(q_n;x)\, z_{n,p}(\breve{q}_n;y) \int_{\frac{[k]_{q_n}}{[n]_{q_n}}}^{\frac{[k+1]_{q_n}}{[n]_{q_n}}} \int_{\frac{[p]_{\breve{q}_n}}{[n]_{\breve{q}_n}}}^{\frac{[p+1]_{\breve{q}_n}}{[n]_{\breve{q}_n}}} f(t,s)\, d_{\breve{q}_n}s\, d_{q_n}t . \quad (1.11)$$

We study the two above operators discusses the convergence of these operators to the function $f(x,y)$ as $n \to \infty$. Then, we establish a Voronovskaya-type asymptotic formulas for the operators G_n and R_n.

2. Notations and preliminaries.

Throughout this paper we use the standard notations of q-calculus. (see [9] and [10])

For $n \in N^0$, $N^0 = \{0,1,2,3,\ldots\}$. The q-analogue is defined as:

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q} & if \quad n \neq 0 , \quad q \in \mathbb{R}^+/\{1\} \\ 0 & if \quad n = 0 \quad . \end{cases}$$

We can write

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} ; n \neq 0 .$$

The q- factorial is defined as:

$$[n]_q ! = \begin{cases} [1]_q [2]_q [3]_q \ldots [n]_q & ; n \in N \\ 1 & ; n = 0. \end{cases}$$

*In this work, we used the notation [n] instead $[n]_q$ where q is value or sequence.

The q-derivative of a function $f(x)$ is defined as:

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x} , x \neq 0 , q \in R^+/\{1\}.$$

The formula for the q-derivative of a product of two functions is defined as:

$D_q(u(x)v(x)) = D_q(u(x))v(x) + u(qx)D_q(v(x))$,

and $D_q^n f = \begin{cases} D_q^{n-1}(Df); & n \neq 0 \\ f \ ; & n = 0 \end{cases}$

The q-analogue of $(t-x)^n$ is defined by: $(t-x)_q^n = (t-x)(t-qx)(t-q^2x)\ldots(t-q^{n-1}x)$

The q-Taylor's formula defined as:

$f(t) = f(x) + \frac{(t-x)_q}{[1]!} D_q f(x) + \frac{(t-x)_q^2}{[2]!} D_q^2 f(x) + \cdots = \sum_{k=0}^{\infty} \frac{(t-x)_q^k}{[k]!} D_q^k f(x)$

.

The q-exponential function define as: $e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}$.

Lemma (2.1). [8]

For the function $z_{n,k}(q_n;x)$ we have

$(i) \sum_{k=0}^{\infty} z_{n,k}(q_n;x) = 1;$

$(ii)\ x\, D_{q_n} z_{n,k}(q_n;x) = ([k]-[n]x) z_{n,k}$

(iii) Suppose that $\varphi_{n,m}(q_n;x) = \sum_{k=0}^{\infty} [k]^m z_{n,k}(q_n;x)$, then $\varphi_{n,m+1}(q_n;x) = xD_{q_n}\varphi_{n,m}(q_n;x) + [n]x\ \varphi_{n,m}(q_n;x)$;

$(iv) \sum_{k=0}^{\infty} [k] z_{n,k}(q_n;x) = [n]x\ ;$

$(v) \sum_{k=0}^{\infty} [k]^2 z_{n,k}(q_n;x) = [n]^2x^2 + [n]x\ ;$

$(vi) \sum_{k=0}^{\infty} [k]^3 z_{n,k}(q_n;x) = [n]^3x^3 + (2+q_n)[n]^2x^2 + [n]x\ .$

Theorem(2.1). [11]

If $M_{n,m}(f(t,u);x,y)$ be a sequence of linear positive operators of 2 dimensional space with the norm $\|.\|$ and the four conditions are hold:

$(i) \lim_{n,m\to\infty} \|M_{n,m}(1;(x,y)) - 1\| = 0;$

$(ii) \lim_{n,m\to\infty} \|M_{n,m}(t;(x,y)) - x\| = 0;$

$(iii) \lim_{n,m\to\infty} \|M_{n,m}(s;(x,y)) - y\| = 0;$

$(iv) \lim_{n,m\to\infty} \|M_{n,m}(t^2+s^2;(x,y)) - (x^2+y^2)\| = 0.$

Then

$\|M_{n,m}(f(t,s);(x,y)) - f(x,y)\| = 0 \ \ as\ n,m \to \infty.$

Then $M_{n,m}(f(t,u);x,y)$ converges to $f(x,y)$ in the same area $as\ n,m \to \infty$.

Definition (2.1). [12]

For $n \in N^0$, the m-th orders of q- moments $T_{n,m}(q_n,x)$ for the operators (1.8) defines as $(T_{n,m})_1(q_n,x) = A((t-x)_{q_n}^m, q_n;x) = \sum_{k=0}^{\infty} z_{n,k}(q_n,x)(t-x)_{q_n}^m$

.

Lemma (2.2). [12]

For the function $(T_{n,m})_1(q_n,x)$ we have:

$(i)(T_{n,0})_1(q_n;x) = 1\ ;$

$(ii)(T_{n,1})_1(q_n;x) = 0\ ;$

$(iii)(T_{n,2})_1(q_n;x) = \frac{1}{[n]}\ x\ ;$

$(iv)(T_{n,3})_1(q_n;x) = \left(\frac{1-q_n^2}{[n]}\right)x^2 + \frac{1}{[n]^2}\ x\ ;$

Definition (2.2).[8]

For $m \in N^0$, the m-th order q-moments $(T_{n,m})_2(q_n;x)$ for the operators (1.9) are defined as:

$(T_{n,m})_2(q_n;x) = B_n((t-x)_{q_n}^m, q_n;x) = [n]\sum_{k=0}^{\infty} S_{n,k}(q_n;x) \int_{[k]/[n]}^{[k+1]/[n]} (t-x)_{q_n}^m\, d_{q_n}t\ .$

Lemma (2.3). [8]

For the function $(T_{n,m})_2(q_n;x)$ we have:

$(i)(T_{n,0})_2(q_n;x) = (q_n-1)[n]x + 1;$

$(ii)(T_{n,1})_2(q_n;x) = \frac{q_n^2+q_n-2}{[2]}\ x + \frac{1}{[2][n]};$

$(iii)(T_{n,2})_2(q_n;x) = \frac{2q_n^2-q_n-1}{[3]}x^2 + \frac{q_n^3+2q_n^2+2q_n-2}{[3][n]}\ x + \frac{1}{[3][n]^2};$

$$(iv)\left(T_{n,3}\right)_2(q_n;x)=\frac{-2q_n^5+q_n^4+2q_n-1}{[4]}x^3+\frac{q_n^4+4q_n^3+q_n^2-4q_n-2}{[4][n]}x^2$$
$$+\frac{q_n^4+3q_n^3+5q_n^2+3q_n-2}{[4][n]^2}x+\frac{1}{[4][n]^3};$$

Lemma (2.4). [8]

For $m\in N$;

$$\frac{[n][m+2]}{[m+2]-1}\left(T_{n,m+1}\right)_2(q_n;x)$$
$$=xD_{q_n}\left(T_{n,m}\right)_2+[m]x\left(T_{n,m-1}\right)_2(q_n;q_nx)+[n]\left\{(1-q_n^m)x-\frac{1}{[n+1]-1}\right\}\left(T_{n,m}\right)_2(q_n;x)$$
$$-\frac{[n]^2}{[m+1]}\left\{(q_n^{-1}-1)\left(T_{n,m+2}\right)_1(q_n;x)\right.$$
$$\left.+\left((q_n^m-q_n^{m-1})x-\frac{1}{[n+1]-1}\right)\left(T_{n,m+1}\right)_1(q_n;x)\right\}.$$

Furthermore, for $m\geq 1$,

1. $\left(T_{n,m}\right)_2(q_n;x)$ Is a polynomial in x of degree m
2. For every $x\in[0,\infty)$, $\left(T_{n,m}\right)_2(q_n;x)=O\left([n]^{-(m)}\right)$.

3. Main results.

Here, we improve the convergence and introduce a Voronvaskaya - *type asymptotic formulas* for the operators G_n and R_n .

3.1.The operators $G_n\left(f(t,s),q_n,\breve{q}_n;(x,y)\right)$:

Our first result shows that the operators $G_n(f,q_n,\breve{q}_n;(x,y))$, see the equation (1.10), converges to the function $f(x,y)$ as $n\to\infty$.

Lemma (3.1.1).

For the operators G_n the following conditions are hold:

(1) $\lim_{n\to\infty}\|G_n(1,q_n,\breve{q}_n;(x,y))-1\|_{C_h}=0;$

(2) $\lim_{n\to\infty}\|G_n(t,q_n,\breve{q}_n;(x,y))-x\|_{C_h}=0;$

(3) $\lim_{n\to\infty}\|G_n(s,q_n,\breve{q}_n;(x,y))-y\|_{C_h}=0;$

(4) $\lim_{n\to\infty}\|G_n(t^2+s^2,q_n,\breve{q}_n;(x,y))-(x^2+y^2)\|_{C_h}=0,$

Proof:

Using Lemma (2.1) and the directs computation, the consequences (1), (2) and (3) are hold. To evaluate consequence (4), we have:

(4) $\lim_{n\to\infty}\|G_n(t^2+s^2,q_n,\breve{q}_n;(x,y))-(x^2+y^2)\|_{C_h}$

$$\lim_{n\to\infty}\left\|\sum_{k=0}^{\infty}z_{n,k}(q_n;x)\left(\frac{[k]}{[n]}\right)^2\sum_{p=0}^{\infty}z_{n,p}(\breve{q}_n;y)+\sum_{k=0}^{\infty}z_{n,k}(q_n;x)\sum_{p=0}^{\infty}z_{n,p}(\breve{q}_n;y)\left(\frac{[p]}{[n]}\right)^2-(x^2+y^2)\right\|_{C_h}$$
$$=\lim_{n\to\infty}\left\|\frac{x}{[n]}+\frac{y}{[n]}\right\|_{C_h}=0.$$

Therefore , by Theorem (2.1) we get:

$\|G_n(f(t,s),q_n,\breve{q}_n;(x,y))-f(x,y)\|_{C_h}=0$ as $n\to\infty$. ■

Theorem (3.1.1).

For $f\in C_h$, and suppose that $\frac{f_{xx}(x,y)}{\partial_q^2x},\frac{f_{yy}(x,y)}{\partial_q^2y}$ and $\frac{f_{xy}(x,y)}{\partial_qx\partial_qy}$ are exist and continuous at a point $x,y\in[0,\infty)$, then:

$$\lim_{n\to\infty}[n]\left[G_n(f,q_n,\breve{q}_n;(x,y))-f(x,y)\right]=\frac{1}{2}xf_{xx}(x,y)+\frac{1}{2}yf_{yy}(x,y)$$

.

Proof:

By q-Taylor's formula [11] for f, about the point (x,y) we have:

$$f(t,s)=f(x,y)+f_x(x,y)(t-x)+f_y(x,y)(s-y)$$
$$+\frac{1}{2}\left\{f_{xx}(x,y)(t-x)^2+2f_{xy}(t-x)(s-y)+f_{yy}(x,y)(s-y)^2\right\}$$
$$+\varphi(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4},$$

where $\varphi(t,s;(x,y)):=\varphi(t,s)$ is a function in the space $C_h([0,\infty)\times[0,\infty))$ and $\varphi(t,s)\to 0$ as $(t,s)\to(x,y)$ thus $\varphi(x,y)=0$.

$$G_n(f(t,s),q_n,\breve{q}_n;(x,y)) = f(x,y) + f_x(x,y)G_n\left((t-x),q,\right. + \frac{1}{2}f_{xx}(x,y)G_n((t-x)^2,q_n;x) + + \frac{1}{2}f_{yy}(x,y)G_n((s-y)^2,\breve{q}_n;y) + G_n\left(\varphi(t,s)\sqrt{(t-x)^4+(s-y)^4}\right.$$

Using Lemma (2.2), we have:

$$G_n(f(t,s),q_n,\breve{q}_n;(x,y)) = f(x,y) + \frac{1}{2}f_{xx}(x,y)\left(\frac{x}{[n]}\right) + \frac{1}{2}f_{yy}(x,y)\left(\frac{y}{[n]}\right) + G_n\left(\varphi(t,s)\sqrt{(t-x)^4+(s-y)^4},q_n,\breve{q}_n;(x,y)\right).$$

Then

$$\lim_{n\to\infty}[n]\left[G_n(f(t,s),q_n,\breve{q}_n;(x,y)) - f(x,y)\right] = \frac{1}{2}xf_{xx}(x,y) + \frac{1}{2}yf_{yy}(x,y) + \lim_{n\to\infty}[n]G_n\left(\varphi(t,s)\sqrt{(t-x)^4+(s-y)^4},q_n,\breve{q}_n;(x,y)\right).$$

To complete the proof, we must show that the term

$$[n]G_n\left(\varphi(t,s)\sqrt{(t-x)^4+(s-y)^4},q_n,\breve{q}_n;(x,y)\right) \to 0 \ as \ n\to\infty.$$

By using Cauchy-Schwartz inequality, we get:

$$\left|G_n\left(\varphi(t,s)\sqrt{(t-x)^4+(s-y)^4},q_n,\breve{q}_n;(x,y)\right)\right| \le \left(G_n(\varphi^2(t,s),q_n,\breve{q}_n;(x,y)\right)^{\frac{1}{2}} \times \left(G_n((t-x)^4,q_n;x) + G_n((s-y)^4,\breve{q}_n;y)\right)^{\frac{1}{2}}.$$

By the properties of $\varphi(t,s) = 0 \ as \ t \to x$ $and \ s \to y$ we get $G_n(1,q_n,\breve{q}_n;(x,y)) = \varphi^2(x,y)$=0

Thus, from Lemma (2.2) we get:

$$\lim_{n\to\infty}\left(G_n((t-x)^4,q_n;x) + G_n((s-y)^4,\breve{q}_n;y)\right)^{\frac{1}{2}} = \lim_{n\to\infty}\left(\left(\frac{1-q_n^2-q_n^3+q_n^5}{[n]}\right)x^3 + \left(\frac{2+2q_n-q_n^3}{[n]^2}\right)x^2 + \frac{1}{[n]^3}\ x + \left(\frac{1-\breve{q}_n^2-\breve{q}_n^3+\breve{q}_n^5}{[n]}\right)y^3 + \left(\frac{2+2\breve{q}_n-\breve{q}_n^3}{[n]^2}\right)y^2 + \frac{1}{[n]^3}\ y\right)^{\frac{1}{2}} \to 0$$

Therefore,

$$\lim_{n\to\infty}[n]G_n\left(\begin{matrix}\varphi(t,u)\sqrt{(t-x)^4+(s-y)^4}, \\ q_n,\breve{q}_n;(x,y)\end{matrix}\right) \to 0.$$

Hence, the proof of the Theorem is complete. ■

3.2. The operators $R_n(f(t,s),q_n,\breve{q}_n;(x,y))$:

We start hear with proof the convergence of the operators $R_n(f,q_n,\breve{q}_n;(x,y))$, see the equation (1.11), and then we prove Voronovskaya- type asymptotic formulas for these operators.

Lemma (3.2.1).

For all $n \in N$ and $x,y \in [0,\infty)$ the following conditions are hold:

(1) $\lim_{n\to\infty}\|R_n(1,q_n,\breve{q}_n;(x,y)) - 1\|_{C_h} = 0;$

(2) $\lim_{n\to\infty}\|R_n(t,q_n,\breve{q}_n;(x,y)) - x\|_{C_h} = 0;$

(3) $\lim_{n\to\infty}\|R_n(s,q_n,\breve{q}_n;(x,y)) - y\|_{C_h} = 0;$

(4) $\lim_{n\to\infty}\left\|R_n(t^2+s^2,q_n,\breve{q}_n;(x,y)) - (x^2+y^2)\right\|_{C_h} = 0.$

Proof:

Using Lemma (2.1) and the direct computation, we have:

(1) $\lim_{n\to\infty}\left\|R_n(1,q_n,\breve{q}_n;(x,y)) - 1\right\|_{C_h}$

$$= \lim_{n\to\infty}\left\|[n]^2\sum_{k=0}^{\infty}z_{n,k}(q_n;x)\sum_{p=0}^{\infty}z_{n,p}(\breve{q}_n;y)\int_{\frac{[k]}{[n]}}^{\frac{[k+1]}{[n]}}\int_{\frac{[p]}{[n]}}^{\frac{[p+1]}{[n]}}1d_{\breve{q}_n}s\,d_{q_n}t - 1\right\|_{C_h}$$

$$= \lim_{n\to\infty}\left\|[n]\sum_{k=0}^{\infty}z_{n,k}(q_n;x)\left(\frac{1+[k](q_n-1)}{n}\right)[n]\sum_{p=0}^{\infty}z_{n,p}(\breve{q}_n;y)\left(\frac{1+[p](\breve{q}_n-1)}{n}\right) - 1\right\|_{C_h}$$

$$= \lim_{n\to\infty} \left\| \left((q_n-1)[n]x+1\right)\left((\breve{q}_n-1)[n]y+1\right)-1 \right\|_{C_h} = 0.$$

(2) $\lim_{n\to\infty} \| R_n(t, q_n, \breve{q}_n; (x,y)) - x \|_{C_h}$

$$= \lim_{n\to\infty} \left\| [n]^2 \sum_{k=0}^{\infty} z_{n,k}(q_n;x) \sum_{p=0}^{\infty} z_{n,p}(\breve{q}_n;y) \int_{\frac{[k]}{[n]}}^{\frac{[k+1]}{[n]}} \int_{\frac{[p]}{[n]}}^{\frac{[p+1]}{[n]}} t\, d_{\breve{q}_n}s\, d_{q_n}t - x \right\|_{C_h}$$

$$= \lim_{n\to\infty} \left\| \left(\frac{(q_n^2-1)}{[2][n]}([n]^2x^2+[n]x) + \frac{2q_n}{[2][n]}[n]x + \frac{1}{[2][n]} \right)(1+(\breve{q}_n-1)[n]y) - x \right\|_{C_h}$$

$$= \lim_{n\to\infty} \left\| (q_{n-1})[n]x^2 + \frac{q_n^2+2q_n-1}{[2]}x + \frac{1}{[2][n]})(1+(\breve{q}_n-1)[n]y) - x \right\|_{C_h} = 0$$

.

By using the same technique of (2) we can get the consequence (3) .

(4) $\lim_{n\to\infty} \| R_n(t^2+s^2, q_n, \breve{q}_n; (x,y)) - (x^2+y^2) \|_{C_h}$

$$= \lim_{n\to\infty} \| [n]^2 \sum_{k=0}^{\infty} z_{n,k}(q_n;x) \sum_{p=0}^{\infty} z_{n,p}(\breve{q}_n;y) \int_{\frac{[k]}{[n]}}^{\frac{[k+1]}{[n]}} \int_{\frac{[p]}{[n]}}^{\frac{[p+1]}{[n]}} (t^2+s^2)\, d_{\breve{q}_n}s\, d_{q_n}t - (x^2+y^2) \|_{C_h}$$

$$= \lim_{n\to\infty} \left\| \left(\frac{(q_n^3-1)}{[3][n]^2}([n]^3x^3 + (2+q_n)[n]^2x^2 + [n]x) + \frac{3q_n^2}{[3][n]^2}([n]^2x^2+[n]x) + \frac{3q_n}{[3][n]^2}([n]x) + \frac{1}{[3][n]^2} \right)((\breve{q}_n-1)[n]y+1) + ((q_n-1)[n]x+1) \right.$$
$$\left. \left(\frac{(\breve{q}_n^3-1)}{[3][n]^2}([n]^3y^3+(2+\breve{q}_n)[n]^2y^2+[n]y) + \frac{3\breve{q}_n^2}{[3][n]^2}([n]^2y^2+[n]y) + \frac{3\breve{q}_n}{[3][n]^2}([n]y) + \frac{1}{[3][n]^2} \right) - (x^2+y^2) \right\|$$

$$= \lim_{n\to\infty} \left\| \left((q_n-1)[n]x^3 + \frac{(q_n^3-1)(2+q_n)+3q_n^2}{[3]}x^2 + \frac{q_n^3+3q_n^2+3q_n-1}{[3][n]}x + \frac{1}{[3][n]^2} \right) ((\breve{q}_n-1)[n]y+1) + ((q_n-1)[n]x+1) \left((\breve{q}_n-1)[n]y^3 + \frac{(\breve{q}_n^3-1)(2+\breve{q}_n)+3\breve{q}_n^2}{[3]}y^2 + \frac{\breve{q}_n^3+3\breve{q}_n^2+3\breve{q}_n-1}{[3][n]}y + \frac{1}{[3][n]^2} \right) - (x^2+y^2) \right\|_{C_h} = 0$$

Then, by Theorem (2.1) we get,

$\| R_n(f(t,s), q_n, \breve{q}_n; x, y) - f(x,y) \|_{C_h} = 0 \;\; as \;\; n \to \infty.$ ∎

Definition (3.2.1).

For $m, r \in N^0$, the u-th order q-moments $(T_{n,m,r})_2(q_n, \breve{q}_n; (x,y))$ for the operators $R_n(f, q_n, \breve{q}_n; (x,y))$ are defined as:

$$(T_{n,m,r})_2(q_n, \breve{q}_n, (x,y)) = [n]^2 \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} z_{n,k}(q_n;x)\, z_{n,p}(\breve{q}_n;y) \int_{\frac{[k]}{[n]}}^{\frac{[k+1]}{[n]}} \int_{\frac{[p]}{[n]}}^{\frac{[p+1]}{[n]}} (t-x)^m (s-y)^r d_{\breve{q}_n}s\, d_{q_n}t$$

Lemma (3.2.2).

For the function $(T_{n,m,r})_2(q_n, \breve{q}_n, (x,y))$, we have

$$\frac{[m+2][r+2][n]^2}{([m+2]-1)([r+2]-1)} (T_{n,m+1})_2(q_n;(x,y)) = \Big[xD_q (T_{n,m})_2(q_n;x) + [m]x(T_{n,m-1})_2(q_n;q_nx) + [n]\left((1-q_n^m)x + \frac{1}{[n+1]-1} \right)(T_{n,m})_2(q_n;x) - \frac{[n]^2}{[m+1]} \Big\{ (q_n^{-1}-1)(T_{n,m+2})_1(q_n;x) + \left((q_n^m - q_n^{m+1})x - \frac{1}{[n+1]-1} \right)(T_{n,m+1})_1(q_n;x) \Big\} \Big]$$

$$\Big[yD_q\left(T_{n,r}\right)_2(\breve{q}_n;y)+[r]y\left(T_{n,r-1}\right)_2(\breve{q}_n;\breve{q}_n y)+[n]\left((1-\breve{q}_n^{r})x+\frac{1}{[n+1]-1}\right)\left(T_{n,r}\right)_2(\breve{q}_n;y)$$
$$-\frac{[n]^2}{[r+1]}\Big\{(\breve{q}_n^{-1}-1)\left(T_{n,r+2}\right)_1(\breve{q}_n;y)$$
$$+\left((\breve{q}_n^{r}-\breve{q}_n^{r+1})y-\frac{1}{[n+1]-1}\right)\left(T_{n,r+1}\right)_1(\breve{q}_n;y)\Big\}\Big]$$

Proof:
Using Lemma (2.4) and the linearity of the operator R_n , the relation above can be easily follows.

Theorem (3.2.1) .

To prove *Voronovskaya* theory we need the moments of $B_n(f,q_n;x)$ from lemma (2.3).

For $f\in C_h$, suppose that $\frac{\partial_q^2 f(x,y)}{\partial_q^2 x}$, $\frac{\partial_q^2 f(x,y)}{\partial_q^2 y}$ and $\frac{\partial_q^2 f(x,y)}{\partial_q x\partial_q y}$ exist and are continuous at a point $(x,y)\in[0\infty)$, then:

$$\lim_{n\to\infty}[n]\left[R_n\left(f,q_n,\breve{q}_n;(x,y)\right)-f(x,y)\right]$$
$$=\frac{1}{2}f_{xx}(x,y)+\frac{1}{2}f_{yy}(x,y)+xf_{xx}(x,y)+\frac{1}{4}f_{xy}(x,y)+yf_{yy}(x,y)$$

Proof:
By Taylor's formula [8] for f, about the point (x,y) we have:

$$f(t,s)=f(x,y)+f_x(x,y)(t-x)+f_y(x,y)(s-y)$$
$$+\frac{1}{2}\left\{f_{xx}(x,y)(t-x)^2+2f_{xy}(t-x)(s-y)+f_{yy}(x,y)(s-y)^2\right\}$$
$$+\varphi(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4},$$

where $\varphi(t,s;(x,y))=\varphi(t,s)$ is a function from the space $C_{h_1,h_2}([0,\infty)\times[0,\infty))$, and $\varphi(t,s)\to 0$ as $(t,s)\to(x,y)$ thus, $\varphi(x,y)=0$.

Using Lemma (2.3) , we have:

$$G_n\left(f(t,s),q_n,\breve{q}_n;(x,y)\right)$$
$$=f(x,y)\left((q_n-1)[n]x+1\right)\left((\breve{q}_n-1)[n]y+1\right)+f_x(x,y)\left(\frac{q_n^2+q_n-2}{[2]}x+\frac{1}{[2][n]}\right)$$
$$+f_y(x,y)\left(\frac{\breve{q}_n^2+\breve{q}_n-2}{[2]}y+\frac{1}{[2][n]}\right)$$
$$+\frac{1}{2}f_{xx}(x,y)\left(\frac{2q_n^2-q_n-1}{[3]}x^2+\frac{q_n^3+2q_n^2+2q_n-2}{[3][n]}x+\frac{1}{[3][n]^2}\right)$$
$$+f_{xy}(x,y)\left(\frac{q_n^2+q_n-2}{[2]}x+\frac{1}{[2][n]}\right)\left(\frac{\breve{q}_n^2+\breve{q}_n-2}{[2]}y+\frac{1}{[2][n]}\right)$$
$$+\frac{1}{2}f_{yy}(x,y)\left(\frac{2\breve{q}_n^2-\breve{q}_n-1}{[3]}y^2+\frac{\breve{q}_n^3+2\breve{q}_n^2+2\breve{q}_n-2}{[3][n]}y+\frac{1}{[3][n]^2}\right)$$
$$+R_n\left(\varphi(t,s)\sqrt{(t-x)^4+(s-y)^4},q_n,\breve{q}_n;(x,y)\right).$$

Then

$$\lim_{n\to\infty}[n]\left[R_n\left(f(t,s),q_n,\breve{q}_n;(x,y)\right)-f(x,y)\right]$$
$$=\frac{1}{2}f_{xx}(x,y)+\frac{1}{2}f_{yy}(x,y)+xf_{xx}(x,y)+\frac{1}{4}f_{xy}(x,y)+yf_{yy}(x,y)$$
$$+\lim_{n\to\infty}[n]R_n\left(\varphi(t,s)\sqrt{(t-x)^4+(s-y)^4},q_n,\breve{q}_n;(x,y)\right).$$

To complete the proof, we must show that the term

$$[n]R_n\left(\varphi(t,s)\sqrt{(t-x)^4+(s-y)^4},q_n,\breve{q}_n;(x,y)\right)\to 0 \ as\ n\to\infty.$$

This fact we can do it in the same technique of the similar term of theorem (3.2.1)
Therefore,

$$\lim_{n\to\infty}[n]R_n\left(\varphi(t,u)\sqrt{(t-x)^4+(s-y)^4},q_n,\breve{q}_n;(x,y)\right)\to 0.$$

Hence, the proof of is complete.

References

1. S.N. Bernstein, **1912**/13, Démonstration du théoréme de Weierstrass fondéesur le calcul de probabilités,Comm.Soc. Math. Kharkow**13** 1-2.
2. E. Voronovskaya, **1932**, Détermination de la formeasymptotiqued 'ápproximation des functions par les polynômes de S.N. Bernstein, C.R. Adad. Sci. USSR 79-85.
3. O. Szãsz, **1950**, Generalization of S. Bernstein's polynomials to the infinite interval,J.Res.Nat. Bur. Standard, 45239-245.

4. A. Lupaş,**1987**, A q- analogue of the Bernstein operators, University of Cluj- Napoca, Seminar on numerical and statistical calculus, No. 9.85 - 92.

5. L. Rempulsk and M. Skorupka ,**1998**, On Szasz-Mirakyan operators of function of two variables, Matematiche. Vol.LIII Fasc. I, PP. 51-60.

6. A. Aral and V. Gupta, **2006**, The q-derivative and applications to q-Szãsz Mirakjan operators, Calcolo **43** No. 3, 151–170.

7. N. Mahmudove,**2010**, Approximation by q-Szãsz operators, arXiv, Math. FA.1

8. H. Kh. Ghadhban, **2012**, On pproximation of Functions by Linear Positive q-Type Operators, M.S.C. thesis, Univ. of Basrah , Iraq.

9. T. Ernst, **2000**, The history of q-calculus and a new method, U.U.D.M.Report , 16, Uppsala, Department of Mathematics, Uppsala University.

10. V. Kac and P. Cheung ,**2002**, Quantum Calculus, Universitext, Springer-Verlag, New York.

11. İ.Büyükyazıcı , Ertan İbikli, **2004**, The approximation properties of generalized Bernstein polynomials of two variables, Applied Mathematics and Computation **156**, 367-380.

12. H. Kh. Ghadhban, **2012**, On Approximation by q-Szãsz Mirakjan type operators, Iraq-Basrah.