The Dynamics of One Harmful Phytoplankton and Two Competing Zooplankton System

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Abstract
In this paper, a mathematical model consisting of harmful phytoplankton and two competing zooplankton is proposed and studied. The existence of all possible equilibrium points is carried out. The dynamical behaviors of the model system around biologically feasible equilibrium points are studied. Suitable Lyapunov functions are used to construct the basins of attractions of those points. Conditions for which the proposed model persists are established. The occurrence of local bifurcation and a Hopf bifurcation are investigated. Finally, to confirm our obtained analytical results and specify the vital parameters, numerical simulations are used for a hypothetical set of parameter values.

Keywords: phytoplankton-zooplankton, stability, local bifurcation, Hopf bifurcation.

ديناميكية نظام يتكون من عالق نباتي واحد ضار واثنين من العوالق الحيوانية المتنافسة

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الخلاصة
في هذا البحث، تم اقتراح ودراسة نموذج رياضي يتكون من عالق نباتي ضار واثنين من العوالق الحيوانية المتنافسة. تم ايجاد جميع نقاط التوازن الممكنة ومن ثم دراسة السلوك الديناميكي حولها. استخدمنا دوال ليابانوف مناسبة لايجاد احواض التجاذب لتلك النقاط. شروط الاصرار لهذا النموذج وجدت. بحثنا امكانية حدوث التفرع المحلي وتفرع هوبف. وأخيرا، لتأكيد النتائج التحليلية التي تم الحصول عليها وتحديد المعلمات الاساسية، استخدمنا المحاكاة العددية لمجموعة من قيم المعلمات الافتراضية.

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1. Introduction:

Termination of planktonic blooms is of great importance to human health, ecosystem, environment, tourism and fisheries. Toxic substances released by plankton play an important role in this context. The effect of toxin-producing plankton (TPP) on zooplankton is observed from the field-collected samples and mathematical modeling. Information from both the studies led us to suggest that TPP may terminate the planktonic blooms by decreasing the grazing pressure of zooplankton and thus acts as a biological control [1].

The role of toxin producing phytoplankton (TPP) in food web can not be ignored. Reduction of grazing pressure of zooplankton due to release of toxic chemical by phytoplankton plays an important role in the species interaction [2]. TPP act as a strong mediator of zooplankton feeding rate, as shown in both field and laboratory-based studies, see [3,4]. Chattopadhyay et al. [1] observed the effect of TPP on zooplankton population and suggested a suitable control mechanism with the help of mathematical modeling and experimental observation. Chattopadhyay and Sarkar 2003 [5], have been investigated the effect of existence of TPP in a food chain system on their dynamical behavior. Their result suggested that chaotic behavior less likely occurs in a real food chain dynamics. Later on, a number of studies have been conducted to investigate the effects of TPP species on the overall dynamics of phytoplankton and zooplankton, see [6,7,8] and the references therein. Recently, Tanmay Chowdhury et al [9], proposed and studied a mathematical model consisting of non-toxic phytoplankton (NTP) - toxic phytoplankton (TPP) – zooplankton with constant and variable zooplankton migration. They concluded that the migratory grazing of zooplankton has a significant role in determining the dynamic stability and oscillation of phytoplankton zooplankton systems. In this paper however, the effect of toxin producing phytoplankton on the dynamics of two competing zooplankton is considered. It is assumed that the distribution of toxic substance follows either Holling type-I form or Holling type-II form.

2. Mathematical model formulation:-

Consider the simple phytoplankton-zooplankton system with Holling type-II functional response which can be written as:

$$\begin{aligned} \frac{dP}{dT} &= rP\left(1-\frac{P}{K}\right)-\frac{mPZ_1}{\gamma+P} \\ \frac{dZ_1}{dT} &= \frac{m_1PZ_1}{\gamma+P}-\mu_1 Z_1 \end{aligned} \qquad (1)$$

Here $P(T)$ and $Z_1(T)$ represent the densities of phytoplankton and zooplankton at time T respectively. While the parameters r, K, m, γ, m_1 and μ_1 are assumed to be positive parameters and can be described as follows: r represents the intrinsic growth rate of phytoplankton; K is the carrying capacity; m represents the maximum attack rate of zooplankton to the phytoplankton P; γ is the half- saturation constant; m_1 represents the zooplankton conversion rate from phytoplankton P; μ_1 is the natural death rate of zooplankton.

Assume that, the phytoplankton P produces a toxin, as a defensive strategy against the predation from zooplankton, which effect negatively on the growth of the zooplankton. Therefore, the above system can be reformulated as:

$$\begin{aligned} \frac{dP}{dT} &= rP\left(1-\frac{P}{K}\right)-\frac{mPZ_1}{\gamma+P} \\ \frac{dZ_1}{dT} &= \frac{m_1PZ_1}{\gamma+P}-\mu_1 Z_1-\theta_1 f(P)Z_1 \end{aligned} \qquad (2)$$

Here $\theta_1>0$ represents the liberation rate of toxic substance by the harmful phytoplankton P; while $f(P)$ represents the distribution of toxic substance which is assumed to be follows either Holling type-I form (called case 1) or Holling type-II form (called case 2) that means:

$$f(P)=\begin{cases} aP & for\ case\ 1 \\ \frac{a_1P}{\gamma_2+P} & for\ case\ 2 \end{cases} \qquad (3)$$

Here $a>0$ and $a_1>0$ represent the maximum zooplankton ingestion rates for the toxic substance produced by phytoplankton P, while $\gamma_2>0$ is the half- saturation constant of the zooplankton Z_1 by the toxic substance.

Now, if we imposed the following additional assumptions on system (2):

1- There exists another zooplankton, denoted by $Z_2(T)$.

2- The second zooplankton Z_2 consumes the food from phytoplankton according to Holling type-II with maximum attack rate $n>0$ and half-saturation constant $\gamma_1>0$, while $n_1>0$ represents the conversion rate of food from P to the Z_2. Further the second zooplankton is decay exponentially in case of absence of phytoplankton with natural death rate $\mu_2>0$. Finally it is assumed that there is inter-specific competition between the first zooplankton and second zooplankton with competition rates $\alpha>0$ and $\beta>0$ respectively.

3- The phytoplankton P produces a toxic substance that effects on the second zooplankton Z_2 too with the same function $f(P)$ that given in equation (3) but different liberation rate $\theta_2>0$.

Therefore, the above two species system (2) can be extension to three species system and reformulated as:

$$\begin{aligned}\frac{dP}{dT}&=rP\left(1-\frac{P}{K}\right)-\frac{mPZ_1}{\gamma+P}-\frac{nPZ_2}{\gamma_1+P}\\ \frac{dZ_1}{dT}&=\frac{m_1PZ_1}{\gamma+P}-\alpha Z_1Z_2-\mu_1Z_1-\theta_1f(P)Z_1 \qquad (4)\\ \frac{dZ_2}{dT}&=\frac{n_1PZ_2}{\gamma_1+P}-\beta Z_1Z_2-\mu_2Z_2-\theta_2f(P)Z_2\end{aligned}$$

Note that system (4) has 15 parameters for case 1 and 16 parameters for case 2 which make the analysis difficult. Therefore, to reduce the number of parameters and then simplifying our system the following dimensionless variables are used

$$t=rT,\ x=\frac{P}{K},\ z_1=\frac{mZ_1}{rK}\ \ and\ \ z_2=\frac{nZ_2}{rK}$$

Therefore, substituting these new variables in system (4) and then simplifying the resulting terms. We obtain the following dimensionless system:

$$\begin{aligned}\frac{dx}{dt}&=x(1-x)-\frac{xz_1}{\omega_1+x}-\frac{xz_2}{\omega_2+x}=xg_1(x,z_1,z_2)\\ \frac{dz_1}{dt}&=\frac{\omega_3xz_1}{\omega_1+x}-\omega_4z_1z_2-\omega_5z_1-\omega_6f(x)z_1\\ &=z_1g_2(x,z_1,z_2)\\ \frac{dz_2}{dt}&=\frac{\omega_7xz_2}{\omega_2+x}-\omega_8z_1z_2-\omega_9z_2-\omega_{10}f(x)z_2\\ &=z_2g_3(x,z_1,z_2)\end{aligned} \qquad \ldots \quad (5)$$

where:

$$f(x)=\begin{cases}\omega_{11}x & for\ case\ 1\\ \frac{a_1x}{\overline{\omega}_{11}+x} & for\ case\ 2\end{cases} \qquad (6)$$

with $\omega_1=\frac{\gamma}{K}$, $\omega_2=\frac{\gamma_1}{K}$, $\omega_3=\frac{m_1}{r}$, $\omega_4=\frac{\alpha K}{n}$, $\omega_5=\frac{\mu_1}{r}$, $\omega_6=\frac{\theta_1}{r}$, $\omega_7=\frac{n_1}{r}$, $\omega_8=\frac{\beta K}{m}$, $\omega_9=\frac{\mu_2}{r}$, $\omega_{10}=\frac{\theta_2}{r}$, $\omega_{11}=aK$, $\overline{\omega}_{11}=\frac{\gamma_2}{K}$.
represent the dimensionless parameters.

Clearly, system (5) contains 11 parameters in case 1 and 12 parameters for case 2 which may make the analysis of system (5) easier. The initial condition for system (5) may be taken as any point in the region $R_+^3=\{(x,z_1,z_2):x\geq 0,z_1\geq 0,z_2\geq 0\}$. Obviously, the interaction functions in the right hand side of system (5) are continuously differentiable functions on R_+^3, hence they are Lipschitizian. Therefore the solution of system (5) exists and is unique. Further, all the solutions of system (5) with non-negative initial condition are uniformly bounded as shown in the following theorem.

Theorem 1. All the solutions of the system (5), which initiate in R_+^3 are uniformly bounded.

Proof. Let $(x(t),z_1(t),z_2(t))$ be any solution of the system (5). Since

$$\frac{dx}{dt}\leq x(1-x)$$

Thus by solving the differential inequality:

$$\lim_{t\to\infty} Sup\, x(t)\leq 1\Rightarrow x(t)\leq 1,\forall t>0$$

Now, consider the function:

$$W_1(x,z_1,z_2)=\omega_3\omega_7x+\omega_7z_1+\omega_3z_2$$

Then the time derivative of $W_1(t)$ along the solution of the system (5) is:

$$\frac{dW_1}{dt}+DW_1\leq D_1$$

where $D=\min\{1,\omega_5,\omega_9\}$, $D_1=2\omega_3\omega_7$.

By comparing the above differential inequality with the associated linear differential equation, we obtain:

$$0<W_1\leq\frac{D_1}{D}(1-e^{-Dt})+w_{10}\,e^{-Dt}$$

Where $W_1(0)=W_{10}$ and hence we get:

$$0<W_1\leq\frac{D_1}{D},\ \text{as}\ t\to\infty$$

Hence, all the solutions of system (5) are uniformly bounded, and then the proof is complete. ■

According to the above theorem system (5) is dissipative system.

3. Existence of equilibrium points and stability analysis.

The system (5) have at most five non-negative equilibrium points, two of them namely $F_0 = (0,0,0)$, $F_x = (1,0,0)$ always exist. While the existence of other equilibrium points are shown in the following:

The second zooplankton free equilibrium point $F_{xz_1} = (\hat{x}, \hat{z}_1, 0)$ exists in $Int.R_+^2$ of $xz_1 -$plane**,** where

$$\hat{z}_1 = (1-\hat{x})(\omega_1 + \hat{x}) \quad (7)$$

while $\hat{x}$ in case 1, represents the positive root to the following equation:

$$e_1 x^2 + e_2 x + e_3 = 0 \quad (8)$$

where $e_1 = \omega_6 \omega_{11} > 0$, $e_3 = \omega_1 \omega_5 > 0$ and $e_2 = \omega_5 - \omega_3 + \omega_1 \omega_6 \omega_{11}$. So by using Descartes rule of signs, Eq. (8) has either no positive root and hence there is no equilibrium point or two positive roots given by:

$$\hat{x}_1, \hat{x}_2 = \tfrac{-e_2}{2e_1} \pm \tfrac{\sqrt{e_2^2 - 4e_1 e_3}}{2e_1} \quad (9a)$$

Clearly $\hat{x}_1$ and $\hat{x}_2$ are positive provided that:

$$e_2 < 0 \Rightarrow \omega_5 + \omega_1 \omega_6 \omega_{11} < \omega_3 \quad (9b)$$

$$e_2^2 > 4 e_1 e_3 \quad (9c)$$

and then, by substituting $\hat{x}_i$, $i = 1,2$ in Eq. (7), there exist two second zooplankton free equilibrium points in the $Int.R_+^2$ of $xz_1 -$plane namely $F_{x_1 z_{11}}$ and $F_{x_2 z_{12}}$**,** provided that

$$\hat{x}_i < 1 \text{ for } i = 1,2. \quad (10)$$

Now for case 2, $\hat{x}$ in Eq. (7) represents a positive root to the following equation:

$$e_4 x^2 + e_5 x + e_6 = 0 \quad (11)$$

where $e_4 = \omega_3 - \omega_5 - a_1 \omega_6$, $e_6 = -\omega_1 \omega_5 \overline{\omega}_{11} < 0$ and $e_5 = \omega_3 \overline{\omega}_{11} - \omega_1 \omega_5 - \omega_5 \overline{\omega}_{11} - a_1 \omega_1 \omega_6$. So by using Descartes rule of signs, Eq. (11) has a positive root given by:

$$\hat{x} = \tfrac{-e_5}{2e_4} + \tfrac{\sqrt{e_5^2 - 4e_4 e_6}}{2e_4} \quad (12a)$$

provided that the following condition holds:

$$e_4 > 0 \Rightarrow \omega_3 > \omega_5 + a_1 \omega_6 \quad (12b)$$

Therefore, by substituting $\hat{x}$ in Eq. (7), system (5) has a unique second zooplankton free equilibrium point in the $Int.R_+^2$ of $xz_1 -$plane denoted by F_{xz_1}**,** provided that

$$\hat{x} < 1 \quad (13)$$

The first zooplankton free equilibrium point $F_{xz_2} = (\tilde{x}, 0, \tilde{z}_2)$ exists in $Int.R_+^2$ of xz_2 - plane

$$\tilde{z}_2 = (1-\tilde{x})(\omega_2 + \tilde{x}) \quad (14)$$

while $\tilde{x}$ in case1, represents the positive root to the following equation:

$$e_7 x^2 + e_8 x + e_9 = 0 \quad (15)$$

here $e_7 = \omega_{10} \omega_{11} > 0$, $e_9 = \omega_2 \omega_9 > 0$ and $e_8 = \omega_9 - \omega_7 + \omega_2 \omega_{10} \omega_{11}$. So by using Descartes rule of signs, Eq. (15) has either no positive root and hence there is no equilibrium point or two positive roots given by:

$$\tilde{x}_1, \tilde{x}_2 = \tfrac{-e_8}{2e_7} \pm \tfrac{\sqrt{e_8^2 - 4e_7 e_9}}{2e_7} \quad (16a)$$

Clearly $\tilde{x}_1$ and $\tilde{x}_2$ are positive provided that

$$e_8 < 0 \Rightarrow \omega_9 + \omega_2 \omega_{10} \omega_{11} < \omega_7 \quad (16b)$$

$$e_8^2 > 4 e_7 e_9 \quad (16c)$$

and then, by substituting $\tilde{x}_i$, $i = 1,2$ in Eq. (14), there exist two first zooplankton free equilibrium points in the $Int.R_+^2$ of $xz_2 -$plane namely $F_{x_1 z_{21}}$ and $F_{x_2 z_{22}}$**,** provided that

$$\tilde{x}_i < 1 \text{ for } i = 1,2. \quad (17)$$

Now for case2, $\tilde{x}$ in Eq. (14) represents the positive root to the following equation:

$$e_{10} x^2 + e_{11} x + e_{12} = 0 \quad (18)$$

where $e_{10} = \omega_7 - \omega_9 - a_1 \omega_{10}$, $e_{12} = -\omega_2 \omega_9 \overline{\omega}_{11} < 0$ and $e_{11} = \omega_7 \overline{\omega}_{11} - \omega_2 \omega_9 - \omega_9 \overline{\omega}_{11} - a_1 \omega_2 \omega_{10}$. So by using Descartes rule of signs, Eq. (18) has a positive root given by:

$$\tilde{x} = \tfrac{-e_{11}}{2e_{10}} + \tfrac{\sqrt{e_{11}^2 - 4e_{10} e_{12}}}{2e_{10}} \quad (19a)$$

provided that the following condition holds:

$$e_{10} > 0 \Rightarrow \omega_7 > \omega_9 + a_1 \omega_{10} \quad (19b)$$

Therefore, by substituting $\tilde{x}$ in Eq. (14), system (5) has a unique first zooplankton free equilibrium point in the $Int.R_+^2$ of $xz_2 -$plane denoted by F_{xz_2}**,** provided that

$$\tilde{x} < 1 \quad (20)$$

Finally the coexistence equilibrium point $F_{xz_1 z_2} = (x^*, z_1^*, z_2^*)$ exists in $Int.R_+^3$**,** for case 1, where

$$z_2^* = \tfrac{1}{\omega_4(\omega_1 + x^*)}\Big[-\omega_6 \omega_{11} x^{*2} + (\omega_3 - \omega_5 - \omega_1 \omega_6 \omega_{11}) x^* - \omega_1 \omega_5\Big] \quad \ldots (21a)$$

and

$$z_1^* = \frac{1}{\omega_8(\omega_2+x^*)}\Big[-\omega_{10}\omega_{11}x^{*2} + (\omega_7-\omega_9-\omega_2\omega_{10}\omega_{11})x^* - \omega_2\omega_9\Big] \quad \ldots (21b)$$

Clearly z_2^* is positive under the following two conditions:

$$\omega_3 > \omega_5 + \omega_1\omega_6\omega_{11} \quad (22a)$$

$$(\omega_3-\omega_5-\omega_1\omega_6\omega_{11})x^* > \omega_1\omega_5 + \omega_6\omega_{11}x^{*2} \quad (22b)$$

and z_1^* is positive under the following two conditions:

$$\omega_7 > \omega_9 + \omega_2\omega_{10}\omega_{11} \quad (23a)$$

$$(\omega_7-\omega_9-\omega_2\omega_{10}\omega_{11})x^* > \omega_2\omega_9 + \omega_{10}\omega_{11}x^{*2} \quad \ldots (23b)$$

However x^* represents the positive root of the following equation:

$$s_1x^3 + s_2x^2 + s_3x + s_4 = 0 \quad (24)$$

where:

$$s_1 = -\omega_4\omega_8 < 0$$
$$s_2 = \omega_4\omega_8(1-\omega_1-\omega_2) + \omega_{11}(\omega_4\omega_{10}+\omega_6\omega_8)$$
$$s_3 = \omega_4[\omega_8(\omega_1+\omega_2(1-\omega_1)) + \omega_9 - \omega_7 + \omega_2\omega_{10}\omega_{11}] + \omega_8(\omega_5-\omega_3+\omega_1\omega_6\omega_{11})$$
$$s_4 = \omega_2\omega_4(\omega_1\omega_8+\omega_9) + \omega_1\omega_5\omega_8 > 0$$

So by using Descartes rule of signs, Eq. (24) has a unique positive root say x^* provided that at least one of the following two conditions hold:

$$s_2 < 0 \quad (25a)$$

$$s_3 > 0 \quad (25b)$$

On the other hand, **in case 2**, where

$$z_2^* = \frac{1`}{\omega_4(\omega_1+x^*)(\overline{\omega}_{11}+x^*)}\big[(\omega_3-\omega_5-a_1\omega_6)x^{*2} + (\omega_3\overline{\omega}_{11}-\omega_1\omega_5-\omega_5\overline{\omega}_{11} - a_1\omega_1\omega_6)x^* - \omega_1\omega_5\overline{\omega}_{11}\big] \quad \ldots (26a)$$

and

$$z_1^* = \frac{1}{\omega_8(\omega_2+x^*)(\overline{\omega}_{11}+x^*)}\big[(\omega_7-\omega_9-a_1\omega_{10})x^{*2} + (\omega_7\overline{\omega}_{11}-\omega_2\omega_9-\omega_9\overline{\omega}_{11} - a_1\omega_2\omega_{10})x^* - \omega_2\omega_9\overline{\omega}_{11}\big] \quad \ldots (26b)$$

Again, it is clear that z_2^* is positive under the conditions:

$$\omega_3 > \omega_5 + a_1\omega_6 \quad (27a)$$

$$\omega_3\overline{\omega}_{11} > \omega_1\omega_5 + \omega_5\overline{\omega}_{11} + a_1\omega_1\omega_6 \quad (27b)$$

$$(\omega_3-\omega_5-a_1\omega_6)x^{*2} + (\omega_3\overline{\omega}_{11}-\omega_1\omega_5 - \omega_5\overline{\omega}_{11} - a_1\omega_1\omega_6)x^* > \omega_1\omega_5\overline{\omega}_{11} \quad \ldots (27c)$$

and z_1^* is positive under the conditions:

$$\omega_7 > \omega_9 + a_1\omega_{10} \quad (28a)$$

$$\omega_7\overline{\omega}_{11} > \omega_2\omega_9 + \omega_9\overline{\omega}_{11} + a_1\omega_2\omega_{10} \quad (28b)$$

$$(\omega_7-\omega_9-a_1\omega_{10})x^{*2} + (\omega_7\overline{\omega}_{11}-\omega_2\omega_9 - \omega_9\overline{\omega}_{11} - a_1\omega_2\omega_{10})x^* > \omega_2\omega_9\overline{\omega}_{11} \quad \ldots (28c)$$

While x^* represents the positive root of the following equation:

$$q_1x^4 + q_2x^3 + q_3x^2 + q_4x + q_5 = 0 \quad (29)$$

here:

$$q_1 = -\omega_4\omega_8 < 0$$
$$q_2 = \omega_4\omega_8(1-\omega_1-\omega_2-\overline{\omega}_{11})$$
$$q_3 = \omega_4\omega_8\big[\omega_1(1-\omega_2-\overline{\omega}_{11}) + \omega_2(1-\overline{\omega}_{11}) + \overline{\omega}_{11}\big] - \omega_4(\omega_7-\omega_9-a_1\omega_{10}) - \omega_8(\omega_3-\omega_5-a_1\omega_6)$$
$$q_4 = \omega_4\omega_8\big[\omega_1(\omega_2+\overline{\omega}_{11}) + \omega_2\overline{\omega}_{11}(1-\omega_1)\big] - \omega_4(\omega_7\overline{\omega}_{11}-\omega_2\omega_9 - \omega_9\overline{\omega}_{11} - a_1\omega_2\omega_{10}) - \omega_8(\omega_3\overline{\omega}_{11}-\omega_1\omega_5-\omega_5\overline{\omega}_{11}-a_1\omega_1\omega_6)$$
$$q_5 = \overline{\omega}_{11}\big[\omega_2\omega_4(\omega_1\omega_8+\omega_9) + \omega_1\omega_5\omega_8\big]$$

So by using Descartes rule of signs, Eq. (29) has a unique positive root say x^* provided that one set of the following sets of conditions hold:

$$q_2 < 0,\ q_4 > 0 \quad (30a)$$

$$q_2 < 0,\ q_3 < 0,\ q_4 < 0 \quad (30b)$$

$$q_2 > 0,\ q_3 > 0,\ q_4 > 0 \quad (30c)$$

In the following, the local dynamical behavior of the system (5) around each of the above equilibrium points is discussed. First the Jacobian matrix of system (5) at each of these points is determined and then the eigenvalues for the resulting matrix are computed and then the obtained results are summarized in the following:

The Jacobian matrix of system (5) at the equilibrium point $F_0 = (0,0,0)$ can be written as $J_0 = J(F_0) = [c_{ij}]_{3\times3}; i,j = 1,2,3$, where $c_{11} = 1$, $c_{22} = -\omega_5$, $c_{33} = -\omega_9$ and zero otherwise. Then the eigenvalues of J_0 are:

$$\lambda_{01} = 1 > 0,\ \lambda_{02} = -\omega_5 < 0 \text{ and } \lambda_{03} = -\omega_9 < 0$$

Therefore, the equilibrium point F_0 is a saddle point.

The Jacobian matrix of system (5) at the equilibrium point $F_x=(1,0,0)$ can be written as $J_x=J(F_x)=[d_{ij}]_{3\times3}\,;i,j=1,2,3$, where $d_{11}=-1$, $d_{12}=\frac{-1}{\omega_1+1}$, $d_{13}=\frac{-1}{\omega_2+1}$, $d_{22}=\frac{\omega_3}{\omega_1+1}-\omega_5-\omega_6 f(1)$, $d_{33}=\frac{\omega_7}{\omega_2+1}-\omega_9-\omega_{10}f(1)$ and zero otherwise.

Hence, the eigenvalues of J_x are:

$\breve{\lambda}_1=-1<0$, $\breve{\lambda}_2=\frac{\omega_3}{\omega_1+1}-\omega_5-\omega_6 f(1)$

$\breve{\lambda}_3=\frac{\omega_7}{\omega_2+1}-\omega_9-\omega_{10}f(1)$

where $f(1)$ is obtained from Eq. (6) by substituting $x=1$. Clearly, F_x is locally asymptotically stable in the R_+^3 if the following two conditions are satisfied

$$\frac{\omega_3}{\omega_1+1}<\omega_5+\omega_6 f(1) \tag{31a}$$

and

$$\frac{\omega_7}{\omega_2+1}<\omega_9+\omega_{10}f(1) \tag{31b}$$

However, F_x is a saddle point in the R_+^3 if at least one of the following two conditions are satisfied:

$$\frac{\omega_3}{\omega_1+1}>\omega_5+\omega_6 f(1) \tag{31c}$$

and

$$\frac{\omega_7}{\omega_2+1}>\omega_9+\omega_{10}f(1) \tag{31d}$$

Before we go further to analyze the dynamical behavior of system (5) in the neighborhood of the second zooplankton free equilibrium point, recall that the system have either two equilibrium points $F_{x_1z_{11}}$ and $F_{x_2z_{12}}$ or there is no equilibrium point in case1.While, it has a unique equilibrium point F_{xz_1} in case 2. Since all these equilibrium points, whenever they exist have the same locally stability conditions which depend on the form of equilibrium points, therefore we assume here F_{xz_1} represent any one of them that belongs to xz_1-plane.

So, the Jacobian matrix of system (5) at the second zooplankton free equilibrium point $F_{xz_1}=(\hat{x},\hat{z}_1,0)$ in xz_1-plane, can be written in the form: $J_{xz_1}=J(F_{xz_1})=[f_{ij}]_{3\times3}\,;i,j=1,2,3$, where

$f_{11}=\hat{x}\left(-1+\frac{\hat{z}_1}{(\omega_1+\hat{x})^2}\right)$, $f_{12}=\frac{-\hat{x}}{\omega_1+\hat{x}}$, $f_{13}=\frac{-\hat{x}}{\omega_2+\hat{x}}$,

$f_{21}=\hat{z}_1\left(\frac{\omega_1\omega_3}{(\omega_1+\hat{x})^2}-\omega_6 f'(\hat{x})\right)$, $f_{23}=-\omega_4\hat{z}_1$,

$f_{33}=\frac{\omega_7\hat{x}}{\omega_2+\hat{x}}-\omega_8\hat{z}_1-\omega_9-\omega_{10}f(\hat{x})$ and zero otherwise. Therefore, the eigenvalues of J_{xz_1} are given by:

$$\hat{\lambda}_{1,2}=\frac{\hat{x}}{2}\left(-1+\frac{\hat{z}_1}{(\omega_1+\hat{x})^2}\right)\pm\frac{\sqrt{\hat{x}^2\left(-1+\frac{\hat{z}_1}{(\omega_1+\hat{x})^2}\right)^2-4\frac{\hat{x}\hat{z}_1}{\omega_1+\hat{x}}\left(\frac{\omega_1\omega_3}{(\omega_1+\hat{x})^2}-\omega_6 f'(\hat{x})\right)}}{2}$$

and

$$\hat{\lambda}_3=\frac{\omega_7\hat{x}}{\omega_2+\hat{x}}-\omega_8\hat{z}_1-\omega_9-\omega_{10}f(\hat{x})$$

where $f(\hat{x})$ is obtained from Eq. (6) by substituting $x=\hat{x}$ and

$$f'(\hat{x})=\frac{d}{dx}f(x)\Big|_{x=\hat{x}} \tag{32}$$

Consequently, F_{xz_1} is locally asymptotically stable in the R_+^3 if the following conditions are satisfied:

$$\hat{z}_1<(\omega_1+\hat{x})^2 \tag{33a}$$

$$\omega_1\omega_3>\omega_6 f'(\hat{x})(\omega_1+\hat{x})^2 \tag{33b}$$

and

$$\frac{\omega_7\hat{x}}{\omega_2+\hat{x}}<\omega_8\hat{z}_1+\omega_9+\omega_{10}f(\hat{x}) \tag{33c}$$

Obviously, conditions (33a) and (33b) guarantee the local stability of F_{xz_1} in the $Int.R_+^2$ of the xz_1-plane. However, F_{xz_1} will be unstable point in the R_+^3 if we reversed any one of the above conditions.

Similarly, it is assumed that, F_{xz_2} represent any one of the first zooplankton free of the equilibrium points those may belong to xz_2-plane. Hence the Jacobian matrix of system (5) at the first zooplankton free equilibrium point $F_{xz_2}=(\tilde{x},0,\tilde{z}_2)$ in xz_2-plane, can be written in the form: $J_{xz_2}=J(F_{xz_2})=[h_{ij}]_{3\times3};i,j=1,2,3$, where

$h_{11}=\tilde{x}\left(-1+\frac{\tilde{z}_2}{(\omega_2+\tilde{x})^2}\right)$, $h_{12}=\frac{-\tilde{x}}{\omega_1+\tilde{x}}$, $h_{13}=\frac{-\tilde{x}}{\omega_2+\tilde{x}}$,

$h_{22}=\frac{\omega_3\tilde{x}}{\omega_1+\tilde{x}}-\omega_4\tilde{z}_2-\omega_5-\omega_6 f(\tilde{x})$

$h_{31}=\tilde{z}_2\left(\frac{\omega_2\omega_7}{(\omega_2+\tilde{x})^2}-\omega_{10}f'(\tilde{x})\right)$,

$h_{32}=-\omega_8\tilde{z}_2$ and zero otherwise. Hence, the eigenvalues of F_{xz_2} are given by:

$$\tilde{\lambda}_{1,3} = \tfrac{\tilde{x}}{2}\left(-1+\tfrac{\tilde{z}_2}{(\omega_2+\tilde{x})^2}\right) \pm \frac{\sqrt{\tilde{x}^2\left(-1+\tfrac{\tilde{z}_2}{(\omega_2+\tilde{x})^2}\right)^2 - 4\,\tfrac{\tilde{x}\tilde{z}_2}{\omega_2+\tilde{x}}\left(\tfrac{\omega_2\omega_7}{(\omega_2+\tilde{x})^2} - \omega_{10} f'(\tilde{x})\right)}}{2}$$

and

$$\tilde{\lambda}_2 = \frac{\omega_3\tilde{x}}{\omega_1+\tilde{x}} - \omega_4\tilde{z}_2 - \omega_5 - \omega_6 f(\tilde{x})$$

where $f(\tilde{x})$ is obtained from Eq. (6) by substituting $x=\tilde{x}$ and

$$f'(\tilde{x}) = \tfrac{d}{dx} f(x)\big|_{x=\tilde{x}} \qquad (34)$$

Consequently, F_{xz_2} is locally asymptotically stable in the R^3_+ if the following conditions are satisfied:

$$\tilde{z}_2 < (\omega_2+\tilde{x})^2 \qquad (35a)$$

$$\omega_2\omega_7 > \omega_{10} f'(\tilde{x})(\omega_2+\tilde{x})^2 \qquad (35b)$$

and

$$\frac{\omega_3\tilde{x}}{\omega_1+\tilde{x}} < \omega_4\tilde{z}_2 + \omega_5 + \omega_6 f(\tilde{x}) \qquad (35c)$$

Obviously, conditions (35a) and (35b) guarantee the local stability of F_{xz_2} in the $Int.R^2_+$ of the xz_2 –plane. While, F_{xz_2} is unstable point in the R^3_+ if we reversed any one of the above conditions.

Finally, the Jacobian matrix of the system (5) at the positive equilibrium point $F_{xz_1z_2} = (x^*, z_1^*, z_2^*)$ in the $Int.R^3_+$ can be written as:

$$J_{xz_1z_2} = J(F_{xz_1z_2}) = \lfloor a_{ij} \rfloor_{3\times 3}; i,j = 1,2,3 \qquad (36)$$

where $a_{11} = \frac{x^*B_3}{B_1^2B_2^2}$, $a_{12} = -\frac{x^*}{B_1} < 0$, $a_{13} = -\frac{x^*}{B_2} < 0$, $a_{21} = \frac{z_1^*B_4}{B_1^2}$, $a_{22} = 0$, $a_{23} = -\omega_4 z_1^* < 0$, $a_{31} = \frac{z_2^*B_5}{B_2^2}$, $a_{32} = -\omega_8 z_2^* < 0$; $a_{33} = 0$ with $B_1 = \omega_1 + x^* > 0$, $B_2 = \omega_2 + x^* > 0$, $B_3 = -B_1^2B_2^2 + z_1^*B_2^2 + z_2^*B_1^2$, $B_4 = \omega_1\omega_3 - \omega_6 B_1^2 f'(x^*)$ and $B_5 = \omega_2\omega_7 - \omega_{10}B_2^2 f'(x^*)$.

Accordingly the characteristic equation of $J_{xz_1z_2}$ can be written as:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \qquad (37)$$

here

$$A_1 = -a_{11}$$
$$A_2 = -a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31}$$
$$A_3 = a_{32}(a_{11}a_{23} - a_{13}a_{21}) - a_{31}a_{12}a_{23}$$

and

$$\Delta = A_1A_2 - A_3 = a_{12}(a_{11}a_{21} + a_{31}a_{23}) + a_{13}(a_{11}a_{31} + a_{32}a_{21})$$

So, by substituting the value of a_{ij}, and then simplifying the resulting terms we obtain:

$$A_1 = -\tfrac{x^*B_3}{B_1^2B_2^2} \qquad (38a)$$

$$A_3 = \tfrac{x^*z_1^*z_2^*}{B_1^2B_2^2}\left[\omega_4\omega_8B_3 - \omega_8B_2B_4 - \omega_4B_1B_5\right] \quad \ldots (38b)$$

$$\Delta = -\tfrac{x^*z_1^*}{B_1B_2^2}\left[\tfrac{x^*}{B_1^4}B_3B_4 - \omega_4 z_2^* B_5\right] - \tfrac{x^*z_2^*}{B_1^2B_2}\left[\tfrac{x^*}{B_2^4}B_3B_5 - \omega_8 z_1^* B_4\right] \quad \ldots (38c)$$

Therefore, in the following theorem, the local stability conditions for the positive equilibrium point $F_{xz_1z_2}$ in the $Int.R^3_+$ are established.

Theorem 2. Assume that $F_{xz_1z_2}$ exists in the $Int.R^3_+$ and the following conditions are satisfied;

$$z_1^*B_2^2 + z_2^*B_1^2 < B_1^2B_2^2 \qquad (39a)$$

$$\tfrac{\omega_1\omega_3}{\omega_6B_1^2} < f'(x^*) < \tfrac{\omega_2\omega_7}{\omega_{10}B_2^2} \qquad (39b)$$

$$\omega_8(\omega_4B_3 - B_2B_4) > \omega_4B_1B_5 \qquad (39c)$$

and

$$B_5 > \max.\left\{\tfrac{x^*B_3B_4}{\omega_4 z_2^* B_1^4}, \tfrac{\omega_8 z_1^* B_4 B_2^4}{x^*B_3}\right\} \qquad (39d)$$

Then it is locally asymptotically stable.

Proof. According to the Routh-Hawirtiz criterion the characteristic equation (37) has roots with negative real parts if and only if $A_1 > 0$, $A_3 > 0$ and $\Delta > 0$. Note that, it is easy to verify that, condition (39a) guarantees that $B_3 < 0$ and hence $A_1 > 0$; while conditions (39b) ensure that $B_4 < 0$ and $B_5 > 0$, hence A_3 will be positive provided that conditions (39a)-(39c) hold. Further, the conditions (39a)-(39b) with condition (39d) guarantee that $\Delta > 0$. Hence, all the roots (eigenvalues) of the $J_{xz_1z_2}$ have negative real parts.

Therefore $F_{xz_1z_2}$ is locally asymptotically stable in the $Int.R_+^3$ and hence the proof is complete. ■

Now, before go further to study the global dynamical behavior of system (5) in the $Int.R_+^3$, we will discuss the dynamical behavior of system (5) in the interior of the boundary planes as shown in the following theorems.

Theorem 3. System (5) has no periodic dynamics in the $Int.R_+^2$ of $xz_1-$and xz_2-planes provided that

$$z_1 < (\omega_1 + x)^2 \tag{40}$$

$$z_2 < (\omega_2 + x)^2 \tag{41}$$

respectively.

Proof. The proof follows directly by using Bendixson-Dulic criterion with Dulic functions $1/xz_1$ and $1/xz_2$ respectively. ■

Keeping the above in view, Since all the solutions of the system (5) are bounded and F_{xz_1} and F_{xz_2} (for case 2) are the unique positive equilibrium points in $Int.R_+^2$ of the xz_1- and xz_2- planes respectively, hence by using the Poincare-Bendixson theorem F_{xz_1} and F_{xz_2} are globally asymptotically stable in the $Int.R_+^2$ of xz_1-plane and xz_2- plane respectively.

4. Global stability of the system.

In this section the global stability of the equilibrium points F_x, F_{xz_1}, F_{xz_2} and $F_{xz_1z_2}$ in R_+^3 are investigated as shown in the following theorems.

Theorem 4. Assume that the equilibrium point F_x is locally asymptotically stable in the R_+^3, and let the following conditions:

$$\frac{\omega_5}{\omega_3} \geq \frac{1}{\omega_1} \tag{42a}$$

$$\frac{\omega_9}{\omega_7} \geq \frac{1}{\omega_2} \tag{42b}$$

hold, then F_x is globally asymptotically stable in the R_+^3.

Proof. Consider the following positive definite function:

$$U_1(x,z_1,z_2) = c_1\left(x-1-\ln x\right) + c_2 z_1 + c_3 z_2$$

Clearly $U_1 : R_+^3 \to R$, and is a C^1 positive definite function, where $c_i, (i=1,2,3)$ are nonnegative constants to be determined. Now, since the derivative of U_1 along the trajectory of system (5) can be written as:

$$\frac{dU_1}{dt} < -c_1(x-1)^2 - (c_1 - c_2\omega_3)\frac{xz_1}{\omega_1 + x} - (c_1 - c_3\omega_7)\frac{xz_2}{\omega_2 + x} - \left(c_2\omega_5 - \frac{c_1}{\omega_1 + x}\right)z_1 - \left(c_3\omega_9 - \frac{c_1}{\omega_2 + x}\right)z_2 - (c_2\omega_4 + c_3\omega_8)z_1z_2$$

So, by choosing the nonnegative constants as $c_1 = 1$, $c_2 = \frac{1}{\omega_3}$ and $c_3 = \frac{1}{\omega_7}$ gives:

$$\frac{dU_1}{dt} \leq -(x-1)^2 - \left(\frac{\omega_5}{\omega_3} - \frac{1}{\omega_1}\right)z_1 - \left(\frac{\omega_9}{\omega_7} - \frac{1}{\omega_2}\right)z_2 - \left(\frac{\omega_4}{\omega_3} + \frac{\omega_8}{\omega_7}\right)z_1z_2$$

Therefore, $\frac{dU_1}{dt} < 0$ under conditions (42a) and (42b), and hence U_1 is strictly Lyapunov function. Therefore, F_x is globally asymptotically stable in the R_+^3. ■

Now, since system (5) in case 1, may have either two equilibrium points or no equilibrium points in the $Int.R_+^3$ of the xz_1- and xz_2- planes respectively. Therefore, in the following two theorems we will study the global dynamics of system (5) in these planes for case 2 only.

Theorem 5. Assume that the second zooplankton free equilibrium point F_{xz_1} is locally asymptotically stable in R_+^3. Then the basin of attraction of F_{xz_1} is given by:

$$B(F_{xz_1}) = \left\{(x,z_1,z_2) \in R_+^3 : x > \hat{x}, z_1 > \hat{z}_1, z_2 \geq 0\right\}$$

provided that:

$$\hat{z}_1 < \omega_1(\omega_1 + \hat{x}) \tag{43a}$$

$$\omega_3(\overline{\omega}_{11} + 1)(\overline{\omega}_{11} + \hat{x}) > a_1\omega_6\overline{\omega}_{11}(\omega_1 + \hat{x}) \tag{43b}$$

Proof. Follows directly by using the candidate Lyapunov function

$$U_2(x,z_1,z_2)=c_1\left(x-\hat{x}-\hat{x}\ln\frac{x}{\hat{x}}\right)+c_2\left(z_1-\hat{z}_1-\hat{z}_1\ln\frac{z_1}{\hat{z}_1}\right)+c_3 z_2$$ ■

Theorem 6. Assume that the first zooplankton free equilibrium point F_{xz_2} is locally asymptotically stable in R_+^3. Then the basin of attraction of F_{xz_2} is given by:

$$B(F_{xz_2})=\left\{(x,z_1,z_2)\in R_+^3: x>\tilde{x}, z_1\geq 0, z_2>\tilde{z}_2\right\}$$

provided that:

$$\tilde{z}_2<\omega_2(\omega_2+\tilde{x}) \quad (44a)$$

$$\omega_7(\overline{\omega}_{11}+1)(\overline{\omega}_{11}+\tilde{x})>a_1\omega_{10}\overline{\omega}_{11}(\omega_2+\tilde{x}) \quad (44b)$$

Proof. Follows directly by using the candidate Lyapunov function

$$U_3(x,z_1,z_2)=c_1\left(x-\tilde{x}-\tilde{x}\ln\frac{x}{\tilde{x}}\right)+c_2 z_1+c_3\left(z_2-\tilde{z}_2-\tilde{z}_2\ln\frac{z_2}{\tilde{z}_2}\right)$$ ■

Theorem 7. Assume that the coexistence equilibrium point $F_{xz_1z_2}$ is locally asymptotically stable in $Int.R_+^3$. Then the basin of attraction of $F_{xz_1z_2}$ is given by:

$$B(E_{xz_1z_2})=\left\{(x,z_1,z_2): x>x^*, z_1>z_1^*, z_2>z_2^*\right\}$$

provided that:

$$\omega_2(\omega_2+x^*)z_1^*+\omega_1(\omega_1+x^*)z_2^*<\omega_1\omega_2(\omega_1+x^*)(\omega_2+x^*) \quad \ldots (45a)$$

$$\omega_3>\omega_6\omega_{11}(\omega_1+x^*) \quad (45b)$$

$$\omega_7>\omega_{10}\omega_{11}(\omega_2+x^*) \quad (45c)$$

Proof. Follows directly by using the candidate Lyapunov function

$$U_4(x,z_1,z_2)=c_1\left(x-x^*-x^*\ln\frac{x}{x^*}\right)+c_2\left(z_1-z_1^*-z_1^*\ln\frac{z_1}{z_1^*}\right)+c_3\left(z_2-z_2^*-z_2^*\ln\frac{z_2}{z_2^*}\right)$$ ■

5. Persistence Analysis.

In this section, the persistence of system (5) is studied. It is well known that the system is said to be persistence if and only if each species persists. Mathematically this is meaning that the solution of system (5) do not have omega limit set in the boundaries of R_+^3 Gard and Hallam [10]. Therefore, in the following theorem, the necessary and sufficient conditions for the uniform persistence of the system (5) are derived.

Theorem 8. Assume that there are no periodic dynamics in the boundary planes xz_1 and xz_2 respectively. Further, if in addition to conditions (31c), (31d) the following conditions are hold.

$$\frac{\omega_7\hat{x}}{\omega_2+\hat{x}}>\omega_8\hat{z}_1+\omega_9+\omega_{10}f(\hat{x}) \quad (46)$$

and

$$\frac{\omega_3\tilde{x}}{\omega_1+\tilde{x}}>\omega_4\tilde{z}_2+\omega_5+\omega_6 f(\tilde{x}) \quad (47)$$

Then, system (5) is uniformly persistence.

Proof: Consider the function $\sigma(x,z_1,z_2)=x^{p_1}z_1^{p_2}z_2^{p_3}$, where $p_i; i=1,2,3$ is an undetermined positive constants. Obviously $\sigma(x,z_1,z_2)$ is a C^1 positive function defined in R_+^3, and $\sigma(x,z_1,z_2)\to 0$ if $x\to 0$ or $z_1\to 0$ or $z_2\to 0$. Consequently we obtain

$$\Psi(x,z_1,z_2)=\frac{\sigma'(x,z_1,z_2)}{\sigma(x,z_1,z_2)}=p_1g_1+p_2g_2+p_3g_3$$

Here $g_i; i=1,2,3$ are given in system (5). Therefore

$$\Psi(x,z_1,z_2)=p_1\left(1-x-\frac{z_1}{\omega_1+x}-\frac{z_2}{\omega_2+x}\right)+p_2\left(\frac{\omega_3 x}{\omega_1+x}-\omega_4 z_2-\omega_5-\omega_6 f(x)\right)+p_3\left(\frac{\omega_7 x}{\omega_2+x}-\omega_8 z_1-\omega_9-\omega_{10}f(x)\right)$$

Now, since it is assumed that there are no periodic attractors in the boundary planes, then the only possible omega limit sets of the system (5) are the equilibrium points F_0, F_x, F_{xz_1} *and* F_{xz_2}. Thus according to the Gard technique [10] the proof is follows and the system is uniformly persists if we can proof that $\Psi(.)>0$ at each of these points. Since

$$\Psi(F_0)=p_1-\omega_5 p_2-\omega_9 p_3 \quad (48a)$$

$$\Psi(F_x)=\left(\frac{\omega_3}{\omega_1+1}-\omega_5-\omega_6 f(1)\right)p_2 + \left(\frac{\omega_7}{\omega_2+1}-\omega_9-\omega_{10} f(1)\right)p_3 \quad (48b)$$

$$\Psi(F_{xz_1})=\left(\frac{\omega_7\hat{x}}{\omega_2+\hat{x}}-\omega_8\hat{z}_1-\omega_9-\omega_{10} f(\hat{x})\right)p_3 \quad \ldots (48c)$$

$$\Psi(F_{xz_2})=\left(\frac{\omega_3\tilde{x}}{\omega_1+\tilde{x}}-\omega_4\tilde{z}_2-\omega_5-\omega_6 f(\tilde{x})\right)p_2 \quad \ldots (48d)$$

where $f(1)$, $f(\hat{x})$ and $f(\tilde{x})$ are obtained from Eq. (6) by substituting $x=1$, $x=\hat{x}$ and $x=\tilde{x}$ respectively. Obviously, $\Psi(F_0)>0$ for suitable choose of $p_1>0$ sufficiently large than $p_i>0$ *for* $i=2,3$. $\Psi(F_x)>0$ for any positive constants $p_i; i=2,3$ provided that conditions (31c) and (31d) hold. However, $\Psi(F_{xz_1})$ and $\Psi(F_{xz_2})$ are positive provided that the conditions (46) and (47) are satisfied respectively. Then strictly positive solution of system (5) do not have omega limit set and hence, system (5) is uniformly persistence. ■

6. The local Bifurcation:-

In this section an investigation for the dynamical behavior of system (5) under the effect of varying one parameter at each time is carried out. The occurrence of local bifurcation in the neighborhood of the equilibrium points of the (5) are studied in the following theorems.

Theorem 9. Assume that condition (31b) holds and the parameter ω_3 passes through the value $\breve{\omega}_3=(\omega_5+\omega_6 f(1))(\omega_1+1)$ where $f(1)$ is obtained from Eq. (6) by substituting $x=1$, then system (5) near the equilibrium F_x has:

1. No saddle-node bifurcation.
2. A transcritical bifurcation but no pitch-fork bifurcation can occur provided that the following condition holds:
$$\frac{\omega_1(\omega_5+\omega_6 f(1))}{\omega_1+1}\neq\omega_6 f'(1) \quad (49)$$
where
$$f'(1)=\frac{d}{dx}f(x)\Big|_{x=1}$$
3. A pitch-fork bifurcation otherwise.

Proof. According to the Jacobian matrix of system (5) at F_x that is given by J_x, it is easy to verify that as $\omega_3=\breve{\omega}_3$, the $J_x(F_x,\breve{\omega}_3)$ has the following eigenvalues:
$\breve{\lambda}_1=-1$, $\breve{\lambda}_2=0$ and $\breve{\lambda}_3=\frac{\omega_7}{\omega_2+1}-\omega_9-\omega_{10} f(1)$

Let $\breve{v}=(\breve{\theta}_1,\breve{\theta}_2,\breve{\theta}_3)^T$ be the eigenvector of $J_x(F_x,\breve{\omega}_3)$ corresponding to the eigenvalue $\breve{\lambda}_2=0$. Then it is easy to check that $\breve{v}=\left[\frac{-\breve{a}_{12}\breve{\theta}_2}{\breve{a}_{11}},\breve{\theta}_2,0\right]^T$, where $\breve{\theta}_2$ represents any nonzero real value. Also, let $\breve{w}=(\breve{h}_1,\breve{h}_2,\breve{h}_3)^T$ represents the eigenvector of $J_x^T(F_x,\breve{\omega}_3)$ that corresponding to the eigenvalue $\breve{\lambda}_2=0$. Straightforward calculation shows that $\breve{w}=(0,\breve{h}_2,0)^T$, where $\breve{h}_2$ is any nonzero real number.

Now, since $\frac{\partial G}{\partial\omega_3}\equiv G_{\omega_3}(X,\omega_3)=[0,\frac{xz_1}{\omega_1+x},0]^T$, where $X=(x,z_1,z_2)^T$ and $G=(G_1,G_2,G_3)^T$ with $G_i; i=1,2,3$ represent the right hand side of system (5). Then we get $G_{\omega_3}(F_x,\breve{\omega}_3)=(0,0,0)^T$ and the following is obtained:

$$\breve{w}^T[G_{\omega_3}(F_x,\breve{\omega}_3)]=(0,\breve{h}_2,0)(0,0,0)^T=0.$$

Thus the system (5) at F_x does not experience any saddle-node bifurcation in view of Sotomayor theorem [11]. Also, since

$$\breve{w}^T[DG_{\omega_3}(F_x,\breve{\omega}_3)\breve{v}]=(0,\breve{h}_2,0)(0,\frac{\breve{\theta}_2}{\omega_1+1},0)^T = \frac{\breve{h}_2\breve{\theta}_2}{\omega_1+1}\neq 0$$

here $DG_{\omega_3}(F_x,\breve{\omega}_3)=\frac{\partial}{\partial X}G_{\omega_3}(X,\omega_3)\Big|_{X=F_x,\omega_3=\breve{\omega}_3}$.

Moreover, we have

$$\breve{w}^T[D^2G(F_x,\breve{\omega}_3)(\breve{v},\breve{v})]= -2\left(\frac{\omega_1(\omega_5+\omega_6 f(1))}{\omega_1+1}-\omega_6 f'(1)\right)\left(\frac{1}{\omega_1+1}\right)\breve{\theta}_2^{\,2}\breve{h}_2$$

where $D^2G(F_x,\breve{\omega}_3)=DJ_x(X,\omega_3)\Big|_{X=F_x,\omega_3=\breve{\omega}_3}$. Clearly, $\breve{w}^T[D^2G(F_x,\breve{\omega}_3)(\breve{v},\breve{v})]\neq 0$ provided that condition (49) holds, and then by Sotomayor theorem, the system (5) possesses a transcritical bifurcation but not pitch-fork bifurcation near F_x where $\omega_3=\breve{\omega}_3$. However, violate condition (49) gives that $\breve{w}^T[D^2G(F_x,\breve{\omega}_3)(\breve{v},\breve{v})]=0$, and hence further computation shows

$$\breve{w}^T[D^3G(F_x,\breve{\omega}_3)(\breve{v},\breve{v},\breve{v})] = -3\left[\frac{2\omega_1(\omega_5+\omega_6 f(1))}{(\omega_1+1)^2} + \omega_6 f''(1)\right]\left(\frac{1}{\omega_1+1}\right)^2 \breve{\theta}_2{}^3\breve{h}_2 \neq 0$$

where:

$$f''(1) = \begin{cases} 0 & for\, case 1 \\ \dfrac{-2a_1\overline{\omega}_{11}}{(\overline{\omega}_{11}+1)^3} & for\, case\, 2 \end{cases}$$

Therefore system (5) possesses a pitch-fork bifurcation near F_x where $\omega_3 = \breve{\omega}_3$. ■

Theorem 10. Assume that conditions (37a)-(37b) hold and the parameter ω_7 passes through the value $\hat{\omega}_7 = \frac{\omega_2+\hat{x}}{\hat{x}}\left(\omega_8\hat{z}_1+\omega_9+\omega_{10}f(\hat{x})\right)$, then system (5) near the equilibrium F_{xz_1} has:

1. No saddle-node bifurcation.
2. A transcritical bifurcation but no pitch-fork bifurcation can occur
   provided that one of the following condition holds:

$$R = \left(\frac{\omega_2\varsigma_2}{\hat{x}}\left(\omega_8\hat{z}_1+\omega_9+\omega_{10}f(\hat{x})\right) - \omega_{10}f'(\hat{x})\varsigma_2^2\right)\left(\frac{\omega_4\varsigma_1^2}{\varsigma_3\varsigma_2^2}\right) + \left(\frac{\omega_8\varsigma_1}{\varsigma_3}\right)\left(-\varsigma_4\omega_4+\frac{\varsigma_3}{\varsigma_2}\right) \neq 0 \qquad \text{…. (50)}$$

3. A pitch-fork bifurcation provided that

$$R = 0 \qquad (51a)$$

$$R_1 = \left(\frac{2\omega_2\left(\omega_8\hat{z}_1+\omega_9+\omega_{10}f(\hat{x})\right)}{\varsigma_2^2\hat{x}} + \omega_{10}f''(\hat{x})\right)\left(\frac{\omega_4\varsigma_1^2}{\varsigma_3}\right)^2 \neq 0 \qquad \text{… (51b)}$$

here: $\varsigma_1 = \omega_1+\hat{x}$, $\varsigma_2 = \omega_2+\hat{x}$, $\varsigma_3 = \omega_1\omega_3 - \omega_6 f'(\hat{x})\varsigma_1^2$, $\varsigma_4 = -\varsigma_1^2+\hat{z}_1$

where $f(\hat{x})$ is obtained from Eq. (6) by substituting $x=\hat{x}$, $f'(\hat{x})$ is given in Eq. (32) and

$$f''(\hat{x}) = \begin{cases} 0 & for\, case 1 \\ \dfrac{-2a_1\overline{\omega}_{11}}{(\overline{\omega}_{11}+\hat{x})^3} & for\, case\, 2 \end{cases}$$

Proof. Follows directly by applying Sotomayor theorem as shown in proof of theorem (11). ■

Theorem 11. Assume that conditions (35a) and (35b) hold and the parameter ω_3 passes through the value $\tilde{\omega}_3 = \frac{\omega_1+\tilde{x}}{\tilde{x}}\left(\omega_4\tilde{z}_2+\omega_5+\omega_6 f(\tilde{x})\right)$, then system (5) near the equilibrium F_{xz_2} has:

1. No saddle-node bifurcation.
2. A transcritical bifurcation but no pitch-fork bifurcation can occur
   provided that one of the following condition holds:

$$L = \left(\frac{\xi_1\omega_1}{\tilde{x}}\left(\omega_4\tilde{z}_2+\omega_5+\omega_6 f(\tilde{x})\right) - \omega_6 f'(\tilde{x})\xi_1^2\right)\left(\frac{\omega_8\xi_2^2}{\xi_3\xi_1^2}\right) + \left(\frac{\omega_4\xi_2}{\xi_3}\right)\left(-\xi_4\omega_8+\frac{\xi_3}{\xi_1}\right) \neq 0 \qquad \text{… (52)}$$

3. A pitch-fork bifurcation provided that

$$L = 0 \qquad (53a)$$

$$L_1 = \left(\frac{2\omega_1\left(\omega_4\tilde{z}_2+\omega_5+\omega_6 f(\tilde{x})\right)}{\xi_1^2\tilde{x}} + \omega_6 f''(\tilde{x})\right)\left(\frac{\omega_8\xi_2^2}{\xi_3}\right)^2 \neq 0 \qquad \text{…. (53b)}$$

here $\xi_1 = \omega_1+\tilde{x}$, $\xi_2 = \omega_2+\tilde{x}$, $\xi_3 = \omega_2\omega_7 - \omega_{10}f'(\tilde{x})\xi_2^2$, $\xi_4 = -\xi_2^2+\tilde{z}_2$

where $f(\tilde{x})$ is obtained from Eq. (6) by substituting $x=\tilde{x}$, $f'(\tilde{x})$ is given in Eq. (34) and

$$f''(\tilde{x}) = \begin{cases} 0 & for\, case 1 \\ \dfrac{-2a_1\overline{\omega}_{11}}{(\overline{\omega}_{11}+\tilde{x})^3} & for\, case\, 2 \end{cases}$$

Proof. Follows directly by applying Sotomayor theorem as shown in proof of theorem (11). ■

7. Hopf bifurcation.

Finally, in order to investigate the Hopf bifurcation of the model system (5), we will follow the Liu approach [12] as shown in the following theorem.

Theorem 12. Assume that the coexistence equilibrium point of system (5) exists and let in addition to conditions (39a)-(39c), the following conditions hold:

$$\frac{x^*B_3B_4}{z_2{}^*B_1{}^4B_5} < \omega_4 < \frac{x^*B_2B_3B_4}{x^*B_3{}^2 - z_1{}^*B_1B_2{}^4B_4} \qquad (54)$$

Then a simple Hopf bifurcation of the model system (5) occurs at

$$\omega_8 \equiv \omega_8{}^* = \frac{D_2}{z_2{}^*B_4B_1^3B_2} + \frac{x^*B_3B_5}{z_1{}^*B_4B_2^4}$$

where $D_2 = x^*B_3B_4 - \omega_4 z_2{}^*B_5B_1{}^4$ and $B_i; i=1,2,3,4,5$ are given in Eq. (36).

Proof. According to the Liu approach a simple Hopf bifurcation occurs if and only if $A_1(\mu_*) > 0,\ A_3(\mu_*) > 0,\ \Delta(\mu_*) = 0$ *and* $\left.\frac{d\Delta}{d\mu}\right|_{\mu=\mu_*} \neq 0$, where μ_* is a critical value of the key parameter and A_i *for* $i = 1,3$ *and* Δ are given in equations (38a), (38b) and (38c). Note that it is clear that $D_2 < 0$ under the condition (54) and hence $\omega_8{}^*$ is positive under the conditions (39a)-(39b). Now, by substituting the value of $\omega_8{}^*$ in these equations we obtain:

$A_1(\omega_8{}^*) = -\frac{x^* B_3}{B_1{}^2 B_2{}^2}$, which is positive due to condition (39a).

$$A_3(\omega_8{}^*) = \frac{x^* z_1{}^* z_2{}^*}{B_1{}^2 B_2{}^2}\left[\omega_8{}^*(\omega_4 B_3 - B_2 B_4) - \omega_4 B_1 B_5\right]$$

$$= \frac{x^* z_1{}^* z_2{}^*}{B_1{}^2 B_2{}^2}\left[\frac{D_2(\omega_4 B_3 - B_2 B_4)}{z_2{}^* B_1^3 B_2 B_4} + B_5\left(\omega_4\left(\frac{x^* B_3{}^2}{z_1{}^* B_2^4 B_4} - B_1\right) - \frac{x^* B_3}{z_1{}^* B_2{}^3}\right)\right]$$

Clearly, $A_3(\omega_8{}^*) > 0$ under the conditions (39a)-(39c) with (54). Moreover, rewrite equation (38c) gives that:

$$\Delta = -\frac{x^{*2} z_1{}^* B_3 B_4}{B_1{}^5 B_2{}^2} + \frac{\omega_4 x^* z_1{}^* z_2{}^* B_5}{B_1 B_2{}^2} - \frac{x^{*2} z_2{}^* B_3 B_5}{B_1{}^2 B_2{}^5} + \omega_8 \frac{x^* z_1{}^* z_2{}^* B_4}{B_1{}^2 B_2}$$

Hence it easy to verify that $\Delta(\omega_8{}^*) = 0$. Finally, since

$$\left.\frac{d\Delta}{d\omega_8}\right|_{\omega_8 = \omega_8{}^*} = \frac{x^* z_1{}^* z_2{}^* B_4}{B_1{}^2 B_2} \neq 0$$

Thus, a simple Hopf bifurcation occurs in system (5) at $\omega_8 = \omega_8{}^*$. ■

8. Numerical analysis.

In this section the global dynamics of system (5), is studied numerically. In both the cases 1 and 2, system (5) is solved numerically for different sets of parameters and different sets of initial conditions, and then the attracting sets and their time series are drown.

For the following set of parameters

$$\omega_1 = 0.1,\ \omega_2 = 0.1, \omega_3 = 0.75,\ \omega_4 = 0.1,\ \omega_5 = 0.60,\ \omega_6 = 0.01, \omega_7 = 0.75,\ \omega_8 = 0.1,\ \omega_9 = 0.60,\ \omega_{10} = 0.01\ \ \omega_{11} = 0.25,\ \overline{\omega}_{11} = 0.1,\ \mathrm{a}_1 = 1 \quad \text{…… (55)}$$

The attracting sets along with their time series of system (5) for case 1 and case 2 are drown in Figure. (1) and Figure. (2) respectively, starting from different sets of initial conditions.

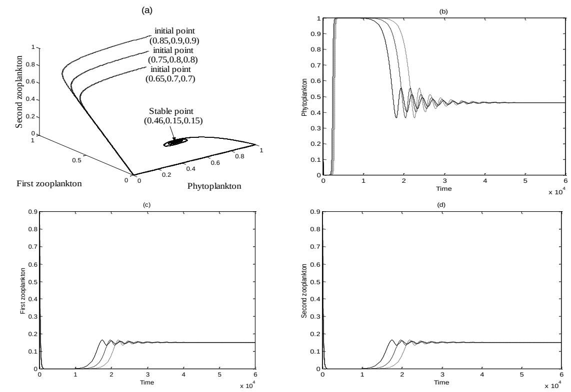

Figure.1-The phase plot of system (5) in case 1. (a) The solution of system (5) approaches asymptotically to stable positive point starting from different initial points. (b) The trajectories of x as a function of time. (c) The trajectories of z_1 as a function of time. (d) The trajectories of z_2 as a function of time.

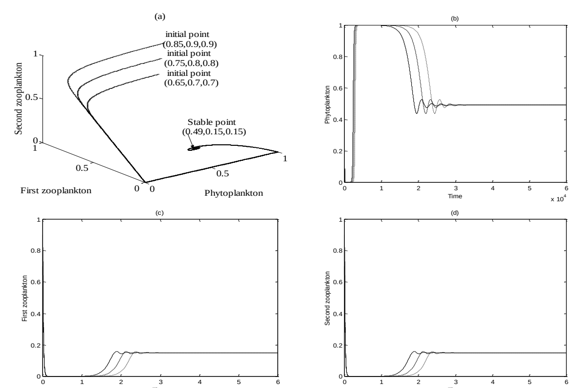

Figure 2-The phase plot of system (5) in case 2. (a) The solution of system (5) approaches asymptotically to stable positive point starting from different initial points. (b) The trajectories of x as a function of time. (c) The trajectories of z_1 as a function of time. (d) The trajectories of z_2 as a function of time.

Note that from now onward, time series figures, we will use **solid line** type for x, **dash line** type for z_1 and **dot line** type for z_2.

However for the above set of data with $\omega_5 = \omega_9 = 0.58$, the system (5) approaches asymptotically to stable limit cycle in both the cases as shown in the following two figures respectively.

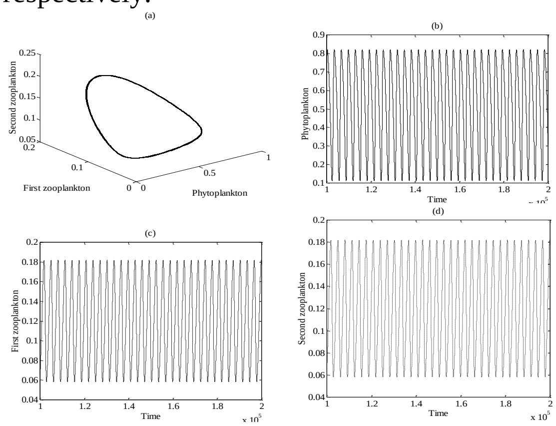

Figure 3-The phase plot of system (5) in case 1 for the data given by Eq. (55) with $\omega_5 = \omega_9 = 0.58$. (a) The solution of system (5) approaches asymptotically to limit cycle. (b) The trajectory of x as a function of time. (c) The trajectory of z_1 as a function of time. (d) The trajectory of z_2 as a function of time.

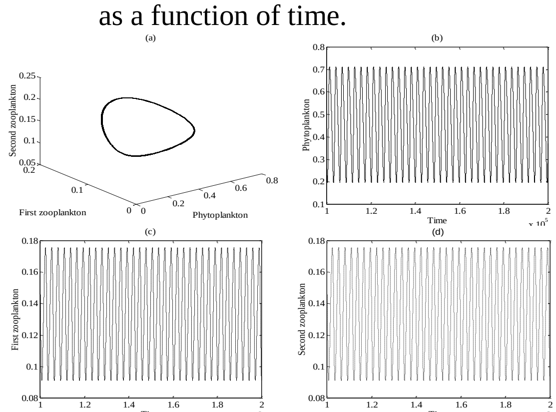

Figure 4-The phase plot of system (5) in case 2 for the data given by Eq. (55) with $\omega_5 = \omega_9 = 0.58$. (a) The solution of system (5) approaches asymptotically to limit cycle. (b) The trajectory of x as a function of time. (c) The trajectory of z_1 as a function of time. (d) The trajectory of z_2 as a function of time

According to the above figures, it is observed that, although two competing species can not survive for all the time simultaneously, the existence of phytoplankton makes the survival of both the competing zooplankton species possible.

Now, further analysis for the role of varying the natural death rates of both zooplanktons, represented by ω_5 *and* ω_9, on the dynamics of system (5) is performed, and the following results are obtained:

For case 1: system (5) has a periodic dynamic in the $Int.R_+^3$ for the data given by Eq. (55) with $\omega_5 = \omega_9 \le 0.59$ see for example Figure. (3), while for $0.6 \le \omega_5 = \omega_9 \le 0.67$ the system (5) has a globally asymptotically stable positive point in the $Int.R_+^3$ see for example Figure. (1). It approaches asymptotically to the equilibrium point $F_x = (1,0,0)$ for the data given by Eq. (55) with $\omega_5 = \omega_9 \ge 0.68$ as shown in Figure. (5). Finally, for all the values of $\omega_5 < \omega_9$ with rest of data as given in Eq. (55), system (5) loses its persistence and the solution approaches asymptotically to either periodic attractor in the $Int.R_+^2$ of xz_1-plane or stable point in the $Int.R_+^2$ of xz_1-plane or to the equilibrium point $F_x = (1,0,0)$ see for explanation Figure. (6a)-(6c), however similar observations have been obtained and the solution of system (5) approaches to one of these three types of attractors in the $Int.R_+^2$ of xz_2-plane for the all values of $\omega_5 > \omega_9$ with the rest of data as given in Eq. (55).

For case 2: it is observed that, the system (5) has a periodic dynamic in the $Int.R_+^3$ when $\omega_5 = \omega_9 \le 0.58$ with the rest of data as in Eq. (55) see for example Fig. (4), while the system (5) has a globally asymptotically stable positive point in the $Int.R_+^3$ for $0.59 \le \omega_5 = \omega_9 \le 0.67$ with the rest of data as in Eq. (55) see for example Fig. (2), and finally the system approaches asymptotically to equilibrium point $F_x = (1,0,0)$ for $\omega_5 = \omega_9 \ge 0.68$ with the rest of data as in Eq. (55). Again as in case 1, it is observed that for all the values of $\omega_5 < \omega_9$ with rest of data as given in Eq. (55) system (5) approaches asymptotically to the $Int.R_+^2$ of xz_1-plane, while it approaches to the $Int.R_+^2$ of xz_2-plane for all values of $\omega_5 > \omega_9$ with rest of data as given in Eq. (55).

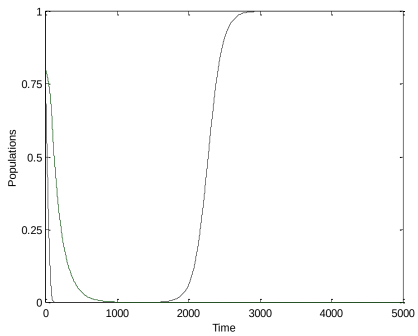

Figure 5-The trajectories of system (5) in case 1, for the data given in Eq. (55) with $\omega_5 = \omega_9 = 0.68$, approaches asymptotically to Stable point $F_x = (1,0,0)$.

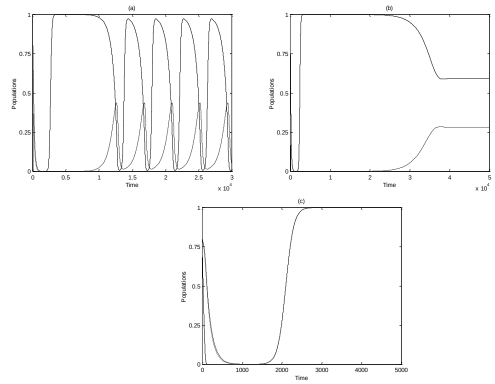

Figure 6-The trajectories of system (5) in case 1, for the data given in Eq. (55), approaches asymptotically to: (a) Periodic dynamics in xz_1 – plane for $\omega_5 = 0.55$ and $\omega_9 = 0.64$; (b) Stable point $(0.59, 0.28, 0)$ in xz_1 – plane for $\omega_5 = 0.64$ and $\omega_9 = 0.66$; (c) Stable point $F_x = (1,0,0)$ for $\omega_5 = 0.7$ and $\omega_9 = 0.75$.

According to the above, the effect of the other parameters on the dynamics of system (5) is also studied in case of varying different couples of parameters simultaneously and the obtained results are summarized in the following tables.

Table 1 - Numerical behavior and persistence of system (5) in case 1, as varying in some parameters keeping the rest of parameters fixed as in Eq. (55).

Parameters varied in system (3.5)	Numerical behavior of system (5)	Persistence of system (5)
$\omega_1 = \omega_2 \le 0.09$	Approaches to periodic dynamic in $Int.R_+^3$	Persists
$0.1 \le \omega_1 = \omega_2 \le 0.23$	Approaches to stable point in $Int.R_+^3$	Persists
$\omega_1 = \omega_2 \ge 0.24$	Approaches to stable point $F_x = (1,0,0)$.	Not persists
$\omega_3 = \omega_7 \le 0.66$	Approaches to stable point F_x.	Not persists
$0.67 \le \omega_3 = \omega_7 \le 0.75$	Approaches to stable point in $Int.R_+^3$	Persists
$\omega_3 = \omega_7 \ge 0.76$	Approaches to periodic dynamic in $Int.R_+^3$	Persists
$\omega_4 = \omega_8 \le 0.07$	Approaches to periodic dynamic in $Int.R_+^3$	Persists
$0.08 \le \omega_4 = \omega_8 \le 7.81$	Approaches to stable point in $Int.R_+^3$	Persists
$\omega_4 = \omega_8 \ge 7.82$	Approaches to stable point F_x.	Not persists
$\omega_6 = \omega_{10} \le 0.32$	Approaches to stable point in $Int.R_+^3$	Persists
$\omega_6 = \omega_{10} \ge 0.33$	Approaches to stable point F_x.	Not persists
$\omega_{11} \le 8.07$	Approaches to stable point in $Int.R_+^3$	Persists
$\omega_{11} \ge 8.08$	Approaches to stable point F_x.	Not Persists

Table 2- Numerical behavior and persistence of system (5) in case 2, as varying in some parameters keeping the rest of parameters fixed as in Eq. (55).

Parameters varied in system (5)	Numerical behavior of system (5)	Persistence of system (5)
$\omega_1=\omega_2\le 0.09$ $0.1\le\omega_1=\omega_2\le 0.22$ $\omega_1=\omega_2\ge 0.23$	Approaches to periodic dynamic in $Int.R^3_+$ Approaches to stable point in $Int.R^3_+$ Approaches to stable point F_x .	Persists Persists Not persists
$\omega_3=\omega_7\le 0.67$ $0.68\le\omega_3=\omega_7\le 0.76$ $\omega_3=\omega_7\ge 0.77$	Approaches to stable point F_x . Approaches to stable point in $Int.R^3_+$ Approaches to Periodic dynamic in $Int.R^3_+$	Not persists Persists Persists
$\omega_4=\omega_8\le 0.03$ $0.04\le\omega_4=\omega_8\le 7.15$ $\omega_4=\omega_8\ge 7.16$	Approaches to periodic dynamic in $Int.R^3_+$ Approaches to stable point in $Int.R^3_+$ Approaches to stable point F_x .	Persists Persists Not persists
$\omega_6=\omega_{10}\le 0.08$ $\omega_6=\omega_{10}\ge 0.09$	Approaches to stable point in $Int.R^3_+$ Approaches to stable point F_x .	Persists Not persists
for all values of $\overline{\omega}_{11}$	Approaches to stable in $Int.R^3_+$	Persists
$a_1\le 8.77$ $a_1\ge 8.78$	Approaches to stable point in $Int.R^3_+$ Approaches to stable point F_x	Persists Not persists

Table 3- Numerical behavior and persistence of system (5) in case 1, as varying in some parameters keeping the rest of parameters fixed as in Eq. (55) with $\omega_5=\omega_9=0.58$.

Parameters varied in system (5)	Numerical behavior of system (5)	Persistence of system (5)
$\omega_1=\omega_2\le 0.11$ $0.12\le\omega_1=\omega_2\le 0.28$ $\omega_1=\omega_2\ge 0.29$	Approaches to periodic dynamic in $Int.R^3_+$ Approaches to stable point in $Int.R^3_+$ Approaches to stable point F_x .	Persists Persists Not persists
$\omega_3=\omega_7\le 0.64$ $0.65\le\omega_3=\omega_7\le 0.73$ $\omega_3=\omega_7\ge 0.74$	Approaches to stable point F_x . Approaches to stable point in $Int.R^3_+$ Approaches to Periodic dynamic in $Int.R^3_+$	Not persists Persists Persists
$\omega_4=\omega_8\le 0.18$ $0.19\le\omega_4=\omega_8\le 9.81$ $\omega_4=\omega_8\ge 9.82$	Approaches to periodic dynamic in $Int.R^3_+$ Approaches to stable point in $Int.R^3_+$ Approaches to stable point F_x .	Persists Persists Not persists
$\omega_6=\omega_{10}\le 0.14$ $0.15\le\omega_6=w_{10}\le 0.39$ $\omega_6=\omega_{10}\ge 0.4$	Approaches to periodic dynamic in $Int.R^3_+$ Approaches to stable point in $Int.R^3_+$ Approaches to stable point F_x .	Persists Persists Not persists
$\omega_{11}\le 3.72$ $3.73\le\omega_{11}\le 10.08$ $\omega_{11}\ge 10.09$	Approaches to periodic dynamic in $Int.R^3_+$ Approaches to stable point in $Int.R^3_+$ Approaches to stable point F_x	Persists Persists Not persists

Table 4- Numerical behavior and persistence of system (5) in case 2, as varying in some parameters keeping the rest of parameters fixed as in Eq. (55) with $\omega_5 = \omega_9 = 0.58$.

Parameters varied in system (5)	Numerical behavior of system (5)	Persistence of system (5)
$\omega_1 = \omega_2 \leq 0.10$ $0.11 \leq \omega_1 = \omega_2 \leq 0.26$ $\omega_1 = \omega_2 \geq 0.27$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point F_x	Persists Persists Not persists
$\omega_3 = \omega_7 \leq 0.65$ $0.66 \leq \omega_3 = \omega_7 \leq 0.73$ $\omega_3 = \omega_7 \geq 0.74$	Approaches to stable point F_x Approaches to stable point in $Int.R_+^3$ Approaches to Periodic dynamic in $Int.R_+^3$	Not persists Persists Persists
$\omega_4 = \omega_8 \leq 0.14$ $0.15 \leq \omega_4 = \omega_8 \leq 9.15$ $\omega_4 = \omega_8 \geq 9.16$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point F_x	Persists Persists Not persists
$\omega_6 = \omega_{10} \leq 0.02$ $0.03 \leq \omega_6 = \omega_{10} \leq 0.10$ $\omega_6 = \omega_{10} \geq 0.11$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point F_x	Persists Persists Not persists
for all values of $\overline{\omega}_{11}$	Approaches to periodic dynamic in $Int.R_+^3$	Persists
$a_1 \leq 2$ $2.01 \leq a_1 \leq 10.98$ $a_1 \geq 10.99$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point F_x	Persists Persists Not persists

9. Conclusions and Discussions.

In this paper, a mathematical model consisting of single harmful phytoplankton interacting with two competing zooplankton species has been proposed and analyzed. It is assumed that the phytoplankton producing a toxin substance as a defensive strategy against the predation by zooplankton. The effect of toxin producing plankton on the dynamical behavior of phytoplankton-zooplankton system is considered. Two different scenarios of the distribution of the toxin substance, through Holling type-I (called case 1) and Holling type-II (called case 2), are studied. In both the cases, the dynamical behavior of system (5) has been investigated locally as well as globally. The conditions for the system (5) to be persists have been derived. The occurrence of local bifurcation as well as Hopf bifurcation in system (5) is investigated. Finally the effects of varying the parameters on the dynamics of system (5) are studied numerically and the trajectories of the system are presented in the form of figures and tables. According to those figures and tables the following conclusions are obtained.

1. Increasing the natural death rates for both the zooplankton (ω_5 and ω_9) simultaneously keeping the rest of parameters as given in Eq. (55) for both the studied cases have stabilizing effect on the dynamics of system (5) and then with further increasing of the parameters values the system faces extinction in both the zooplankton and the solution approaches to $F_x = (1,0,0)$.
2. Increasing the half saturation constants (ω_1 *and* ω_2) simultaneously or inter-specific competition parameters between both the zooplankton species (ω_4 *and* ω_8) simultaneously, keeping the other parameters fixed as given in Eq. (55) with $\omega_5 = \omega_9 = 0.6$ (the system (5) has

a stable positive point) and $\omega_5 = \omega_9 = 0.58$ (the system (5) has periodic dynamics), for both the cases described above have the same effect as that of the natural death rates given in (1).

3. Increasing the liberation rates constants of toxin substance from phytoplankton (ω_6 *and* ω_{10}) simultaneously with the other parameters as given in Eq. (55) have extinction effect on the system. However increasing the values of these parameters simultaneously with the other parameters as given in Eq. (55) and $\omega_5 = \omega_9 = 0.58$ have stabilizing effect at first and then lead to extinction in both the zooplankton species.
4. Increasing the conversion rates parameters (ω_3 and ω_7) simultaneously keeping the rest of parameters as given in Eq. (55), when the system (5) has asymptotically stable point and periodic dynamics, for both the studied cases causes coexistence of all the species at the stable positive point. However increasing these parameters further causes losing to the stability and the system (5) goes to periodic dynamics.
5. Increasing the maximum ingestion rates of two zooplankton to the toxin produced by phytoplankton (ω_{11}), in case 1, keeping other parameters fixed as in Eq. (55) has extinction effect and the system approaches to $F_x = (1,0,0)$, however for the data in Eq. (55) with $\omega_5 = \omega_9 = 0.58$ it has stabilizing effect at first and then extinction of both the zooplankton species and the system approaches to $F_x = (1,0,0)$.
6. Varying the half saturation constant of zooplankton to the toxin substance using Holling type-II for the distribution of the toxin substance does not has any effect on the dynamics of system (5).
7. Finally, increasing the maximum zooplankton ingestion rate to the toxin substance produced by phytoplankton (a_1), in case 2, keeping other parameters fixed as in Eq. (55) has extinction effect and the system approaches to $F_x = (1,0,0)$, however for the data in Eq. (55) with $\omega_5 = \omega_9 = 0.58$ it has stabilizing effect at first and then extinction of both the zooplankton species and the system approaches to $F_x = (1,0,0)$.

References:

1. Chattopadhyay J., Sarkar R.R, Mandal, S., **2002a**. Toxin-producing plankton may act as a biological control for planktonic blooms-field study and mathematical modeling, J. of Theoretical Biology, 215, p. 333–344.

2. Kirk K., Gilbert J., **1992.** Variation in herbivore response to chemical defense: zooplankton foraging on toxic cynobacteria, Ecology, 73(6), p. 2208-2213.

3. Ives J.D., **1987.** Possible mechanism underlying copepod grazing responses to levels of toxicity in red tide dinoflagellates, J. Exp. Mar. Biol. Ecol. 112, p. 131–145.

4. Nielsen T.G., Kiörboe T., Björnsen P.K., **1990.** Effects of a Chrysochromulina polylepis sub surface bloom on the plankton community, Mar. Ecol. Prog. Ser. 62, p. 21–35.

5. Chattopadhyay J., Sarkar R.R., **2003.** Chaos to order: preliminary experiments with a population dynamics models of three trophic levels, Ecological Modelling 163, p. 45–50.

6. Roy S., Alam S., Chattopadhyay J., **2006.** Competitive effects of toxin-producing phytoplankton on the over all plankton populations in the Bay of Bengal, Bull. Math. Biol. 68, p. 2303-2320.

7. Roy S., Chattopadhyay J., **2007.** Toxin allelopathy among phytoplankton species prevents competitive exclusion, J. Biol. Syst. 15, p. 1-21.

8. Sarkar R.R., Chattopadhayay J., **2003**. The role of environmental stochasticity in a toxic phytoplankton–non-toxic phytoplankton–zooplankton system, Environmetrics 14, p. 775-792.

9. Tanmay Chowdhury, Shovonlal Roy, Chattopadhayay J., **2008**. Modeling migratory grazing of zooplankton on toxic and non toxic phytoplankton, Applied Math. and computation 197, p. 659-671.

10. Gard T.C. and Hallam T, G., **1979.** Persistence in food web. Lotka-Volttera food chains, Bull. Math. Biol., 41, p. 877-89

11. Perko, L., **2000.** *Differential Equations and Dynamical Systems*, third Edition, Springer Verlag, Berlin.

12. Liu W.M., **1994.** Criterion of Hopf Bifurcations without using Eigenvalues, J. of Mathematical Analysis and Application, 182**,** p. 250-256.