Stability and Bifurcation of Epidemic Model

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Abstract

In this paper a mathematical model that describes the flow of infectious disease in a population is proposed and studied. It is assumed that the disease divided the population into four classes: susceptible individuals (*S*), vaccinated individuals (*V*), infected individuals (*I*) and recover individuals (*R*). The impact of immigrants, vaccine and external sources of disease, on the dynamics of *SVIRS* epidemic model is studied. The existence, uniqueness and boundedness of the solution of the model are discussed. The local and global stability of the model is studied. The occurrence of local bifurcation as well as Hopf bifurcation in the model is investigated. Finally the global dynamics of the proposed model is studied numerically.

Keywords: Epidemic models, Stability, Vaccinated, Immigrants, external sources, Local and Hopf bifurcation.

الاستقرارية والتفرع لنموذج وبائي

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الخلاصة

في هذا البحث تم عرض ودراسة نموذج رياضي يصف انتشار الامراض المعدية في المجتمع السكاني، افترضنا ان الامراض تقسم المجتمع السكاني الى اربعة اقسام هم افراد معرضين للاصابة و ملقحين و مصابين بالمرض و معافين من المرض مع احتمال اصابتهم بالمرض مره ثانية . درسنا تأثير المهاجرين والتطعيم والمصادر الخارجية للاصابة بالامراض على ديناميكية النموذج الوبائي *SVIRS* . تمت مناقشة وجود و وحدانية وقيود الحل للنموذج المقترح. قمنا بدراسة السلوك المحلي والشامل له. كذلك بحثنا التفرعات المحلية وتفرع هوبف. واخيرا من اجل تأكيد نتائجنا وتحديد تأثير معلمات النموذج الوبائي المقترح على السلوك الديناميكي له اجرينا محاكاة عددية له.

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1.Introduction

Mathematical models have become important tools to study and analyze the spread and control of infectious disease. Most the proposed mathematical models those describe the transmission of infectious disease have been derived from the classical susceptible – infective – recover (*SIR*) model, which is suggested originally by Kermack and Mckenderick [1]. In that model the susceptible individuals become infective by contact with infected individuals and then the infected individuals may recover and transfer to removal individuals at a specific rate. Numbers of mathematical models were developed to study and analyze the spread of infectious diseases in order to prevent or minimize the transmission of them through quarantine and other measures see for example [2-5] and the references there in.

On the other hand, since the resistance against an infectious disease represents protection that reduces an individual's risk of contracting the disease, therefore many epidemiological models involving vaccination (*V*) have been proposed and studied, see for example [6-8] and the reference there in.

Keeping the above in view, there are many infectious diseases spread within the population by direct contact between susceptible and infective individuals, they may spread through external sources in the environment such as (air, water, insects, etc…). Therefore, recently Das et al. [9] have been proposed and studied a mathematical model consisting of eco-epidemiological model involving external sources of disease. In this paper we proposed and studied a mathematical model consisting of *SVIRS* epidemic model involving immigrant individuals, some of them may arrive infected with the disease, and vaccine in which it is assumed that the disease transmitted by contact as well as external sources in the environment.

2. The mathematical model:

Consider a simple epidemiological model in which the total population (say *N(t)*) at time t is divided in to three sub classes the susceptible individuals *S(t)*, infected individuals *I(t)* and recover individuals *R(t)*. Such model can be represented as follows:

$$\begin{aligned}\frac{dS}{dt} &= \Lambda - \beta IS - \mu S \\ \frac{dI}{dt} &= \beta IS - (\mu + \alpha)I \\ \frac{dR}{dt} &= \alpha I - \mu R\end{aligned} \quad (1)$$

Here $\Lambda > 0$ is the recruitment rate of the population, $\mu > 0$ is the natural death rate of the population, $\beta > 0$ is the infected rate (incidence rate) of the susceptible individuals due to direct contact with the infected individuals and $\alpha > 0$ is the natural recovery rate of the infected individuals.

Now, since there are many infectious diseases (Alanfelonzha, bird's Anfelonzha and typhoid etc.) spread in the environment by different factors including insects, contact or other vectors, therefore, we assumed that the disease in the above model will transmitted between the population individuals by contact as well as external sources of disease in the environment with an external incidence rate $\beta_\circ \geq 0$. Also it is assumed that the lifetime of removal individual's immunity may not continue forever and then the removal individuals return to be susceptible class with a constant rate $\gamma \geq 0$ (also known as losing removal individual's immunity rate). Further, there is a constant flow, say $A > 0$, of a new members arriving into the population with the fraction p of A arriving infected ($0 \leq p \leq 1$). Then the above system (1) can be rewritten in the form:

$$\begin{aligned}\frac{dS}{dt} &= \Lambda + (1-p)A - (\beta_\circ + \beta I)S - \mu S + \gamma R \\ \frac{dI}{dt} &= pA + (\beta_\circ + \beta I)S - (\mu + \alpha)I \\ \frac{dR}{dt} &= \alpha I - (\gamma + \mu)R\end{aligned} \quad (2)$$

Keeping the above in view, in order to study the effect of vaccination on the system (2) let *V(t)* represented the vaccinated individuals in the population at time t , and then the following assumptions are made:

- The susceptible class is vaccinated at per capita rate $\psi \geq 0$.
- The infection can invade the susceptible class or vaccinated class depending on vaccine efficiency.
- The vaccine reduces the possibility of infection by a factor of σ, which is known as intensity vaccine immunity rate, where $0 \leq \sigma \leq 1$.
- The vaccine may not give a permanent immunity for susceptible individuals, so the vaccine may disappear and then the individuals loss the immunity with rate $0 \leq \theta \leq 1$.

Accordingly, the flow of disease in system (2) along with the above assumptions can be representing in the following block diagram:

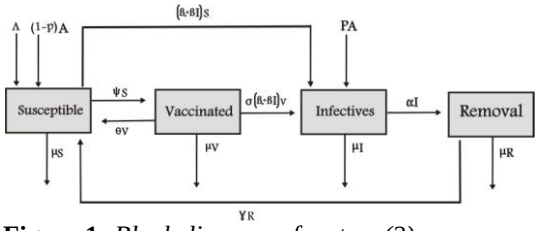

Figure 1- *Block diagram of system (2).*

Therefore system (2) can be modified to:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda + (1-p)A - (\beta_\circ + \beta I)S - (\mu + \psi)S + \theta V + \gamma R \\
\frac{dV}{dt} &= \psi S - \sigma(\beta_\circ + \beta I)V - (\mu + \theta)V \\
\frac{dI}{dt} &= pA + (\beta_\circ + \beta I)S + \sigma(\beta_\circ + \beta I)V - (\mu + \alpha)I \\
\frac{dR}{dt} &= \alpha I - (\mu + \gamma)R
\end{aligned} \tag{3}$$

Clearly for $\sigma = 0$ the vaccine is completely affective. While, $\sigma = 1$ stand for the situation where the vaccine is totally ineffective. On the other hand, $\theta = 0$ denotes to the case when immunity is life-long while $\theta = 1$ corresponds to the case where there is absolutely no vaccine induced immunity. Therefore at any point of time *t* the total number of population becomes $N = S(t) + V(t) + I(t) + R(t)$.

Obviously, due to the biological meaning of the variables *S(t), V(t), I(t), and R(t),* system (3) has the domain

$$\Re_+^4 = \left\{(S,V,I,R) \in \Re^4, S \geq 0, V \geq 0, I \geq 0, R \geq 0\right\}$$

which is positively invariant for system (3). Clearly, the interaction functions on the right hand side of system (3) are continuously differentiable. In fact they are Liptschizan function on $\Re_+^4$. Therefore the solution of system (3) exists and is unique. Further, all solutions of the system (3) with non-negative initial conditions are uniformly bounded as shown in the following theorem.

Theorem (1): All the solutions of system (3), which are initiate in $\Re_+^4$, are uniformly bounded.

Proof: Let (*S(t), V(t), I(t), R(t))* be any solution of the system (3) with non-negative initial condition *(S(0), V(0), I(0), R(0)),* since *N(t)= S(t)+V(t)+I(t)+R(t),* then :

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dV}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$$

Which gives

$$\frac{dN}{dt} + \mu N = \Lambda + A$$

Now, by solving the above linear differential equation, we get that the total population is asymptotically constant by:

$$N(t) = \frac{\Lambda + A}{\mu}$$

Hence all the solutions of system (3) that initiate in $\Re_+^4$, are confined in the region:

$$\zeta = \left\{(S,V,I,R) \in \Re_+^4 : N \leq \frac{\Lambda + A}{\mu} + \varepsilon; \varepsilon \geq 0\right\}$$

which is complete the proof. ■

3. Existence of Equilibrium points of system(3)

In this section, the existence of all possible equilibrium points of system (3) is discussed. Clearly, if $I = 0$, then the system (3) has an equilibrium point called a disease free equilibrium point and denoted by $E_\circ = (S_\circ, V_\circ, 0, 0)$ where:

$$\left.\begin{aligned}
S_\circ &= \frac{(\Lambda + A)(\mu + \theta)}{\mu(\mu + \theta + \psi)} \\
V_\circ &= \frac{\psi(\Lambda + A)}{\mu(\mu + \theta + \psi)}
\end{aligned}\right\} \tag{4}$$

However, if $I \neq 0$ then the system (3) has an endemic equilibrium point denoted by $E_1 = (S_1, V_1, I_1, R_1)$ where S_1, V_1, I_1 and R_1 represent the positive solution of the following set of equations:

$$\begin{aligned}
&\Lambda + (1-p)A - (\beta_\circ + \beta I)S - (\mu+\psi)S + \theta V + \gamma R = 0\\
&\psi S - \sigma(\beta_\circ + \beta I)V - (\mu+\theta)V = 0\\
&pA + (\beta_\circ + \beta I)S + \sigma(\beta_\circ + \beta I)V - (\mu+\alpha)I = 0\\
&\alpha I - (\mu+\gamma)R = 0
\end{aligned} \tag{5}$$

Straightforward computation to solve the above system of equations gives that:

$$\left.\begin{aligned}
S_1 &= \frac{[\sigma(\beta_\circ + \beta I_1) + (\mu+\theta)]\ L}{Z}\\
V_1 &= \frac{\psi\ L}{Z}\\
R_1 &= \frac{\alpha I_1}{\mu+\gamma}
\end{aligned}\right\} \tag{6}$$

here :

$L = \{(\mu+\gamma)[\Lambda + (1-p)A] + \alpha\gamma I_1\}$

$Z = (\beta_\circ + \beta I_1 + \mu)\{(\mu+\gamma)[\sigma(\beta_\circ + \beta I_1) + \mu] + \theta(\mu+\gamma)\} + \psi(\mu+\gamma)[\sigma(\beta_\circ + \beta I_1) + \mu]$

While I_1 is a positive root for the following third order equation:

$$D_1 I_1^3 + D_2 I_1^2 + D_3 I_1 + D_4 = 0 \tag{7}$$

here:

$$\begin{aligned}
D_1 &= \sigma\mu\beta^2(\mu+\alpha+\gamma) < 0\\
D_2 &= (\Lambda + A)[\sigma\beta^2(\mu+\gamma)] + \beta\alpha\gamma[2\sigma\beta_\circ + \sigma\psi + \mu + \theta] - (\mu+\alpha)(\mu+\gamma)\left[\theta + \beta(2\sigma\beta_\circ + \sigma\mu + \sigma\psi + \mu)\right]\\
D_3 &= (\mu+\theta)[pA\beta(\mu+\gamma) + \alpha\beta_\circ\gamma] + \sigma\alpha\gamma\beta_\circ(\beta_\circ + \psi) + (\mu+\gamma)\{\beta[\sigma pA\mu + (\Lambda + A)(2\sigma\beta_\circ + \sigma\psi + \mu + \theta)] - [pA\beta(\mu+\theta) + (\mu+\alpha)[(\sigma\beta_\circ + \mu)(\beta_\circ + \mu + \psi) + \theta(\beta_\circ + \mu)]]\}\\
D_4 &= pA(\beta_\circ + \mu)(\mu+\gamma)\left[\sigma\beta_\circ + \mu + \theta\right] + pA\psi(\mu+\gamma)(\sigma\beta_\circ + \mu) + \beta_\circ(\mu+\gamma)\left[\Lambda + (1-p)A\right]\left[\sigma\beta_\circ + \sigma\psi + \mu + \theta\right] > 0
\end{aligned}$$

Clearly, equation (7) has a unique positive root given by I_1 and then E_1 exists uniquely in Int. $\Re_+^4$ if and only if at least one of the following two conditions hold.

$$(\Lambda + A)[\sigma\beta^2(\mu+\gamma)] + \beta\alpha\gamma[2\sigma\beta_\circ + \sigma\psi + \mu + \theta] < (\mu+\alpha)(\mu+\gamma)\left[\theta + \beta(2\sigma\beta_\circ + \sigma\mu + \sigma\psi + \mu)\right] \tag{8a}$$

$$\beta[\sigma pA\mu + (\Lambda + A)(2\sigma\beta_\circ + \sigma\psi + \mu + \theta)] > pA\beta(\mu+\theta) + (\mu+\alpha)[(\sigma\beta_\circ + \mu)(\beta_\circ + \mu + \psi) + \theta(\beta_\circ + \mu)] \tag{8b}$$

4. Local stability analysis of system (3)

In this section, the local stability analysis of the equilibrium points $E_\circ$ and E_1 of the system (3) is studied as shown in the following theorems.

Theorem (2): The disease free equilibrium point $E_\circ = (S_\circ, V_\circ, 0, 0)$ of system (3) is locally asymptotically stable if the following sufficient condition is satisfied:

$$\beta(S_\circ + \sigma V_\circ) < \mu + \alpha \tag{9}$$

Proof: The Jacobian matrix of system (3) at $(E_\circ)$ can be written as:

$$J(E_\circ) = \begin{bmatrix} -(\mu+\psi) & \theta & -\beta S_\circ & \gamma \\ \psi & -(\mu+\theta) & -\sigma\beta V_\circ & 0 \\ 0 & 0 & \beta(S_\circ + \sigma V_\circ) - (\mu+\alpha) & 0 \\ 0 & 0 & \alpha & -(\mu+\gamma) \end{bmatrix} = [a_{ij}]_{4\times 4}$$

Clearly, $J(E_\circ)$ has the following eigenvalues:

$$\begin{aligned}
\lambda_{S,V} &= -\frac{(2\mu+\psi+\theta)}{2} \pm \frac{1}{2}\sqrt{(2\mu+\psi+\theta)^2 - 4\mu(\mu+\psi+\theta)}\\
\lambda_I &= \beta(S_\circ + \sigma V_\circ) - (\mu+\alpha)\\
\lambda_R &= -(\mu+\gamma)
\end{aligned}$$

here λ_k , $k = S, V, I, R$ represents the eigenvalue in *k*-direction.

Obviously, λ_S and λ_V have negative real parts, while $\lambda_R < 0$. Therefore $E_\circ$ is locally asymptotically stable if and only if the eigenvalue $\lambda_I < 0$, which is satisfied provided that condition (9) holds and hence the proof is complete. ■

Theorem (3): Assume that, the endemic equilibrium point $E_1 = (S_1, V_1, I_1, R_1)$ of system (3) exists in the Int. $\Re_+^4$. Then it is locally asymptotically stable if the following condition is satisfied:

$$\mu > 2\beta(S_1 + \sigma V_1) \tag{10}$$

Proof: The Jacobian matrix of system (3) at the endemic equilibrium point E_1 that denoted by $J(E_1)$ can be written:

$$\begin{bmatrix} -(\beta_\circ + \beta I_1) - (\mu+\psi) & \theta & -\beta S_1 & \gamma \\ \psi & -(\sigma\beta_\circ + \sigma\beta I_1 + \mu + \theta) & -\sigma\beta V_1 & 0 \\ \beta_\circ + \beta I_1 & \sigma(\beta_\circ + \beta I_1) & \beta(S_1 + \sigma V_1) - (\mu+\alpha) & 0 \\ 0 & 0 & \alpha & -(\mu+\gamma) \end{bmatrix} = [b_{ij}]_{4\times 4}$$

Now, according to *Gersgorin theorem* if the following condition holds:

$$|b_{ii}| > \sum_{\substack{i=1 \\ i\neq j}}^{4} |b_{ij}|$$

Then all eigenvalues of $J(E_1)$ exists in the region:

$$\wp = \bigcup \left\{ U^* \in C : \left| U^* - b_{ii} \right| < \sum_{\substack{i=1 \\ i \neq j}}^{4} \left| b_{ij} \right| \right\}$$

Therefore, according to the given condition (10) all the eigenvalues of $J(E_1)$ exists in the left half plane and hence, E_1 is locally asymptotically stable. ■

5. Global stability analysis of system (3)

In this section, the global dynamics of system (3) is studied with the help of Lyapunov function as shown in the following theorems.

Theorem (4): Assume that, the disease free equilibrium point $E_\circ$ of system (3) is locally asymptotically stable. Then the basin of attraction of $E_\circ$, say $B(E_\circ) \subset \Re_+^4$, satisfy the following conditions:

$$\left(\frac{\theta}{S} + \frac{\psi}{V} \right)^2 < 4 \left(\frac{\mu + \psi}{S} \right) \left(\frac{\theta + \mu}{V} \right) \quad (11a)$$

$$\frac{pAS_\circ}{S} + (\beta_\circ + \beta I)(S_\circ + \sigma V_\circ) < \left(\frac{\gamma S_\circ}{S} + \mu \right) R + \mu I \quad (11b)$$

Proof: Consider the following positive definite function:

$$W_1 = \left(S - S_\circ - S_\circ \ln \frac{S}{S_\circ} \right) + \left(V - V_\circ - V_\circ \ln \frac{V}{V_\circ} \right) + I + R$$

Clearly, $W_1 : R_+^4 \to R$ is a continuously differentiable function such that $W_1(S_\circ, V_\circ, 0, 0) = 0$, and $W_1(S,V,I,R) > 0$ $\forall (S,V,I,R) \neq (S_\circ, V_\circ, 0, 0)$. Further we have:

$$\frac{dW_1}{dt} = \left(\frac{S - S_\circ}{S} \right) \frac{dS}{dt} + \left(\frac{V - V_\circ}{V} \right) \frac{dV}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$$

By simplifying this equation we get:

$$\frac{dW_1}{dt} = -\frac{(\mu + \psi)}{S}(S - S_\circ)^2 + \left(\frac{\theta}{S} + \frac{\psi}{V} \right)(S - S_\circ)(V - V_\circ) - \frac{(\mu + \theta)}{V}(V - V_\circ)^2 + \frac{pAS_\circ}{S} + (\beta_\circ + \beta I)(S_\circ + \sigma V_\circ) - \left[\left(\frac{\gamma S_\circ}{S} + \mu \right) R + \mu I \right]$$

Therefore, according to condition (11a) it is obtain that:

$$\frac{dW_1}{dt} \leq -\left[\sqrt{\frac{\mu + \psi}{S}}(S - S_\circ) - \sqrt{\frac{\mu + \theta}{V}}(V - V_\circ) \right]^2 + \frac{pAS_\circ}{S} + (\beta_\circ + \beta I)(S_\circ + \sigma V_\circ) - \left[\left(\frac{\gamma S_\circ}{S} + \mu \right) R + \mu I \right]$$

Obviously $\frac{dW_1}{dt} < 0$ for every initial points satisfying condition (11b) and then W_1 is a Lyapunov function provided that conditions (11a)-(11b) hold. Thus $E_\circ$ is globally asymptotically stable in the interior of $B(E_\circ)$, which means that $B(E_\circ)$ is the basin of attraction and that complete the proof. ■

Theorem (5): Let the endemic equilibrium point E_1 of system (3) is locally asymptotically stable. Then it is globally asymptotically stable provided that:

$$\beta(S_1 + \sigma V_1) < \mu + \alpha \quad (12a)$$

$$[\beta(S_1 + I_1) + \beta_\circ]^2 < \frac{4}{9}[\beta_\circ + \beta I + \mu + \psi][\mu + \alpha - \beta(S_1 + \sigma V_1)] \quad (12b)$$

$$[\theta + \psi]^2 < \frac{2}{3}[\beta_\circ + \beta I + \mu + \psi][\mu + \theta + \sigma(\beta_\circ + \beta I)] \quad (12c)$$

$$\gamma^2 < \frac{2}{3}[\beta_\circ + \beta I + \mu + \psi][\mu + \gamma] \quad (12d)$$

$$[\sigma(\beta_\circ + \beta(I_1 - V_1))]^2 < \frac{2}{3}[\mu + \theta + \sigma(\beta_\circ + \beta I)][\mu + \alpha - \beta(S_1 + \sigma V_1)] \quad (12e)$$

$$\alpha^2 < \frac{2}{3}[\mu + \alpha - \beta(S_1 + \sigma V_1)][\mu + \gamma] \quad (12h)$$

Proof: Consider the following positive definite function:

$$W_2 = \frac{(S - S_1)^2}{2} + \frac{(V - V_1)^2}{2} + \frac{(I - I_1)^2}{2} + \frac{(R - R_1)^2}{2}$$

Clearly, $W_2 : R_+^4 \to R$ is a continuously differentiable function such that $W_2(S_1, V_1, I_1, R_1) = 0$, and $W_2(S,V,I,R) > 0$ $\forall (S,V,I,R) \neq (S_1, V_1, I_1, R_1)$. Further, we have:

$$\frac{dW_2}{dt} = (S - S_1)\frac{dS}{dt} + (V - V_1)\frac{dV}{dt} + (I - I_1)\frac{dI}{dt} + (R - R_1)\frac{dR}{dt}$$

By simplifying this equation we get:

$$\frac{dW_2}{dt} = -\frac{1}{3}q_{11}(S-S_1)^2 - \frac{1}{3}q_{33}(I-I_1)^2 + q_{13}(S-S_1)(I-I_1) - \frac{1}{3}q_{11}(S-S_1)^2 - \frac{1}{2}q_{22}(V-V_1)^2 + q_{12}(S-S_1)(V-V_1) - \frac{1}{3}q_{11}(S-S_1)^2 - \frac{1}{2}q_{44}(R-R_1)^2 + q_{14}(S-S_1)(R-R_1) - \frac{1}{2}q_{22}(V-V_1)^2 - \frac{1}{3}q_{33}(I-I_1)^2 + q_{23}(V-V_1)(I-I_1) - \frac{1}{3}q_{33}(I-I_1)^2 - \frac{1}{2}q_{44}(R-R_1)^2 + q_{34}(I-I_1)(R-R_1)$$

With

$q_{11} = \beta_\circ + \beta I + \mu + \psi$, $q_{13} = \beta S_1 + \beta_\circ + \beta I_1$, $q_{33} = \mu + \alpha - \beta(S_1 + \sigma V_1)$, $q_{12} = \theta + \psi$, $q_{22} = \mu + \theta + \sigma(\beta_\circ + \beta I)$, $q_{44} = \mu + \gamma$, $q_{14} = \gamma$, $q_{23} = \sigma(\beta_\circ + \beta I_1 - \beta V_1)$, $q_{34} = \alpha$

Therefore, according to the conditions (12a)-(12h) we obtain that:

$$\frac{dW_2}{dt} \le -\left[\sqrt{\frac{q_{11}}{3}}(S-S_1) - \sqrt{\frac{q_{33}}{3}}(I-I_1)\right]^2 - \left[\sqrt{\frac{q_{11}}{3}}(S-S_1) - \sqrt{\frac{q_{22}}{2}}(V-V_1)\right]^2 - \left[\sqrt{\frac{q_{11}}{3}}(S-S_1) - \sqrt{\frac{q_{44}}{2}}(R-R_1)\right]^2 - \left[\sqrt{\frac{q_{22}}{2}}(V-V_1) - \sqrt{\frac{q_{33}}{3}}(I-I_1)\right]^2 - \left[\sqrt{\frac{q_{33}}{3}}(I-I_1) - \sqrt{\frac{q_{44}}{2}}(R-R_1)\right]^2$$

Clearly, $\frac{dW_2}{dt} < 0$, and then W_2 is a Lyapunov function provided that the given conditions hold. Therefore, E_1 is globally asymptotically stable. ■

6. The local bifurcation analysis of system (3)

In this section, the occurrence of local bifurcations (such as saddle-node, transcritical and pitchfork) near the equilibrium points of system (3) is studied in the following theorem.

Theorem (6): System (3) has a transcritical bifurcation near the disease free equilibrium point $E_\circ$, but neither saddle-node bifurcation, nor pitchfork bifurcation can accrue at the parameter

$$\mu_\circ = \beta(S_\circ + \sigma V_\circ) - \alpha \qquad (13)$$

Proof: It is easy to verify that the Jacobian matrix of system (3) at $(E_\circ, \mu_\circ)$ can be written as:

$$J_{\mu_\circ} = Df(E_\circ, \mu_\circ) = \begin{bmatrix} -(\mu_\circ + \psi) & \theta & -\beta S_\circ & \gamma \\ \psi & -(\mu_\circ + \theta) & -\sigma\beta V_\circ & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & -(\mu_\circ + \gamma) \end{bmatrix}$$

Clearly, the third eigenvalue λ_I in I-direction is zero $(\lambda_I = 0)$, further the eigenvector (say $K = (k_1, k_2, k_3, k_4)^T$) corresponding to $\lambda_I = 0$ satisfy the following:

$J_{\mu_\circ} K = \lambda K$ then $J_{\mu_\circ} K = 0$ From which we get that:

$$-(\mu_\circ + \psi)k_1 + \theta k_2 - \beta S_\circ k_3 + \gamma k_4 = 0 \qquad (14a)$$

$$\psi k_1 - (\mu_\circ + \theta)k_2 - \sigma\beta V_\circ k_3 = 0 \qquad (14b)$$

$$\alpha k_3 - (\mu_\circ + \gamma)k_4 = 0 \qquad (14c)$$

So by solving the above system of equations we get:

$k_1 = -xk_3$; $k_2 = -yk_3$; $k_4 = zk_3$

Where:

$$x = \frac{\{\sigma\beta\theta(\mu_\circ + \gamma)V_\circ + (\mu_\circ + \theta)(\beta S_\circ(\mu_\circ + \gamma) + \alpha\gamma)\}}{2\mu_\circ}$$

$$y = \frac{\{\psi[\sigma\beta\theta(\mu_\circ + \gamma)V_\circ + (\mu_\circ + \theta)(\beta S_\circ(\mu_\circ + \gamma) + \alpha\gamma)] + 2\sigma\beta\mu_\circ V_\circ\}}{2\mu_\circ(\mu_\circ + \theta)}$$

$$z = \frac{\alpha}{(\mu_\circ + \gamma)}$$

Here k_3 be any non zero real number. Thus

$$K = \begin{bmatrix} -xk_3 \\ -yk_3 \\ k_3 \\ zk_3 \end{bmatrix}$$

Similarly the eigenvector $W = (w_1, w_2, w_3, w_4)^T$ that corresponding to $\lambda_I = 0$ of $J^T_{\mu_\circ}$ can be written:

$$J^T_{\mu_\circ}\cdot W=\begin{bmatrix}-(\mu_\circ+\psi) & \psi & 0 & 0\\ \theta & -(\mu_\circ+\theta) & 0 & 0\\ -\beta S_\circ & -\sigma\beta V_\circ & 0 & \alpha\\ \gamma & 0 & 0 & -(\mu_\circ+\gamma)\end{bmatrix}\cdot\begin{bmatrix}w_1\\ w_2\\ w_3\\ w_4\end{bmatrix}=0$$

This gives:

$$W=\begin{bmatrix}0\\ 0\\ w_3\\ 0\end{bmatrix}$$

Here w_3 is any non-zero real number. Now rewrite system (3) in a vector form as:

$$\frac{dX}{dt}=f(X)$$

Where $X=(S,V,I,R)^T$ and $f=(f_1,f_2,f_3,f_4)^T$ with $f_i, i=1,2,3,4$ are given in system (3), and then determine $\frac{df}{d\mu}=f_\mu$ we get that:

$$f_\mu=\begin{bmatrix}-S\\ -V\\ -I\\ -R\end{bmatrix}\quad\text{then}\quad f_\mu(E_\circ,\mu_\circ)=\begin{bmatrix}-S_\circ\\ -V_\circ\\ 0\\ 0\end{bmatrix}$$

Therefore:

$$W^T\cdot f_\mu(E_\circ,\mu_\circ)=0$$

Consequently, according to Sotomayor Theorem [10] the system (3) has no saddle-node bifurcation near $E_\circ$ at $\mu_\circ$.

Now in order to investigate the accruing of other types of bifurcation, the derivative of f_μ with respect to vector X, say $Df_\mu(E_\circ,\mu_\circ)$, is computed

$$Df_\mu(E_\circ,\mu_\circ)=\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}$$

So

$$W^T\cdot\left[Df_\mu(E_\circ,\mu_\circ)\cdot K\right]=-k_3w_3\neq 0$$

Again, according to Sotomayor theorem, if in addition to the above, the following holds

$$W^T\cdot\left[D^2f(E_\circ,\mu_\circ)\cdot(K,K)\right]\neq 0$$

here $Df(E_\circ,\mu_\circ)$ is the Jacobian matrix at $E_\circ$ and $\mu_\circ$, then the system (3) possesses a transcritical bifurcation but no pitch-fork bifurcation can occur. Now since we have that:

$$\left[D^2f(E_\circ,\mu_\circ)\cdot(K,K)\right]=\begin{bmatrix}2x\beta k_3^2\\ 2y\beta k_3^2\\ -\beta(x+\sigma y)(1+k_3)k_3\\ 0\end{bmatrix}$$

Therefore:

$$W^T\cdot\left[D^2f(E_\circ,\mu_\circ)\cdot(K,K)\right]=-\beta(x+\sigma y)(1+k_3)k_3w_3\neq 0$$

Then the system (3) has a transcritical bifurcation at $E_\circ$ when the parameter μ passes through the bifurcation value $\mu_\circ$. ■

7. Numerical analysis of system (3):

In this section, the global dynamics of system (3) is studied numerically. The objectives of this study are confirming our obtained analytical results and understand the effects of immigration, existence of vaccine and existence of the external sources for disease on the dynamic of *SVIRS* epidemic model. Consequently, first system (3) is solved numerically for different sets of initial conditions and for different sets of parameters. It is observed that, for the following set of hypothetical parameters that satisfies stability condition (9) of disease free equilibrium point, system (3) has a globally asymptotically stable disease free equilibrium point as shown in following figure.

$$\begin{aligned}&\Lambda=400,\ A=100,\ p=0,\ \beta=0.0005,\\ &\beta_\circ=0,\ \mu=0.1,\psi=0.5,\ \theta=0.05,\\ &\sigma=0.01,\ \alpha=0.8,\gamma=0.5\end{aligned}\qquad(15)$$

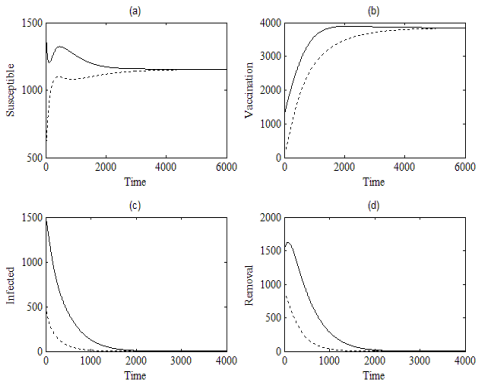

Figure 2- *Time series of the solution of system (3). (a) trajectories of S, (b) trajectories of V, (c) trajectories of I and (d) trajectories of R. The solid line refers to the trajectory started at (1500,1200,1500,1500) while dotted line refers to trajectory started at (500,400,500,900).*

Clearly, Figure (2) show that the solution of system (3) approaches asymptotically to the disease free equilibrium point $E_\circ = (1153.3846,0,0)$ starting from two different initial points and this is confirming our obtained analytical results. However, for the data given by equation (15) with $\beta = 0.001$. The trajectories of system (3) starting from different sets of initial data are drawn in Figures(3a)-(3d).

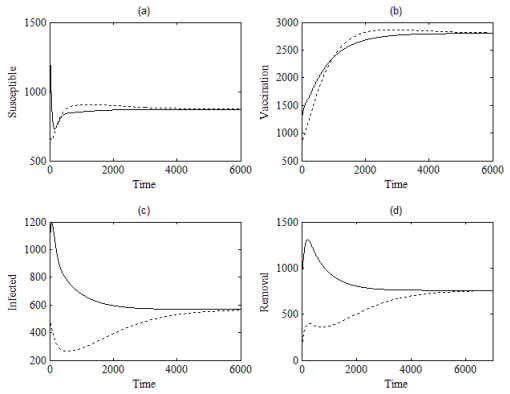

Figure 3-*Time series of the solution of system (3). (a) trajectories of S , (b) trajectories of V, (c) trajectories of I and (d) trajectories of R.* the solid line refers to the trajectory started at *(1500,1200,1000,900)* while the dotted line refers to the trajectory started at *(700,800,500,100).*

Obviously, Figure (3) Show clearly the convergence of system (3) to the endemic equilibrium point $E_1 = (871,2800,568,758)$ asymptotically from two different initial points. This is indicates to occurrence of a transcritical bifurcation near the disease free equilibrium point at a specific value of $\beta \in (0.0005,0.001)$, so $E_\circ$ became unstable and the solution of system (3) approaches to E_1. In addition to that, the above two figures refer to that increasing the contact rate between S and I causes destabilizing to disease free equilibrium point and the system approaches instead to the endemic point.

Now the effect of increasing the incidence rate of disease resulting from external sources in the environment on the dynamics of system (3) is studied by solving the system numerically for the parameters values $\beta_0 = 0.1$, 0.5, 1 respectively, keeping other parameters fixed as given in equation (15), and then the trajectories of system (3) are drawn in Figures (4a)-(4c) respectively.

Note that, in the next figures (4-9), we will use the following representations: *Solid line for describing trajectory of **S**; dashed line for describing trajectory of **V**; dash dot line for describing trajectory of **I**; dotted line for describing trajectory of **R** and starting at (2000, 1500, 1000, 1250).*

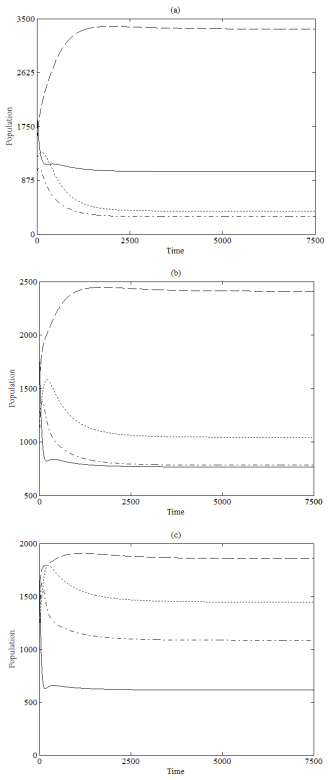

Figure 4- *Time series of the solution of system (3). (a) for* $\beta_0 = 0.1$, *(b) for* $\beta_0 = 0.5$, *(c) for* $\beta_0 = 1$.

According to Figure (4), as the incidence rate of disease resulting from external sources increases *(through increasing* β_0*)*, the disease free equilibrium point of system (3) becomes unstable point and the trajectory of system (3) approaches asymptotically to the endemic equilibrium point. In fact as β_0 increases it is observed that the number of susceptible and vaccinated individuals decrease and the number of recover and infected individuals increases.

Similar results are obtained, as those shown in case of increasing β_0, in case of increasing the density of arriving infected immigrant individuals, that is means

increasing p and keeping other parameters fixed as given in (15).

The effect of varying the vaccine coverage rate on the dynamical behavior of system (3) is studied too. The system is solved numerically for different values of $\psi = 0.01, 0.2, 0.9$, keeping other parameters fixed as given in equation (15) and then the trajectories of system (3) are drawn in Figures (5a)-(5c) respectively.

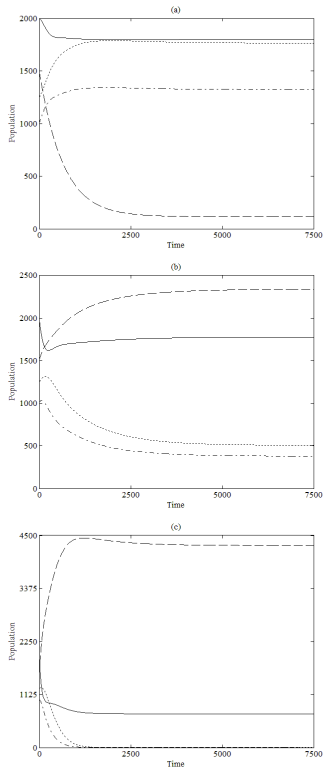

Figure 5- *Time series of the solution of system (3). (a) for* $\psi = 0.01$, *(b) for* $\psi = 0.2$, *(c) for* $\psi = 0.9$.

From the above figure it is clear that as the rate of vaccine coverage increases the endemic equilibrium point of system (3) becomes unstable point and the trajectory of the system approaches asymptotically to the disease free equilibrium point attendant that increasing in vaccined individuals and decreasing in susceptible individuals.

The effect of varying the lifetime of vaccine immunity, on the dynamical behavior of system (3) is investigated. The system (3) is solved numerically for the values $\theta = 0.1$, 0.2, 1, keeping the rest of parameters fixed as given in equation (15), and then the trajectories of system (3) are drawn in Figures (6a)-(6c). In this case, it is observed that increasing θ (that is decreasing the lifetime of vaccine immunity) destabilizes the disease free equilibrium point and then the solution of system (3) approaches to endemic equilibrium point attended that increasing in the susceptible, infected and recover individuals while the number of vaccinated individuals decreases.

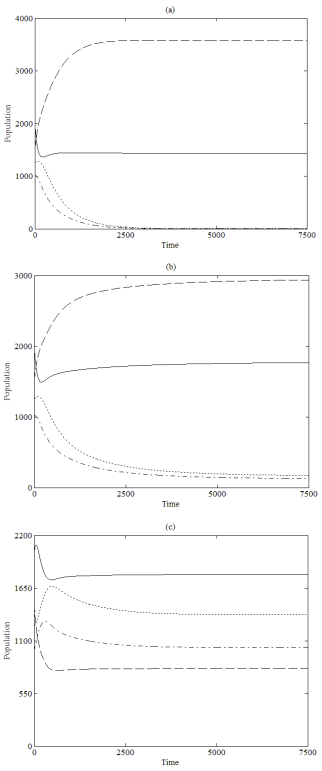

Figure 6-*Time series of the solution of system (3). (a) for* $\theta = 0.1$, *(b) for* $\theta = 0.2$, *(c) for* $\theta = 1$.

In the following, system (3) is solved numerically for the following values of recovery rates $\alpha = 0.1$, 0.3 , 0.6, keeping other parameters fixed as given in equation (15), and then the trajectories of system (3) are drawn in Figures (8a)-(8c) respectively.

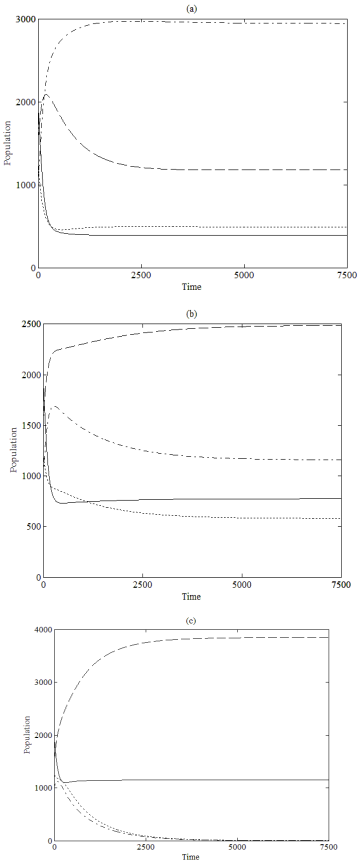

Figure 7- *Time series of the solution of system (3). (a) for* $\alpha = 0.1$, *(b) for* $\alpha = 0.3$, *(c) for* $\alpha = 0.6$.

It is obvious from Figure (7) that, as the recovery rate increases from 0.1 to 0.6 the endemic equilibrium point of system (3) becomes unstable point and the trajectory of system (3) approaches asymptotically to the disease free equilibrium point. But the number of susceptible and vaccinated individuals increases while the number of the infected and recover individuals decreases.

Now the effect of changing the lifetime of removal individual's immunity on the dynamical behavior of system (3) is also studied by changing the value of parameter γ at different values while the other parameters sill fixed. It is observed that changing the parameter γ has no effect on the dynamical behavior of system (3).

Finally, the effect of the natural death rate on the dynamics of system (3) is investigated numerically. It is observed that, decreases the parameter μ less than 0.1 keeping other parameters fixed as in (15) causes transferring in the stability of system (3) from disease free equilibrium point to endemic equilibrium point as shown in Figure (8). However, , increases the parameter μ more than 0.1 keeping other parameters fixed as in (15) with $\beta = 0.001$ causes transferring in the stability of system (3) from endemic equilibrium point to disease free equilibrium point as shown in Figure (9). Therefore, the death rate due to the disease plays a vital role as bifurcation parameter of system (3).

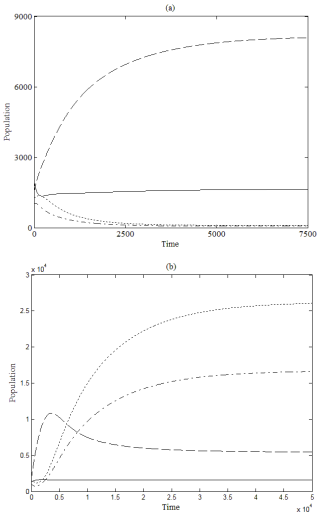

Figure 8- *Time series of the solution of system (3) for the data given by (15) with varying* μ. *(a) for* $\mu = 0.05$, *(b) for* $\mu = 0.01$.

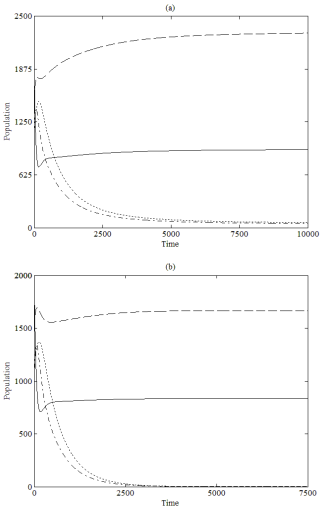

Figure 9- *Time series of the solution of system (3) for the data given by (15) with* $\beta = 0.001$ *and varying* μ. *(a) for* $\mu = 0.15$, *(b) for* $\mu = 0.2$.

8. Conclusion and discussion:

In this paper, mathematical model has been proposed and analyzed. The objective is to study the effect of immigrants, existence and nonexistence vaccine, and then existence

of external sources of the disease in the environment on the dynamical behavior of *SVIRS* epidemic model. The existence and the stability analysis of all possible equilibrium points are studied analytically as well as numerically. It is observed that system (3) has transcritical bifurcation near the disease free equilibrium point, but neither saddle node nor pitchfork bifurcation can accrue. Further the system (3) do not has Hopf bifurcation near the endemic equilibrium point. Finally according to the numerically simulation the following results are obtained:

1. The system (3) do not has periodic dynamic, instead it they approach either to the disease free equilibrium point or else to endemic equilibrium point.

2. As the number of the infected immigrant individuals and the incidence rate of disease (external incidence rate or contact incidence rate) increase, the asymptotic behavior of the system (3) transfer from approaching to disease free equilibrium point to the endemic equilibrium point.

3. As the lifetime of vaccine immunity decreases (the losing vaccine immunity rate (θ) increases), then the disease free equilibrium point of system (3) becomes unstable and the solution will approaches to the endemic equilibrium point. Further, similar result is obtained in system (3) when the natural death rate decreases.

4. As the recovery rates in the system (3) increase then the solution in the system will be transfer from stability at endemic equilibrium point to stability at disease free equilibrium point. Further, similar result is obtained in case of system (3) when the vaccine coverage rate increases.

5. Finally, changing the lifetime of removal individual's immunity in the system (3) do not has vital effect on the dynamical behavior of this.

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