A Study on the Bases of Space of Vector Valued Entire Multiple Dirichlet Series

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Abstract:

Let $f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} e^{(s_1\lambda_m + s_2\mu_n)}$, $(s_j = \sigma_j + it_j, j = 1,2)$, $\{\lambda_m\}_1^{\infty}$ and $\{\mu_n\}_1^{\infty}$ being an increasing sequences of positive numbers and $a_{m,n} \in E$ where E is Banach algebra, represent a vector valued entire Dirichlet functions in two variables. The space Γ of all such entire functions having order at most equal to ρ is considered in this paper. A metric topology using the growth parameters of f is defined on Γ and its various properties are obtained. The form of linear operator on the space Γ is characterized and proper bases are also characterized in terms of growth parameters ρ .

Keyword: Vector Valued Dirichlet Series, Banach Algebra, Entire Function.

دراسة عن اساسات فضاء متجه القيم لسلسلة درشلت الكلية المتعددة

مشتاق شاكر و نجم رعد نجم
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الخلاصة

لـيكن $f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} e^{(s_1\lambda_m + s_2\mu_n)}$ ، $(s_j = \sigma_j + it_j, j = 1,2)$ ، $\{\lambda_m\}_1^{\infty}$ و $\{\mu_n\}_1^{\infty}$ متتاليـات متزايـدة مـن الارقـام الموجبـه و $a_{m,n} \in E$ حيـث E هـو بنـاخ الجبـرا ، يمثـل دوال درشلت الكلية متجهة القيمه لمتغيرين. درسنا في هذا البحث Γ فضـاء جميع الدوال الكلية ذوات الرتبـه ρ . لقد عرّفنـا الفضـاء المتري باستخدام نمو متغيرات الدالـة f والمعرّفة علـى الفضـاء Γ ، وحصّلنا على شكل المؤثر الخطي على الفضاء Γ والممثل بالأساس الفعلي بدلالة نمو الوسيط ρ .

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Introduction

1. Let

$f(s_1,s_2)=\sum_{m,n=1}^{\infty} a_{m,n}e^{(s_1\lambda_m+s_2\mu_n)}$,

$s_j=\sigma_j+it_j$, $j=1,2$ where (σ_j,t_j are real variables) (1.1)

Where $a_{m,n}'s$ belong to a commutative Banach algebra E with identity element ω with $\|\omega\|=1$ and $\lambda_m's\in R$ satisfy the conditions $0<\lambda_1<\lambda_2<...<\lambda_m<...$ $\lambda_m\to\infty$ as $m\to\infty$ also $\mu_n's\in R$ satisfy the conditions $0<\mu_1<\mu_2<...<\mu_n<...$ $\mu_n\to\infty$ as $n\to\infty$ and

$$\lim_{m,n\to\infty}\sup\frac{\log\|a_{m,n}\|}{\lambda_m+\mu_n}=-\infty \qquad (1.2)$$

$$\lim_{m+n\to\infty}\sup\frac{\log(m+n)}{\lambda_m+\mu_n}=D<+\infty \qquad (1.3)$$

Then, the vector valued Dirichlet series in (1.1) represents an entire function $f(s_1,s_2)$ (see [1] and [2]). In [2] G.S. Srivastava and Archna Sharma defined the growth parameters such as order, type of vector valued Dirichlet series in two variables

They also obtained the results for coefficient characterization of order and type. The concepts of order and type of entire function (also for analytic function) represented by vector valued Dirichlet series of one complex variable were first introduced in 1983 by B.L. Srivastava [3]. They also obtained the coefficient characterizations of order and type. The space Y of all entire functions represented by vector valued Dirichlet series $f(s)$ of one complex variable having order at most equal to ρ were first introduced in 2012 by G.S. Srivastava and Archan Sharma [4]. A metric topology using the growth parameters of f is defined on Y and its various properties are also obtained by them. They also obtained the form of linear operator on the space Y is characterization and proper bases are also characterization in terms of growth parameters ρ . In this paper we have extended and improve the above results to the entire function represented by vector valued Dirichlet series of several complex variables. For the sake of simplicity, we consider the functions of two complex variables. Through our results can be easily extended to functions of several complex variables.

Let for entire functions defined as above by (1.1)

$$M(\sigma_1,\sigma_2)=\sup_{-\infty<t_j<\infty}\left\{\|f(\sigma_1+it_1,\sigma_2+it_2)\|,-\infty\le t_j\le\infty,\ j=1,2\right\}$$

.

Then $M(\sigma_1,\sigma_2)$ is called maximum modulus of $f(s_1,s_2)$ on the tube Re $s_j=\sigma_j$ $j=1,2$.

Jain and Gupta [5] defined the order ρ $(0\le\rho\le\infty)$ of $f(s_1,s_2)$ as:

$$\rho=\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{\log\log M(\sigma_1,\sigma_2)}{\log(e^{\sigma_1}+e^{\sigma_2})} ,$$

$$(0\le\rho\le\infty)$$

Let us denote by Γ the linear space of all vector valued entire multiple Dirichlet functions f of finite order less than or equal to ρ . Then every function $f\in\Gamma$, satisfying

$$\lim_{m+n\to\infty}\sup\frac{m+n}{\lambda_m+\mu_n}=D'<\infty \qquad (1.4)$$

is characterized by the condition

$$\lim_{m,n\to\infty}\sup\frac{\log(\lambda_m^{\lambda_m}.\mu_n^{\mu_n})}{\log\|a_{m,n}\|^{-1}}=\rho \qquad (1.5)$$

This is equivalent to the condition

$$\|a_{m,n}\|^{1/(\lambda_m\cdot\mu_n)}(\lambda_m^{1/\mu_n}.\mu_n^{1/\lambda_m})^{1/(\rho+\varepsilon)}\to 0 \quad \text{as } m,n\to\infty \text{ for each } \varepsilon>0. \qquad (1.6)$$

Now for each $f\in\Gamma$, for $\delta>0$, we define the quantity

$$\|f,\rho+\delta\|=\sum_{m,n=1}^{\infty}\|a_{m,n}\|(\lambda_m^{\lambda_m}.\mu_n^{\mu_n})^{1/(\rho+\delta)} \qquad (1.7)$$

In view of (1.6), $\|f,\rho+\delta\|$ is well defined and defines a norm on Γ . Let us denote by $\Gamma(\rho,\delta)$. The space Γ, equipped with the norm $\|f,\rho+\delta\|$. This norm induces a metric topology on Γ . We can define the equivalent metric

$$d(f,g)=\sum_{q,t=1}^{\infty}\frac{1}{2^{q+t}}.\frac{\left\|f-g;\rho+(q,t)^{-1}\right\|}{1+\left\|f-g;\rho+(q,t)^{-1}\right\|} \quad ,$$

$f,g\in\Gamma$. (1.8)

Through this paper, we shall assume that Γ is equipped with the topology generated by the metric d.

2. Following Suzanne D. and Poitiers [6], we give some definitions. A Sequence of function $\{\alpha_{m,n}\}\subseteq\Gamma$ is said to be linearly independent if $\sum_{m,n=1}^{\infty}c_{m,n}\alpha_{m,n}=0$ implies that $c_{m,n}=0 \ \forall m,n$, for all sequence of complex number $\{c_{m,n}\}$ for which $\sum_{m,n=1}^{\infty}c_{m,n}\alpha_{m,n}$ converges in Γ. A subspace Γ_0 of Γ is said to be spanned by sequence $\{\alpha_{m,n}\}\subseteq\Gamma$ if Γ_0 consists of all linear combinations $\sum_{m,n=1}^{\infty}c_{m,n}\alpha_{m,n}$ such that $\sum_{m,n=1}^{\infty}c_{m,n}\alpha_{m,n}$ converges in Γ. A sequence $\{\alpha_{m,n}\}\subseteq\Gamma$ which is linearly independent and spans a subspace Γ_0 of Γ is said to be base in Γ_0. In particular, if $e_{m,n}\in\Gamma$, $e_{m,n}(s_1,s_2)=\omega e^{s_1\lambda_m+s_2\mu_n}$, $m,n\ge 1$, then $\{e_{m,n}\}$ is base in Γ. A sequence $\{\alpha_{m,n}\}\subseteq\Gamma$ will be called a 'proper base' if it is a bases and it satisfies for all sequences $\{a_{m,n}\}\subseteq E$, convergence of $\sum_{m,n=1}^{\infty}a_{m,n}\alpha_{m,n}$ in Γ implies the convergence of $\sum_{m,n=1}^{\infty}a_{m,n}e_{m,n}$ in Γ.

First we shall prove

Theorem 1. The space Γ is complete with respect to the metric d.

Proof: Let $\{f_\alpha\}$ be a Cauchy sequence in Γ, where $f_\alpha(s_1,s_2)=\sum_{m,n=1}^{\infty}a_{m,n}^{(\alpha)}e^{s_1\lambda_m+s_2\mu_n}$. Hence it is a Cauchy sequence in $\Gamma(\rho,\delta)$. Therefore, for given positive number ε and δ there exists a positive integer $N_0=N_0(\varepsilon,\delta)$ such that

$$\left|f_{m,n}^{(\alpha)}-f_{m,n}^{(\beta)}\right|<\varepsilon \qquad \forall\alpha,\beta\ge N_0.$$

$$\left|f_{m,n}^{(\alpha)}-f_{m,n}^{(\beta)}\right|^{1/(\lambda_m+\mu_n)}<\varepsilon \qquad ,\alpha,\beta\ge 0$$

, $m+n\ne 0$, $m,n\ge 0$.

Denoting by

$$f_\alpha(s_1,s_2)=\sum_{m,n=1}^{\infty}a_{m,n}^{(\alpha)}e^{s_1\lambda_m+s_2\mu_n}$$

$$, f_\beta(s_1,s_2)=\sum_{m,n=1}^{\infty}a_{m,n}^{(\beta)}e^{s_1\lambda_m+s_2\mu_n} .$$

We have therefore

$$\sum_{m,n=1}^{\infty}\left\|a_{m,n}^{(\alpha)}-a_{m,n}^{(\beta)}\right\|(\lambda_m^{\lambda_m}.\mu_n^{\mu_n})^{1/(\rho+\delta)}<\varepsilon$$

$\forall\alpha,\beta>N_0$. (2.1)

Therefore for each fixed $m,n=1,2,3,...$, $\{a_{m,n}^{(\alpha)}\}$ is a Cauchy sequence in the Banach space E. Hence there exists a sequence $\{a_{m,n}\}\subseteq E$ such that

$$\lim_{\alpha\to\infty}a_{m,n}^{(\alpha)}=a_{m,n} \quad , m,n\ge 1.$$

Now letting $\beta\to\infty$ in (2.1), we have for $\alpha\ge N_0$,

$$\sum_{m,n=1}^{\infty}\left\|a_{m,n}^{(\alpha)}-a_{m,n}\right\|(\lambda_m^{\lambda_m}.\mu_n^{\mu_n})^{1/(\rho+\delta)}\le\varepsilon \qquad (2.2)$$

Now we choose δ_1, $0<\delta_1<\delta$. Then we have

$$\|a_{m,n}\|\le\left\|a_{m,n}^{(N_0)}-a_{m,n}\right\|+\left\|a_{m,n}^{(N_0)}\right\|$$
$$\le\varepsilon\lambda_m^{-\lambda_m/(\rho+\delta)}.\mu_n^{-\mu_n/(\rho+\delta)}+\lambda_m^{-\lambda_m/(\rho+\delta_1)}.\mu_n^{-\mu_n/(\rho+\delta_1)}$$

.

Hence by (1.6) we have

$$\|a_{m,n}\|(\lambda_m^{\lambda_m}.\mu_n^{\mu_n})^{1/(\rho+\delta)}\le\varepsilon+\lambda_m^{-\lambda_m(\delta-\delta_1)/(\rho+\delta)(\rho+\delta_1)}$$
$$.\mu_n^{-\mu_n(\delta-\delta_1)/(\rho+\delta)(\rho+\delta_1)}$$

.

Since ε arbitrary and $\delta_1<\delta$ therefore we obtain

$$\lim_{m,n\to\infty}\|a_{m,n}\|(\lambda_m^{\lambda_m}.\mu_n^{\mu_n})^{1/(\rho+\delta)}=0, \text{ for each } \delta>0$$

Thus $f(s_1,s_2)=\sum_{m,n=1}^{\infty}a_{m,n}e_{m,n}\in\Gamma$. Therefore $f_\alpha\to f$ in Γ. Hence Γ is complete. Here, Γ

is a normed linear metric space and Γ is complete with respect to the metric d and hence it is a Frechet space also. This proves Theorem 1.
Next we prove

Theorem 2. A continuous linear function $\varphi : \Gamma(\rho,\delta) \to E$ is of the form

$$\varphi(f) = \sum_{m,n=1}^{\infty} a_{m,n} c_{m,n} \ ,$$

$$f = \sum_{m,n=1}^{\infty} a_{m,n} e_{m,n}$$

if and only if $\left|c_{m,n}\right| (\lambda_m^{-\lambda_m} . \mu_n^{-\mu_n})^{1/(\rho+\delta)}$ is bounded for all $m \geq 1, n \geq 1$.

Proof: Let a linear function φ on $\Gamma(\rho,\delta)$ be given by

$$\varphi(f) = \sum_{m,n=1}^{\infty} a_{m,n} c_{m,n} \ , \ f = \sum_{m,n=1}^{\infty} a_{m,n} e_{m,n} \in \Gamma \ ,$$

where $c_{m,n} = \varphi(e_{m,n})$. Let φ be continuous. Hence there exists a positive constant k such that

$$\|\varphi(f)\| \leq k \|f, \rho + \delta\|, \quad \text{for all } f \in \Gamma .$$

Assuming $f = e_{m,n} = \omega e^{s_1 \lambda_m + s_2 \mu_n} \in \Gamma$, this implies that

$$\left|c_{m,n}\right| \leq k(\lambda_m^{\lambda_m} . \mu_n^{\mu_n})^{1/(\rho+\delta)}, \ m,n \geq 1.$$

Conversely, let f be as above and $\varphi(f) = \sum_{m,n=1}^{\infty} a_{m,n} c_{m,n}$, where $\left|c_{m,n}\right| (\lambda_m^{-\lambda_m} . \mu_n^{-\mu_n})^{1/(\rho+\delta)}$ is bounded. Here $\varphi(f)$ is well define since

$$\left\| \sum_{m,n=1}^{\infty} a_{m,n} c_{m,n} \right\| \leq \sum_{m,n=1}^{\infty} \left\| a_{m,n} c_{m,n} \right\| \leq$$

$$k \sum_{m,n=1}^{\infty} \left\| a_{m,n} \right\| (\lambda_m^{\lambda_m} . \mu_n^{\mu_n})^{1/(\rho+\delta)} \leq +\infty \ .$$

Therefore φ is continuous linear function on $\Gamma(\rho,\delta)$.

We next prove

Theorem 3. Let ρ, E and $\alpha_{m,n} \in \Gamma$ be as above. Then the following are equivalent:

(i) There exists a continuous linear transformation $T : \Gamma \to \Gamma$ with

$$T(e_{m,n}) = \alpha_{m,n}$$
$$m = 1,2,3,\ldots , n = 1,2,3,\ldots \ ,$$

(ii) For each $\delta > 0$,

$$\lim_{m,n\to\infty} \sup \frac{\log\left\|\alpha_{m,n}; \rho + \delta\right\|}{\log(\lambda_m^{\lambda_m} . \mu_n^{\mu_n})} < \frac{1}{\rho} \ . \ (2.3)$$

Proof: Let T be a continuous linear transformation from Γ into Γ with $T(e_{m,n}) = \alpha_{m,n}$, $m = 1,2,3,\ldots , n = 1,2,3,\ldots$ Then for any given $\delta > 0$, there exists $\delta_1, \delta_2 > 0$ and a constants $K_1 = K_1(\delta_1), K_2 = K_2(\delta_2)$ depending on δ_1 and δ_2 respectively such that

$$\left\|T(e_{m,n}); \rho + \delta\right\| \leq k_1 k_2 \left\|e_{m,n}; \rho + (\delta_1, \delta_2)\right\|$$

or

$$\left\|\alpha_{m,n}; \rho + \delta\right\| \leq k_1 k_2 . \lambda_m^{\lambda_m/(\rho+\delta_1)} . \mu_n^{\mu_n/(\rho+\delta_2)} \ .$$

Hence

$$\lim_{m,n\to\infty} \sup \frac{\log\left\|\alpha_{m,n}; \rho + \delta\right\|}{\log(\lambda_m^{\lambda_m} . \mu_n^{\mu_n})} < \frac{1}{\rho} .$$

Conversely, let the sequence $\{\alpha_{m,n}\}$ satisfy (2.3) and let $\alpha \in \Gamma$ with $\alpha(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} e_{m,n}$.

Then it follows that

$$\lim_{m,n\to\infty} \left\|a_{m,n}\right\|^{1/(\lambda_m . \mu_n)} . \lambda_m^{1/\mu_n(\rho+\delta_1)} . \mu_n^{1/\lambda_m(\rho+\delta_2)} = 0 ,$$

for each $\delta_1, \delta_2 > 0$.

Or

$$\left\|a_{m,n}\right\|^{1/(\lambda_m . \mu_n)} . \lambda_m^{1/\mu_n(\rho+\delta_1)} . \mu_n^{1/\lambda_m(\rho+\delta_2)} < 1,$$

for all $m,n \geq N_0$.

Further, for a given $\nu, \eta > \delta$, we can find $N_1 = N_1(\nu), N_2 = N_2(\eta)$ from (2.3) , such that for

$$\frac{\log\left\|\alpha_{m,n}; \rho + \delta\right\|}{\log(\lambda_m^{\lambda_m} . \mu_n^{\mu_n})} < \frac{1}{\rho + (\nu, \eta)} ,$$

for all $m \geq N_1, n \geq N_2$.

Or

$\|\alpha_{m,n};\rho+\delta\| < \lambda_m^{\lambda_m/(\rho+\nu)}.\mu_n^{\mu_n/(\rho+\eta)}$, for $m \ge N_1, n \ge N_2$.

Choose $m \ge \max(N_0, N_1)$, $n \ge \max(N_0, N_2)$ .Then

$$\|a_{m,n}\|.\|\alpha_{m,n};\rho+\delta\| < \lambda_m^{-\lambda_m/(\rho+\delta)}.\mu_n^{-\mu_n/(\rho+\delta)}.\lambda_m^{\lambda_m/(\rho+\nu)}.\mu_n^{\mu_n/(\rho+\eta)}$$

$$= \lambda_m^{-\lambda_m(\nu-\delta)/(\rho+\delta)(\rho+\nu)}.\mu_n^{-\mu_n(\eta-\delta)/(\rho+\delta)(\rho+\eta)}$$

Since $\nu,\eta > \delta$, the series $\sum_{m,n=1}^{\infty}\|a_{m,n}\|\|\alpha_{m,n};\rho+\delta\|$ converges for each $\delta > 0$.

Since $\|a_{m,n}\alpha_{m,n}\| \le \|a_{m,n}\|.\|\alpha_{m,n}\|$, $\sum_{m,n=1}^{\infty}a_{m,n}\alpha_{mn}$ converges absolutely in Γ and since Γ is complete we find that $\sum_{m,n=1}^{\infty}a_{m,n}\alpha_{mn}$ converges to an element of Γ. Hence there exists a transformation $T:\Gamma\to\Gamma$, such that $T(\alpha) = \sum_{m,n=1}^{\infty}a_{m,n}\alpha_{mn} = \beta$ (say) for each $\alpha\in\Gamma$.

We observe that T is linear and $T(e_{m,n}) = \alpha_{m,n}$. Now we have only to prove the continuity of T.

From (2.3), give a $\delta > 0$, there exists $\delta_1, \delta_2 > 0$ such that

$$\frac{\log\|\alpha_{m,n};\rho+\delta\|}{\log(\lambda_m^{\lambda_m}.\mu_n^{\mu_n})} < \frac{1}{\rho+(\delta_1,\delta_2)} \text{ , for}$$

all $m,n \ge N = N(\delta,(\delta_1,\delta_2))$.

$\|\alpha_{m,n};\rho+\delta\| < k_1k_2.\lambda_m^{\lambda_m/(\rho+\delta_1)}.\mu_n^{\mu_n/(\rho+\delta_2)}$, for all $m,n \ge 0, k_1 = k_1(\delta_1), k_2 = k_2(\delta_2)$.

Hence

$$\|T(\alpha);\rho+\delta\| \le \sum_{m,n=1}^{\infty}\|a_{m,n}\|.\|\alpha_{m,n};\rho+\delta\|$$

$$\le \sum_{m,n=1}^{\infty}\|a_{m,n}\|\, k_1k_2.\lambda_m^{\lambda_m/(\rho+\delta_1)}.\mu_n^{\mu_n/(\rho+\delta_2)}$$

$$= k_1k_2\|\alpha_{m,n};\rho+(\delta_1,\delta_{2)}\|.$$

Hence T is continuous. This proves Theorem 3.

We now give the characterization of proper bases.

Theorem 4: Let $\{a_{m,n}\}\subseteq E$ and $\{\alpha_{m,n}\}\subseteq\Gamma$ be given sequences. The following three conditions are equivalent:

(i) Convergence of $\sum_{m,n=1}^{\infty}a_{m,n}e_{m,n}$ in Γ implies the convergence of $\sum_{m,n=1}^{\infty}a_{m,n}\alpha_{m,n}$ in Γ

(ii) The convergence of $\sum_{m,n=1}^{\infty}a_{m,n}e_{m,n}$ in Γ implies that $\lim_{m,n\to\infty}(a_{m,n}\alpha_{m,n}) = 0$ in Γ

(iii) $\limsup_{m,n\to\infty}\frac{\log\|\alpha_{m,n};\rho+\delta\|}{\log(\lambda_m^{\lambda_m}.\mu_n^{\mu_n})} < \frac{1}{\rho}$, for each $\delta > 0$.

Proof: First suppose that (i) hold. Then for any sequence $\{a_{m,n}\}$, where $a_{m,n}'s$ belong to Banach algebra E, $\sum_{m,n=1}^{\infty}a_{m,n}e_{m,n}$ converges in Γ implies that $\sum_{m,n=1}^{\infty}a_{m,n}\alpha_{m,n}$ converges in Γ which in turn implies that $a_{m,n}\alpha_{m,n}\to 0$ as $m,n\to\infty$. Hence (i)$\Rightarrow$(ii).

Now we assume that (ii) is true but (iii) is false. This implies that for some $\delta > 0$, there exists sequences $\{m_k\},\{n_l\}$ of positive integers such that

$$\frac{\log\|\alpha_{m_k,n_l};\rho+\delta\|}{\log(\lambda_{m_k}^{\lambda_{m_k}}.\mu_{n_l}^{\mu_{n_l}})} \ge \frac{1}{\rho+(k^{-1},l^{-1})} ,$$

$$\forall m_k, k = 1,2,.. \text{ and } n_l, l = 1,2,...$$

Define a sequence $\{a_{m,n}\}\subseteq E$, as

$$a_{m,n} = \begin{cases} \omega.\lambda_{m_k}^{-\lambda_{m_k}/(\rho+k^{-1})}.\mu_{n_l}^{-\mu_{n_l}/(\rho+l^{-1})} & ,m=m_k, k=1,2,...,n=n_l, l=1,2,... \\ 0 & ,m \neq m_k, n \neq n_l \end{cases} \quad (2.4)$$

Then, we have

$$\|a_{m_k,n_l}\|.\lambda_{m_k}^{\lambda_{m_k}/(\rho+\delta)}.\mu_{n_l}^{\mu_{n_l}/(\rho+\delta)}$$

$$= \lambda_{m_k}^{\lambda_{m_k}(k^{-1}-\delta)/(\rho+\delta)(\rho+k^{-1})}.\mu_{n_l}^{\mu_{n_l}(l^{-1}-\delta)/(\rho+\delta)(\rho+l^{-1})},$$

for sufficiently large k and l with $k^{-1}, l^{-1} < \delta$. Hence

$$\lim_{k,l\to\infty}\sup\|a_{m_k,n_l}\|.\lambda_{m_k}^{\lambda_{m_k}/(\rho+\delta)}.\mu_{n_l}^{\mu_{n_l}/(\rho+\delta)} = 0$$

,

and therefore $\sum_{m,n=1}^{\infty} a_{m,n}e_{m,n}$ converges by (1.6). But

$$\|a_{m_k,n_l}\|.\|\alpha_{m_k,n_l};\rho+\delta\| \geq 1.$$

$a_{m,n}\alpha_{m,n}$ does not tend to zero as $m,n \to \infty$ which contradicts (ii). Hence (ii)$\Rightarrow$(iii). In course of the proof of Theorem 1 above , we have already proved that (iii)$\Rightarrow$(i). Thus the proof of Theorem 4 is complete.

Theorem 5: Let $\{a_{m,n}\} \subseteq E$ and $\{\alpha_{m,n}\} \subseteq \Gamma$. The following three properties are equivalent:

(a) $\lim_{m,n\to\infty}(a_{m,n}\alpha_{m,n}) = 0$ in Γ implies that $\sum_{m,n=1}^{\infty} a_{m,n}e_{m,n}$ converges in Γ.

(b) Convergence of $\sum_{m,n=1}^{\infty} a_{m,n}\alpha_{m,n}$ in Γ implies that $\sum_{m,n=1}^{\infty} a_{m,n}e_{m,n}$ converges in Γ.

(c) $$\lim_{\delta\to 0}\left\{\lim_{m,n\to\infty}\inf \frac{\log\|\alpha_{m,n};\rho+\delta\|}{\log(\lambda_m^{\lambda_m}.\mu_n^{\mu_n})}\right\} \geq \frac{1}{\rho}.$$

Proof: It is evident that (a)$\Rightarrow$(b).We now prove that (b)$\Rightarrow$(c).To prove this, we suppose that (b) hold but (c) does not hold . Hence

$$\lim_{\delta\to 0}\left\{\lim_{m,n\to\infty}\inf \frac{\log\|\alpha_{m,n};\rho+\delta\|}{\log(\lambda_m^{\lambda_m}.\mu_n^{\mu_n})}\right\} < \frac{1}{\rho}$$

Since $\|\alpha_{m,n};\rho+\delta\|$ increases as δ decreases, it follows that for each $\delta > 0$

$$\lim_{m,n\to\infty}\inf \frac{\log\|\alpha_{m,n};\rho+\delta\|}{\log(\lambda_m^{\lambda_m}.\mu_n^{\mu_n})} < \frac{1}{\rho}.$$

Hence, if ν,η be a fixed small positive number, then for each $r,h > 0$, we can find a positive numbers m_r, n_h such that $\forall r,h > 0$, we have $m_{r+1} > m_r$, $n_{h+1} > n_h$ and

$$\frac{\log\|\alpha_{m_r,n_h};\rho+(rh)^{-1}\|}{\log(\lambda_{m_r}^{\lambda_{m_r}}.\mu_{n_h}^{\mu_{n_h}})} < \frac{1}{\rho+(\nu,\eta)} \quad (2.5)$$

Now we choose a positive number $\nu_1 < \nu$ and $\eta_1 < \eta$, and define a sequence $\{a_{m,n}\} \subseteq E$ as

$$a_{m,n} = \begin{cases} \omega.\lambda_{m_r}^{-\lambda_{m_r}/(\rho+\nu_1)}.\mu_{n_h}^{-\mu_{n_h}/(\rho+\eta_1)} & m=m_r, r=1,2,...,n=n_h, h=1,2,... \\ 0 & m \neq m_r, n \neq n_h \end{cases} \quad (2.6)$$

Then, for any $\delta > 0$

$$\sum_{m,n=1}^{\infty}\|a_{m,n}\|.\|\alpha_{m,n};\rho+\delta\|$$

$$= \sum_{r,h=1}^{\infty}\|a_{m_r,n_h}\|.\|\alpha_{m_r,n_h};\rho+\delta\|.$$

For any given $\delta > 0$, omit from the above series those finite number of terms, which correspond to those numbers m_r, n_h for which $1/r > \delta$ and $1/h > \delta$. The remainder of series in (2.6) is dominated by $\sum_{r,h=1}^{\infty}\|a_{m_r,n_h}\|.\|\alpha_{m_r,n_h};\rho+(rh)^{-1}\|$.Now by (2.5) and (2.6), we find that

$$\|a_{m_r,n_h}\|.\|\alpha_{m_r,n_h};\rho+(rh)^{-1}\|$$

$$\leq \lambda_{m_r}^{-\lambda_{m_r}/(\rho+\nu_1)}.\mu_{n_h}^{-\mu_{nh}/(\rho+\eta_1)}.\lambda_{m_r}^{\lambda_{m_r}/(\rho+\nu)}.\mu_{n_h}^{\mu_{n_h}/(\rho+\eta)}$$

$$\leq \lambda_{m_r}^{\lambda_{m_r}(\nu_1-\nu)/(\rho+\nu_1)(\rho+\nu)}.\mu_{n_h}^{\mu_{nh}(\eta_1-\eta)/(\rho+\eta_1)(\rho+\eta)}$$

Since $\nu_1 < \nu, \eta_1 < \eta$ hence the above series $\sum_{r,h=1}^{\infty} \|a_{m_r,n_h}\| . \|\alpha_{m_r,n_h}; \rho + (rh)^{-1}\|$ is convergent. For this sequence $\{a_{m,n}\}$, $\sum_{m,n=1}^{\infty} a_{m,n}\alpha_{m,n}$ converges in $\Gamma(\rho,\delta)$ for each $\delta > 0$ and hence converges in Γ But we have,

$$\begin{aligned} &\|a_{m,n}\| . \lambda_m^{\lambda_m/(\rho+\delta)} . \mu_n^{\mu_n/(\rho+\delta)} \\ &= \|a_{m_r,n_h}\| . \lambda_{m_r}^{\lambda_{m_r}/(\rho+\delta)} . \mu_{n_h}^{\mu_{n_h}/(\rho+\delta)} \\ &= \lambda_{m_r}^{-\lambda_{m_r}/(\rho+\nu_1)} . \mu_{n_h}^{-\mu_{n h}/(\rho+\eta_1)} \\ &\quad . \lambda_{m_r}^{\lambda_{m_r}/(\rho+\delta)} . \mu_{n_h}^{\mu_{n_h}/(\rho+\delta)} \\ &= \lambda_{m_r}^{\lambda_{m r}(\nu_1-\delta)/(\rho+\nu_1)(\rho+\delta)} . \mu_{n_h}^{\mu_{n h}(\eta_1-\delta)/(\rho+\eta_1)(\rho+\delta)} \end{aligned}$$

Now we choose $\nu_1 > \delta$ and $\eta_1 > \delta$ then from above, $\|a_{m,n}\| . \lambda_m^{\lambda_m/(\rho+\delta)} . \mu_n^{\mu_n/(\rho+\delta)}$ does not tend to zero for this δ. Hence $\sum_{m,n=1}^{\infty} a_{m,n} e_{m,n}$ does not converge and this is a contradiction. Therefore (b)$\Rightarrow$(c). Now we prove that (c)$\Rightarrow$(a). We assume (c) is true but (a) is not true. Then there exists a sequences $\{a_{m,n}\} \subseteq E$ for which $a_{m,n}\alpha_{m,n} \to 0$ in Γ but $\sum_{m,n=1}^{\infty} a_{m,n} e_{m,n}$ does not converge in Γ. This implies that

$$\lim_{m,n\to\infty} \sup \frac{\log(\lambda_m^{\lambda_m} . \mu_n^{\mu_n})}{\log\|a_{m,n}\|^{-1}} > \rho .$$

Hence there exists a positive number ε and a sequence $\{m_k\}$, $\{n_l\}$ of positive integers such that

$$\frac{\log(\lambda_m^{\lambda_m} . \mu_n^{\mu_n})}{\log\|a_{m,n}\|^{-1}} \geq (\rho + \varepsilon) \quad ,$$

$\forall m = m_k, n = n_l$ (2.7)

We choose another positive number $\nu < \varepsilon/2, \eta < \varepsilon/2$, by assumption we can find a positive number δ i.e. $\delta = \delta(\nu,\eta)$ such that

$$\lim_{m,n\to\infty} \inf \frac{\log\|\alpha_{m,n}; \rho + \delta\|}{\log(\lambda_m^{\lambda_m} . \mu_n^{\mu_n})} > \frac{1}{\rho + (\nu,\eta)}$$

Hence there exists $N = N(\nu,\eta)$, such that

$$\frac{\log\|\alpha_{m,n}; \rho + \delta\|}{\log(\lambda_m^{\lambda_m} . \mu_n^{\mu_n})} \geq \frac{1}{\rho + 2(\nu,\eta)} \quad .$$

$\forall m = m_k, n = n_l$. (2.8)

Therefore, from (2.7) and (2.8) we have for $m = m_k, n = n_l$

$$\begin{aligned} &\|a_{m,n}\alpha_{m,n}; \rho + \delta\| = \|a_{m,n}\| . \|\alpha_{m,n}; \rho + \delta\| \\ &\geq \lambda_{m_k}^{-\lambda_{m k}/(\rho+\varepsilon)} . \mu_{n_l}^{-\mu_{n l}/(\rho+\varepsilon)} . \lambda_{m_k}^{\lambda_{m_k}/(\rho+2\nu)} . \mu_{n_l}^{\mu_{n_l}/(\rho+2\eta)} \\ &= \lambda_{m_k}^{\lambda_{m k}(\varepsilon-2\nu)/(\rho+\varepsilon)(\rho+2\nu)} . \mu_{n_l}^{\mu_{n l}(\varepsilon-2\eta)/(\rho+\varepsilon)(\rho+2\eta)} \\ &\to \infty \text{ as } k,l \to \infty \end{aligned}$$

Since $\varepsilon > 2\nu, \varepsilon > 2\eta$. Then $\{a_{m,n}\alpha_{m,n}\}$ does not tend to zero in $\Gamma(\rho,\delta)$ for the δ chosen above. Hence $\{a_{m,n}\alpha_{m,n}\}$ does not tend to zero in Γ and this contradiction. Thus (c)$\Rightarrow$(a).This proves Theorem 5.

Corollary . A base $\{\alpha_{m,n}\}$ in a closed subspace Γ_0 of Γ is a proper base if and only if it satisfies the condition (iii) and (c) of Theorem 4 and Theorem 5 respectively.

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