Fully Principally Extending Module

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Abstract
In this work, We introduce the concepts of an FP-Extending, FP-Continuous and FP-Quasi-Continuous which are stronger than P-Extending, P-Continuous and P-Quasi-Continuous. characterizations and properties of FP-Extending, FP-Continuous and FP-Quasi-Continuous are obtained . A module M is called FP-Extending (FP-Continuous, FP-Quasi-Continuous) if every submodule is P-Extending (P-Continuous, P-Quasi-Continuous) .

Keyword: Extending module, P-Extending, P-Continuous, P-Quasi-Continuous, fully invariant submodule, stable submodule, uniform module, closed submodule.

مقاسات التوسع–FP

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الخلاصة:
قدمنا في هذا البحث مفاهيم جديدة هي مقاس التوسع–FP، مقاس التوسع المستمر–FP، مقاس التوسع شبه المستمر–FP. يقال عن مقاس على الحلقة R انه مقاس توسع–FP (مقاس التوسع المستمر–FP، مقاس التوسع شبه المستمر–FP) اذا كان كل مقاس جزئي من هو توسع–P(مقاس التوسع المستمر–P، مقاس التوسع شبه المستمر–P). ميزنا تلك المقاسات ودرسنا خصيصاتها ووالعلاقة فيما بينها وعلاقتها بالاصناف الاخرى من المقاسات.

Introduction

Throughout this paper all rings have an identity and modules are unitary. Let R be a ring and M be a left R-module, a submodule N of M is essential if every nonzero submodule of M intersects N nontrivially, we use $N \overset{e}{\rightarrow} M$ to denote that N is essential submodule of M. An R-module M is uniform if every submodule of M is essential in M . Also, a submodule N of M is closed in M if it has no proper essential extensions in M [1]. An R-module M is said to be Extending if every closed submodule of M is a direct summand [2]. A submodule N of an R-module M is called a fully invariant if $f(N) \subseteq N$ for each $f \in End_R(M)$ [3], an R-module M is called duo if every submodule of M is fully invariant [4]. A submodule N of an R-module M is called stable if $f(N) \subseteq N$ for each R-homomorphism $f: N \rightarrow M$, an R-module M fully stable if every submodule of M is stable [5]. For a module M consider the following conditions:

$(\boldsymbol{PC_1})$: Every cyclic submodule of M is essential in a direct summand of M.

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(PC_2): For each $a, b \in M$, if $aR \approx bR$ and $bR \overset{\oplus}{\to} M$, then $aR \overset{\oplus}{\to} M$.

(PC_3): For each $a, b \in M$ such that $aR \overset{\oplus}{\to} M$ and $bR \overset{\oplus}{\to} M$ with $aR \cap bR = 0$, then $aR \oplus bR \overset{\oplus}{\to} M$.

E. A. Shallal in [6], defined and studied the concept of Pointwise Extending, Pointwise Continuous and Pointwise Quasi-Continuous. Later, the concepts of Pointwise Extending, Pointwise Continuous and Pointwise Quasi-Continuous was introduced by M. A. Kamal; O. A. Elmnophy in [7] in another names Principally Extending, Principally Continuous and Principally Quasi-Continuous. An R-module M is called Principally Extending (for short P-Extending) if it satisfies the condition (PC_1), a Principally Continuous module (for short P-Continuous) if it satisfies $(PC_1)$and (PC_2), and a Principally Quasi-Continuous module (for short P-Quasi-Continuous) if it satisfies (PC_1) and (PC_3). These classes of modules are studied extensively in [7]. We refer to [1], [8], [9], [3] and [10] for background on Extending and (Quasi-) Continuous module.

In this work, We introduce a new concepts, that is, FP-Extending, FP-Continuous and FP-Quasi-Continuous which are stronger than P-Extending, P-Continuous and P-Quasi-Continuous.

Definition(1):

An R-module M is called Fully Principally Extending module (for short FP-Extending) if every submodule of M is P-Extending .

Definition(2):

An R-module M is called Fully Principally Continuous module (for short FP-Continuous) if every submodule of M is P-Continuous.

Definition(3):

An R-module M is called Fully Principally Quasi-Continuous module (for short FP-Quasi-Continuous) if every submodule of M is P-Quasi-Continuous.

Every submodule (hence direct summand) of FP-Extending (FP-Continuous, FP-Quasi-Continuous) module is FP-Extending (FP-Continuous, FP-Quasi-Continuous) module.

It is clear that if R-module M is FP-Extending (FP-Continuous, FP-Quasi-Continuous) module, then M is P-Extending (P-Continuous, P-Quasi-Continuous) module.

Examples(4):

(1) Every regular R-module is FP-Extending.

(2) Every uniform module is FP-Extending (FP-Quasi-Continuous). In particular Z, Z/pZ, Q as Z-module. But Z_6 as Z-module is FP-Extending which is not uniform.

(3) Every Z_n over Z is FP-Extending (FP-Continuous, FP-Quasi-Continuous) for each positive integer $n \geq 2$.

(4) In the ring $R = Z/(4)$ every cyclic submodule is Continuous, then FP-Continuous.

(5) Every semisimple artinian ring is FP-Extending (FP-Continuous, FP-Quasi-Continuous) module .

(6) An R-module M is called a Q-module if every R-module is quasi-injective [11], therefore every Q-module is FP-Extending (FP-Continuous, FP-Quasi-Continuous) module, the Z-module Q is FP-Extending (FP-Quasi-Continuous) but not Q-module.

Proposition(5):

Every FP-Continuous R-module is FP-Quasi-Continuous R-module.

Proof:

Suppose M is FP-Continuous R-module and let N submodule of M, then N is P-Continuous module, therefore N is P-Quasi-Continuous, hence M is FP-Quasi-Continuous.

The converse is not true in general [5].

Proposition(6):

Every FP-Quasi-Continuous is FP-Extending.

Proof: Suppose M is FP-Quasi-Continuous and let N submodule of M, then N is P-Quasi-Continuous module, therefore N is P-Extending, hence M is FP-Extending.

The converse is not true [12]

Examples(7):

(1) The Z-module Z is uniform module and hence FP-Quasi-Continuous but Z is not FP-Continuous because 2Z is isomorphic to Z while 2Z is not a direct summand of Z, therefore Z not P-Continuous and hence Z is not FP-Continuous.

(2) The Z-module $M = Z_2 \oplus Z_4$ is FP-Extending but not FP-Quasi-Continuous, Since $(0)\oplus(2) \approx Z_2 \oplus (0) \overset{\oplus}{\to} Z_2 \oplus Z_4$,but $(0)\oplus(2)$ is not direct summand of $Z_2 \oplus Z_4$, therefore $M$is not P-Quasi-Continuous and hence not FP-Quasi-Continuous.

Like Extending modules a direct sum of FP-Extending (FP-Continuous, FP-Quasi-Continuous) module need not be FP-Extending (FP-Continuous, FP-Quasi-Continuous) module [see example(8)].

Example(8):

(1) For a prime p, the Z-module $M = Z \oplus Z_p$. Since Z is uniform and Z_p is a simple Z-module, therefore Z and Zp are FP-Extending (FP-Quasi-Continuous), but M is not FP-Extending (FP-Quasi-Continuous) module because 2Z is closed but not direct summand, hence not P-Extending.

(2) The Z-module $M = (Z/Z_p) \oplus Q$ has $M_1 = (Z/Z_p) \oplus 0$ and $M_2 = 0 \oplus Q$ both are uniform, thus they are FP-Extending (FP-Quasi-Continuous) module but M is not FP-Extending (FP-Quasi-Continuous) module because $Z_p(1 + Z_p, 1)$ is a closed submodule of M which is not direct summand (see [1]).

(3) The Z-module Z_2 and Z_4 are FP-Continuous (FP-Quasi-Continuous) module but the Z-module $M = Z_2 \oplus Z_4$ is not FP-Continuous (FP-Quasi-Continuous) module (example(7),2).

Since every submodule (hence direct summand) of FP-Extending (FP-Continuous, FP-Quasi-Continuous) module is FP-Extending (FP-Continuous, FP-Quasi-Continuous) module ,we have :

Proposition(9):

If $M \oplus M$ is FP-Extending (FP-Continuous, FP-Quasi-Continuous) module, then M is FP-Extending (FP-Continuous, FP-Quasi-Continuous) .

Proposition(10):

Any fully invariant submodule of P-Extending R-module is P-Extending.

Proof:

Suppose that M is P-Extending and N be fully invariant submodule of M. If aR submodule of N, then aR is a submodule of M, since M is P-Extending, then there exists a direct summand A of M such that $aR \overset{e}{\to} A$. That is $M = A \oplus B$ where B any submodule of M. Since N is a fully invariant submodule of M, then $N = (N \cap A) \oplus (N \cap B)$ [3,lemma 1.1]. That is $N \cap A$ is a direct summand of N, since $aR \overset{e}{\to} A$ and $N \overset{e}{\to} N$, then $aR = (aR \cap N) \overset{e}{\to} (N \cap A)$. Hence N is P-Extending.

Corollary(11):

Every duo P-Extending R-module is FP-Extending.

It is known that every stable submodule of any module is fully invariant [7].

Corollary(12):

Every stable submodule of any module of P-Extending R-module is P-Extending.

Corollary(13):

Every stable submodule of any module of P-Extending module is FP-Extending.

Corollary(14):

Let M be a fully stable R-module. Then M is FP-Extending if an only if M is P-Extending.

Corollary(15):

Let M be a fully stable R-module. Then the following statements are equivalent :

(1) P-Continuous.
(2) P-Quasi-Continuous.
(3) P-Extending .
(4) FP-Extending.

Proof:

(1)⟹(2) and (2)⟹(3) trivial.

(3)⟹(4) from corollary(14).

(4)⟹(1) Let M is a FP-Extending R-module, then M is P-Extending and from Corollary(12), every submodule is P-Extending. Since every fully stable module has (PC_2) [5], then every submodule is P-Continuous.

S. A. G. Al-Saadi in [12], defined and studied the concept of Strongly Extending modules, where an R-module M is called Strongly Extending, if every submodule of M is essential in a stable direct summand, he show that an R-module M is uniform if and only if M is indecomposable module and Strongly Extending. Hence we have the following.

Proposition(15):

For an indecomposable module M, the following are equivalent:

(1) M is FP-Extending.
(2) M is uniform.
(3) M is Strongly Extending.
(4) M is Extending.
(5) M is P-Extending

Proof:

(1)⟹(2): For each submodule $N \neq (0)$ of M, then $\exists x \neq 0 \in N$, since M is P-Extending, then there exist a direct summand K of M such that $xR \overset{e}{\to} K$. Since M is indecomposable ,then (0) and M are only direct summand of M . So K=M, then $xR \overset{e}{\to} M$. Hence $N \overset{e}{\to} M$.

(2)⟹(3) and (3)⟹(4) : see [12]

(4)⟹(5): see [7,lemma 2.14]

(5)⟹(1): Let N be submodule of P-Extending module M, for each submodule xR of N, xR is a submodule of M, then xR essential in a direct summand of M, so $xR \overset{e}{\to} M$. Therefore $xR \overset{e}{\to} N \overset{\oplus}{\to} N$, hence N is P-Extending.

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