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Weakly 2-quasi-prime sub-modules

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Abstract

Let R be a commutative ring with identity, and let M be a left R-module. We define a proper sub-module N of an R-module M to be a weakly 2-quasi-prime sub-module if whenever $0 \neq r_1r_2m \in N, r_1r_2 \in R, m \in M$, then either ${r_1}^2m \in N$ or $\in {r_2}^2m \in N$. This concept is an expansion of the idea of a 2-quasi-prime sub-module, where a proper sub-module N of an R-module M is said to be a 2-quasi-prime sub-module if for all $a, b \in R,\ x \in M$ and $abx \in N$ then either $a^2x \in N$ or $b^2x \in N$. Various properties of weakly 2-quasi-prime sub-modules are considered.

Keywords: prime sub-module, weakly prime sub-module, 2-quasi-prime sub-module, weakly 2-quasi-prime sub-module, suitable sub-module

المقاسات شبه الاولية الضعيفه الظاهرية من النمط −2

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الخلاصة

لتكن R حلقه ابدالية ذا محايد وليكن M مقاسا ايسر على R . تعرف ان مقاسا جزئيا فعليا N في M يكون شبه اولي ضعيفا ظاهريا من النمط 2 اذا كان لكل $0 \neq r_1r_2m \in N, r_1r_2 \in R, m \in M$, يؤدي الى ${r_1}^2m \in N$ or $\in {r_1}^2m \in N$. في الحقيقة ان هذا المفهوم هو تعميم لمفهوم مقاسات الجزئية الاولية الظاهرية من النمط −2 , اذ ان المقاس الجزئي الفعلي N من المقاس M على الحلقة R يسمى مقاسا جزئيا اوليا ظاهريا من النمط −2 اذا كان لكل $x \in M$, $a, b \in R$ و $abx \in N$ يؤدي الى $a^2x \in N$ or $b^2x \in N$. خواص مختلفة عن المقاسات الجزئية الاولية الضعيفة الظاهرية من النمط −2 قد اعطيت.

1. Introduction:

Throughout this paper, R be a commutative ring with identity and M be a unity R-module. A sub-module N of M is prime sub-module if whenever $r \in R, m \in M, rm \in N$, implies $m \in N$ or $r \in [N: M]$, where $[N: M] = \{ r \in R,\ \ rM \subseteq N\}$, see [1], [2]. In 1999, the quasi-prime sub-module was introduced and studied by Muntaha, see [3], such that a sub-module N of M is a quasi-prime sub-module if $r_1r_2\, m \in N$, for $r_1, r_2 \in R$, $m \in M$ implies $r_1\, m \in N$ or, $r_2\, m \in N$. In [4] F. D. Jasem and A. A. Elewi introduced a 2-prime sub-module,

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when $rm \in N, r \in R, m \in M$, then either $m \in N$ or, $r^2 \in [N: M]$, and then then N is a 2-pirme sub-module, which is an extension of a prime sub-module. The idea of 2-quasi-pirme sub-module was introduced by F. D. Jasem and A. A. Elewi [5], where a 2-quasi-prime sub-module N is defined as a valid sub-module of an R-module M, if for all $a, b \in R,\ x \in M$ and $abx \in N$ then either $a^2x \in N$ or, $b^2x \in N$.

In 2022 by Marziye Jamali and Reza Jahani-Nezhad in [6], defined the concept of a weakly prime sub-module. Recall that a proper sub-module N of M is a weakly prime sub-module if whenever $0 \neq rm \in N, r \in R, m \in M$, then either $m \in N$ or, $r \in [N: M]$. Clearly, every prime sub-module is a weakly prime sub-module. To make a 2-prime sub-module more general, [7] introduced the conception of a weakly 2-prime sub-module, where sub-module N of an R-module M is a weakly 2-prime if and only if for every$0 \neq rm \in N, r \in R, m \in M$, then either $m \in N$ or $r^2 \in [N: M]$.

In this paper, we give a definition of a weakly 2-quasi-prime sub-module as follows: a proper sub-module N of an R-module M is a weakly 2-quasi prime sub-module if whenever $0 \neq r_1r_2m \in N, r_1r_2 \in R, m \in M$, then either ${r_1}^2m \in N$ or, $\in {r_2}^2m \in N$. As well as we many properties for this kind of sub-modules have been proven such as, a proper sub-module N of an R-module M is a weakly 2-quasi-prime sub-module exclusively if $[N_R: (m)]$ is a weakly 2-prime ideal, for all $m \in M, m \notin N$.

2. Weakly 2-quasi-prime sub-modules

In this section we present the idea of a weakly 2-quasi-prime sub-module as an extension of a 2-quasi-prime sub-module, where a valid sub-module N of M is a 2-quasi-prime sub-module if for all $a, b \in R,\ x \in M$ and $abx \in N$ then either $a^2x \in N$ or, $b^2x \in N$, vice versa (see, [5]).

Definition 2.1:

A proper sub-module N of an R-module M is weakly 2-quasi prime if, whenever $0 \neq r_1r_2m \in N, r_1r_2 \in R, m \in M$, then either ${r_1}^2m \in N$ or $\in {r_2}^2m \in N$.

An ideal I of a ring R is called weakly 2 –prime ideal if it is weakly 2-quasi-prime sub-module of R.

Remarks and examples 2.2:

1. Every sub-module that is a 2-quasi-prime is also a weakly 2-quasi-prime sub-module

Proof: Let N be a 2-quasi-prime sub-modules of an R-module M, and let $0 \neq r_1r_2m \in N$. Then by assumption either ${r_1}^2m \in N$ or, $\in {r_2}^2m \in N$ and so N is a weakly 2-quasi-prime sub-module.

2. The converse of (1) is not true in general for example: The zero sub-module of the Z-module Z_{12} is a weakly 2-quasi-prime sub-module but it is not 2-quasi-prime sub-module, because $3.2.\bar{2} \in \{\bar{0}\}$, but $3^2.\bar{2} \notin \{0\}$ and $2^2.\bar{2} \notin \{0\}$.

3. By, [5] every quasi prime sub-module is a 2-quasi-prime sub-module and. By (1), we obtain every quasi prime is a sub-module that is a weakly 2-quasi-prime. But the convers is not true, also by having the same example in (2) we have $\{0\}$ in Z_{12} it is not a quasi-prime sub-module, since $3.2.\bar{2} \in \{\bar{0}\}$, but $3^2.\bar{2} \notin \{0\}$ and $2^2.\bar{2} \notin \{0\}$.

4. Each prime sub-module is a weakly 2-quasi-prime sub-module, since by [3] every prime sub-module is a quasi-prime sub-module and by (1) we obtain the result.

The convers of (4) not always holds true for example: The sub-module $, N = 2Z \oplus (0)$of the Z-module $Z \oplus \bar{Z}$ is not prime sub-module, but it is weakly 2-quasi- prime sub-module, given that it is quasi-prime and by (3) we get the result.

5. The sub-module Z of the Z-module$Q$ is not weakly 2-quasi-prime sub-module, since $3.2.\frac{1}{6} \in Z$, but $3^2.\frac{1}{6} \notin Z$ and$2^2.\frac{1}{6} \notin Z$.
6. Every weakly 2- prime sub-module is a weakly 2-quasi-prime sub-module.

Proof: Let N be a weakly 2-prime and let $0 \neq r_1 r_2 \in R, m \in M$ such that $0 \neq r_1 r_2 m \in N$, so $r_1(r_2 m) \in N$, by assumption either $r_2 m \in N$ or, ${r_1}^2 m \in [N: M]$. Thus either ${r_2}^2 m \in N$ or ${r_1}^2 m \subseteq N$, i.e., ${r_2}^2 m \in N$ or, $\in {r_1}^2 m \in N$.

The convers of (5) not always holds true for example: the Z-module $Z \oplus \bar{Z}, N = 2Z \oplus (0)$, N is a weakly 2-quasi-pirme sub-module. But N is not weakly 2-prime sub-module. Since $(0,0 \neq 2(3,0) \in N$ and$(3,0) \notin N$, $2^2 \notin [2Z \oplus (0):_Z Z \oplus Z]$.

7. The sub-module of a weakly 2-quasi-prime sub-module need not be a weakly 2-quasi-prime sub-module, for instant if $N = (\bar{2}) < Z_{12}$, then N is weakly 2-quasi-prime sub-module of Z_{12} , but $L = (\bar{6}) < Z_{12}$ is not a weakly 2-quasi-prime sub-module, since $0 \neq 2.3.\bar{1} \in (\bar{6})$, but $2^2.\bar{1} \notin (\bar{6})$ and $3^2.\bar{1} \notin (\bar{6})$.

A semi prime sub-modules is not relation to a weakly 2-quasi-prime sub-module

8. Semi prime sub-module is not required to be a weakly 2-quasi-prime sub-module.

Not that $6Z$ is semi prime Z-sub-module of Z, but it is not a weakly 2-quasi-prime sub-module, since $0 \neq 2.3.\bar{1} \in 6Z$, but $2^2.\bar{1} \notin 6Z$ and $3^2.\bar{1} \notin 6Z$.

Also, a weakly 2-quasi-prime sub-module is not required to be a semi prime sub-modules.

Take $8Z$of the Z-module Z as an example is a weakly 2-quasi-prime sub-module, but it is not a semi prime sub-modules.

9. Every maximal sub-module of an R-module M is a weakly 2-quasi-prime sub-module.

But the convers is not true, since $(0)$is a weakly 2-quasi-prime sub-module of the Z-module Z, yet it is not a maximal in Z.

Now we will give a characterization for a weakly 2-quasi-prime sub-modules.

Theorem 2.3:

A proper sub-module N of an R-module M is a weakly 2-quasi-prime sub-module if and only if $[N_R: (m)]$ is an ideal that is weakly 2-prime, for all$m \in M$,$m \notin N$.

Proof:

Let N be a weakly 2-quasi-prime sub-module and let$0 \neq ab \in [N_R: (m)]$, such that$m \in M$, $m \notin N$ and$a, b \in R$. Hence $ab(m) \subseteq N$ and thus $abrm \in N$. Since N is a weakly 2-quasi-prime sub-module, so either $a^2 \in N$ or$b^2 r^2 m \in N$. Thus either $a^2 \in [N_R: (m)]$ or $b^2 \in [N_R: (m)]$. Therefore, $[N_R: (m)]$ is an ideal that is weakly 2-prime.

For the convers: suppose $[N_R: (m)]$ is an ideal that is weakly 2-prime, where$m \in M$, $m \notin N$. To show that N is a weakly 2-quasi-prime sub-module: Take$0 \neq abm \in N$, such that$a, b \in R$, $m \in M$. Thus $0 \neq a, b \in [N_R: (m)]$ and by assumption either $a^2 \in [N_R: (m)]$ or $b^2 \in [N_R: (m)]$, so either $a^2 m \in N$ or $b^2 m \in N$. Therefore N is a weakly 2-quasi-prime sub-module.

The following lemma is necessary to prove the next sub-modules:

Lemma 2.4:

Let C and D be two sub-modules of an $R-$module M, for every $d \in D$, where $[C_R: (d)]$ is an ideal which is a weakly 2-prime, then $[C_R: D]$ is an ideal which is a weakly 2-quasi-prime.

Proof:

Let $r_1, r_2 \in R$, such that $0 \neq r_1 r_2 \in [C_R: D]$. Thus $r_1, r_2 d \in C$, for every $d \in D$, so $0 \neq r_1, r_2 \in [C_R: (d)]$ ……(1). By assumption $[C_R: (d)]$ is a weakly 2-prime ideal, thus either ${r_1}^2 \in [C_R: (d)]$ or ${r_2}^2 \in [C_R: (d)]$, for every $d \in D$. This implies that either ${r_1}^2 d \in C$ or ${r_2}^2 d \in C$. Suppose that ${r_1}^2 \notin [C_R: (d)]$ and$r{_2}^2 \notin [C_R: (d)]$. Therefore, there exists $t_1, t_2 \in D$, such that ${r_1}^2 t_1 \in D$ and ${r_1}^2 t_2 \in C$. Thus ${r_1}^2 \notin [C_R:(t_1)]$ and ${r_2}^2 \notin [C_R:(t_2)]$, by (1) $r_1, r_2 \in [C_R:(t_1)]$. But $[C_R:(t_1)]$ is a weakly 2-prime ideal, thus ${r_1}^2 \in [C_R:(t_1)]$, so${r_1}^2 t_1 \in C$. In the same way we can prove that $0 \neq r_1, r_2 \in [C_R:(t_2)]$, implies that ${r_2}^2 t_2 \in C$. On the other hand, by (1) $0 \neq r_1, r_2 \in [C_R:(t_1 + t_2)]$, so either ${r_1}^2 \in [C_R:(t_1 + t_2)]$ or ${r_2}^2 \in [C_R:(t_1 + t_2)]$. Hence, either ${r_1}^2(t_1 + t_2) \in C$ or ${r_2}^2(t_1 + t_2) \in C$. This means either ${r_1}^2 t_1 + {r_1}^2 t_2 = C_1 \in C$ or ${r_2}^2 t_1 + {r_2}^2 t_2 = C_2 \in C$. Thus either ${r_1}^2 t_1 = C_1 - {r_1}^2 t_2 \in C$ or ${r_2}^2 t_2 = C_2 - {r_2}^2 t_1 \in C$, which is a contradiction. So, either $r_1 \in [C_R:D]$ or $r_2 \in [C_R:D]$. Thus $[C_R:(d)]$ is an ideal which is a weakly 2-prime.

Proposition 2.5:

Let M be an R-module over a ring R and $K \subset M$. Then K is a weakly 2-quasi-pirme of M if and only if $[K_R:L]$ is an ideal of R that is weakly 2-prime.

Proof:

Take K be a weakly 2-quasi-prime of M, thus by using Proposition 2.3, we get $[K_R:(x)]$ is an ideal of R that is a weakly 2-prime, for every $x \in M$, $x \notin K$. Also, by using the same Proposition 2.3, we get $[K_R:(x)]$ is an ideal which is weakly 2-prime, for each $x \in L$. So, by Lemma 2.4, we get the result.

Now, for the other side take $0 \neq r_1 r_2 \in K$, where $r_1, r_2 \in R$ and $m \in M$, so $r_1 r_2 \in [K_R:(m)]$ and $m \in L \subseteq M$, by assumption $[K_R:(m)]$ is an ideal which is a weakly 2-prime, then either${r_1}^2 \in [K_R:(m)]$ or ${r_2}^2 \in [K_R:(m)]$. This means that ${r_1}^2 \in K$ or ${r_2}^2 \in K$. Hence, K is a weakly 2-quasi-prime of M.

Corollary 2.6:

Let M be an R −module over a ring R and N be a proper sub-module of M. If N is a weakly 2-quasi-prime sub-module of M. Then $[N_R:M]$ is a weakly 2-prime.

The convers of pervious corollary is not true in general:

Note: consider the Z-module Q and let Z be a sub-module of Q. Notice that $[Z_R:Q] = (0)$ is an ideal that is 2-prime of Z. But Z is not a weakly2-quasi-prime, since $3.5.\frac{1}{15} \in Z$, but $3^2.\frac{1}{15} \notin Z$ and $5^2.\frac{1}{15} \notin Z$.

Notice that, the intersection of two weakly 2-quasi-prime sub-modules may not be a weakly 2-quasi-prime sub-module as the following remark shows:

Remark 2.7:

The two sub-modules $2Z$ and $3Z$ of the Z-module Z are weakly 2-quasi-prime sub-module (since they are 2 -prime sub-module), but $2Z \cap 3Z = 6Z$ is not weakly 2-quasi-prime sub-module, since $2.3.1 \in 6Z$, but $2^2.1 \notin 6Z$ and $3^2.1 \notin 6Z$.

The following proposition describe the condition that makes the intersection of two weakly 2-quasi-prime sub-modules be a weakly 2-quasi-prime sub-module:

Proposition 2.8:

Let H and F be two sub-modules of an R-module L. If H is a weakly 2-quasi-prime sub-module of L and H is not contained in F. Then $H \cap F$ is a weakly 2-quasi-prime sub-module of L.

Proof:

Because $H \not\subset F$. Then $H \cap F$ is a proper sub-module of H. Take $x, y \in R$ and $f \in F$, such that $0 \neq xyF \in H \cap F$. Thus, $0 \neq xyh \in F$. By assumption H is a weakly 2-quasi-prime sub-module of L, so either $x^2 f \in F$ or $y^2 f \in F$. But $f \in F$ and$x^2 f \in H$, so that $x^2 f \in H \cap F$ or $y^2 f \in H \cap F$. Therefore, $H \cap F$ is a weakly 2-quasi-prime sub-module of$L$.

Now, let W be an R-module and K be sub-module of W, S is multiplicative set of R, then $K(S) = \{x \in W: \exists t \in S \text{ such that } tx \in K\}$, [8].

Proposition 2.9:

Let M be an R −module and N be a proper sub-module of M. If $I = [N_R: M]$ be an ideal of R that is a weakly 2 -prime and $N(T) = N$, where $(T = R - I)$, then N is a weakly 2-quasi-prime sub-module of M.

Proof:

Let $x, t \in R$ and $m \in M$, such that $0 \neq xtm \in N$, notice that $R - I$ is multiplicatively closed, [9]. Now, suppose $s^2.m \notin N$, thus $s \notin [N_R: M] = I$, so $s \in T$. This implies that $tm \in N(T)$. By assumption$N(T) = N$, therefore, $t^2 m \in N$, which is means that N is a weakly 2-quasi-prime sub-module of M.

Note that the convers of the above proposition is not true as shown by the following example: Let $M = Z_8, N = \langle\bar{4}\rangle$, it is clear that N is a weakly 2-quasi-prime sub-module. Thus $[\langle\bar{4}\rangle_R: Z_8,] = 4Z$, which is a weakly 2 -prime ideal. But$T = Z - 4Z$,
$N(T) = \{x \in M: \exists t \in T \text{ such that } tx \in N\} = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \neq N$.

3. Some properties of weakly 2-quasi-prime sub-modules

Some basic results and properties of the concept weakly 2-quasi-prime sub-modules have been given in this section.

Recall that an R-module M is called multiplication module for every sub-module N of M, if there exists an ideal I of R such that $IM = N$, [10].

Now, we will give another characterization for a weakly 2-quasi-prime sub-module in class of multiplication modules.

Proposition 3.1:

Let M be a multiplication of an R-module and N be a proper sub-module of M.Then the following statements have the same meaning.

1. N is a weakly 2-quasi-prime sub-module.
2. The ideal $[N_R: M]$ of R is an ideal that is a weakly 2-prime.
3. For a weakly 2-pirme ideal *I* of *R*, *N*=*IM*.

Proof:

1 ⟹)2 Take N be a weakly 2-quasi-prime sub-module of M. To prove that $[N_R: M]$ is an ideal that is a weakly 2-prime: let $t, k \in R$, such that $t, k \in [N_R: M]$, so $tkm \in N$, for every $m \in M$. But N is a sub-module that is a weakly2-quasi-prime, so either $t^2 \in [N: (m)]$ or $k^2 \in [N: (m)]$. Then $[N_R: M]$ is an ideal of R that is a weakly 2-prime, by Lemma 2.4

2⟹ 3) It is clear that by using Corollary 2.6.

3⟹ 1) It is clear that by using Remarks and example 2.2.5.

Proposition 3.2

Let M and M' be two R −modules and let $f: M \to M'$ be an R-epimorphism. If N is a sub-module of M'that is a weakly2-quasi-prime. Then $f^{-1}(N)$ is a sub-module of M that is a weakly 2-quasi-prime.

Proof:

To prove $f^{-1}(N)$ is a sub-module of M that is a weakly 2-quasi-prime, we went to prove $[f^{-1}(N)\ _R: M]$ is a weakly 2-quasi-prime ideal. $N \leq M'$, such that $f^{-1}(N) \subsetneq A$. Let $0 \neq ab \in [f^{-1}(N)\ _R: A]$ and also $abA \subseteq f^{-1}(N)$. Thus $f(abA) \subseteq f(f^{-1}(N)$, so $abf(A) \subseteq N$. Therefore, $0 \neq ab \in [N_R: f(A)]$. But N is a sub-module of M' that is a weakly 2-quasi-prime, then either $a^2 \in [N_R: f(A)]$ or $b^2 \in [N_R: f(A)]$, thus either $a^2 f(A) \subseteq N$ or $b^2 f(A) \subseteq N$, i.e., either $a^2 A \subseteq f^{-1}(N)$ or $b^2 A \subseteq f^{-1}(N)$. Therefore, either $a^2 \in [f^{-1}(N)\ _R: A]$ or $b^2 \in$

$[f^{-1}(N)\ _R: A]$, thus $[f^{-1}(N)\ _R: A]$ is an ideal that is a weakly 2-quasi-prime, so $f^{-1}(N)$ is a sub-module of M is also weakly 2-quasi-prime .

Proposition 3.3:

Let $f: M \to M'$ be an R-epimorphism, such that $Kerf \subseteq A$, where A is a sub-module of M that is a weakly 2-quasi-prime, then $f(A)$ is a sub-module of M'that is a weakly 2-quasi-prime.

Proof:

To prove $f(N)$ is a sub-module of M'that is a weakly 2-quasi-prime, we prove that $[f(A)_R: N']$ is an ideal of R that is a weakly 2-quasi-prime, for all $N' \subseteq M'$and $N' \subseteq f(A)$. Since f is an epimorphism, then $ff'(N') = N'$. Let$f(N) = N'$, it follows that$f(N) \supsetneq f(A)$. Now, to prove that $[f(A)_R: f(N)]$ is a weakly 2-quasi-prime ideal of R, let $a, b \in R$ such that $0 \neq ab \in [f(A)_R: f(N)]$, so $0 \neq abf(N) \subseteq f(A)$. Thus, for each$x \in N$, $abf(x) \in f(A)$, so $f(abx) = f(s)$, for some $s \in A$, then $0 \neq abx - s \in Kerf \subseteq A$ and hence $abx \in A$, for each $x \in N$. Hence, $ab \in [A_R: N]$. But $[A_R: N]$ is a weakly 2-quasi-prime ideal of$R$, so either $a^2 \in [A_R: N]$ or $b^2 \in [A_R: N]$, thus either $a^2N \subseteq A$ or $b^2N \subseteq A$, so either $a^2f(N) \subseteq f(A)$ or $b^2f(N) \subseteq f(A)$. Therefore, either $a^2 \in [f(A)_R: f(N)]$ or $b^2 \in [f(A)_R: f(N)]$ is an ideal of R that is a weakly 2-quasi-prime and hence then $f(A)$ is a sub-module of M' that is weakly 2-quasi-prime.

Corollary 3.4:

Let A and B be two sub-modules of R-module M and$A \subseteq B$. Then $\frac{B}{A}$ is a sub-module of $\frac{M}{A}$ that is a weakly 2-quasi-prime if and only if B is a sub-module of M that is a weakly 2-quasi-prime.

Proof:

Let $f: M \to M/A$ be a natural mapping, then the result follows by Proposition 3.3.

Proposition 3.5:

Let B a sub-module of M that is a weakly 2-quasi-prime, such that B_S is a proper M_S, then B_S is a sub-module of M_S that is a weakly 2-quasi-prime.

Proof:

Let $\frac{r_1}{t_1}, \frac{r_2}{t_2} \in R_S$ and $\frac{m}{t} \in M_S$. Suppose that $0 \neq \frac{r_1}{t_1} . \frac{r_2}{t_2} . \frac{m}{t} \in B_S$, so there exists $b \in B$ and $s \in S$ such that $\frac{r_1}{t_1} . \frac{r_2}{t_2} . \frac{m}{t} = \frac{b}{s}$, such that $(r_1 r_2 ms - t_1 t_2 tb)m = 0$, which implies that $r_1 r_2 ms \in B$. But B is a sub-module of M that is a weakly 2-quasi-prime, thus either ${r_1}^2 ms \in B$ or ${r_2}^2 ms \in B$. Hence, either $\frac{{r_1}^2 ms}{{t_1}^2 t_s} \in B_S$ or $\frac{{r_2}^2 ms}{{t_2}^2 t_s} \in B_S$. That means $\frac{{r_1}^2 m}{{t_1}^2 t} \in B_S$ or$\frac{{r_2}^2 m}{{t_2}^2 t} \in B_S$, therefore, B_S is a sub-module of M_S that is a weakly 2-quasi-prime.

Recall that $Hom_R(M_1, M_2)$ is the set of all R-homomorphism from $M_1$to M_2 [5].

Proposition 3.6:

Let M_1 and M_2 be two R-modules and let $A$is a weakly 2-quasi-prime sub-module of M_2 such that $Hom_R(M, A)$ is a proper of $Hom_R(M_1, M_2)$, then $Hom_R(M_1, A)$is a weakly 2-quasi-prime sub-module of $Hom_R(M_1, M_2)$.

Proof:

To show that $Hom_R(M_1, A)$ is a weakly 2-quasi-prime sub-module of $Hom_R(M_1, M_2)$, let $0 \neq r_1 r_2 f \in Hom_R(M_1, A)$, then for each $x \in M_1$, $r_1 r_2 f(x) \in A$. But A is a weakly2-

quasi-prime sub-module of M_2, then either ${r_1}^2 f(x) \in A$ or ${r_2}^2 f(x) \in A$, thus implies that either ${r_1}^2 f \in Hom_R(M_1, A)$ or ${r_2}^2 f \in Hom_R(M_1, A)$.

Conclusions:

1-Every sub-module that is a 2-quasi-prime is also a weakly 2-quasi-prime sub-module.

2-Every weakly 2- prime sub-module is a weakly 2-quasi-prime sub-module.

3-A proper sub-module$N$ of an R-module$M$ is a weakly 2-quasi-prime sub-module if and only if $[N_R:(m)]$ is an ideal that is weakly 2-prime, for all$m \in M, m \notin N$.

4-Let M be an R-module over a ring R and N be a proper sub-module of M. If N is a weakly 2-quasi-prime sub-module of M. Then $[N_R: M]$ is a weakly 2-prime.

5-Let M and M' be two R −module and let $f: M \rightarrow M'$ be an R-epimorphism. If N is a sub-module of $'$ that is a weakly 2-quasi-prime. Then $f^{-1}(N)$ is a sub-module of M that is a weakly 2-quasi-prime.

6- Let A and B be two sub-modules of R-module M and$A \subseteq B$. Then $\frac{B}{A}$ is a sub-module of $\frac{M}{A}$ that is a weakly 2-quasi-prime if and only if B is a sub-module of M that is a weakly 2-quasi-prime.

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