Boundary Controllability of Nonlinear System in Quasi-Banach Spaces

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Abstract:
Sufficient conditions for boundary controllability of nonlinear system in quasi-Banach spaces are established. The results are obtained by using the strongly continuous semigroup theory and some techniques of nonlinear functional analysis, such as, fixed point theorem and quasi-Banach contraction principle theorem. Moreover, we given an example which is provided to illustrate the theory.
Keywords : Boundary controllability, quasi-Banach spaces, semigroup theory, fixed point theorem.

القابلية على السيطرة لمسألة السيطرة غير الخطية ذات الشرط الحدودي في فضاءات شبه بناخ

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الخلاصة:
تم إثبات المبرهنة التي تتعامل مع الشروط الكافية للقابلية على السيطرة لمسألة السيطرة غير الخطية ذات الشرط الحدودي في فضاءات شبه بناخ وذلك باستخدام نظرية شبه الزمرة المستمرة بقوة وبعض الطرائق التقنية ضمن التحليل الدالي غير الخطي مثل نظرية النقطة الصامدة ومبدأ التقليص لشبه فضاء بناخ. علاوة على ذلك، تم اعطاء مثال يوضح قيمة النظرية أعلاه.

1. Introduction:

Many scientific and engineering problems can by modeled by partial differential equations, integral equations, or coupled ordinary and partial differential equations that can be described as differential equations in infinite-dimensional spaces using semigroups. So, the study of controllability results of such problems in infinite-dimensional spaces is important. For the motivation of abstract system and the controllability of linear system, one can refer to [1,2]. Controllability of nonlinear system represented by ordinary differential equations in Banach spaces has been extensively studied by several authors. Han in [3] studied the boundary controllability of differential equations with nonlocal condition. Al-Moosawy [4] discussed the controllability and optimality of the mild solution for semilinear problems in Banach spaces, by using semigroup theory and Banach contraction principle theorem. In [5] studied the boundary controllability of integrodifferential system in Banach space. The controllability for some control problems in quasi-Banach spaces has been studied in [6] by using semigroup theory an some techniques of nonlinear functional analysis.

Since every Banach space is quasi-Banach spaces, but the converse is not true [7]. One could find a reasonable justification to accomplish the study of this paper. The purpose of this paper is to extend the study of the boundary controllability of nonlinear system in any quasi-Banach spaces by using the quasi-Banach fixed point theorem.

2. Definitions and theorems :

This section contains some definitions and theorems that will be used in the sequel.

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Definition 2.1 [8] : let $0 < p < \infty$. Then the collection of all measurable functions f for which $|f|^p$ is integrable will be denoted by $L_p(\mu)$. For each $f \in L_p(\mu)$, let

$\|f\|_p = (\int |f|^p d\mu)^{1/p}$. The number $\|f\|_p$ is called the L_p − norm of f.

Note 2.1 [7] : The space $L_p(\mu)$ for $0 < p < 1$ is a vector space, but not a normed space (thus not a Banach space).

Definition 2.2 [7] : A real-valued function $_q\|.\|$ defined on a vector space V over a field F is called a quasi-norm if it satisfies the following properties :

1. $_q\|x\| \geq 0 \quad \forall x \in V$ and $_q\|x\| = 0 \quad \Leftrightarrow x = 0$.

2. $_q\|\alpha x\| = |\alpha|\, _q\|x\| \quad \forall x \in V,\ \alpha \in F$.

3. $_q\|x+y\| \leq c\,(_q\|x\| + {}_q\|y\|) \quad \forall x, y \in V$, where $c \geq 1$ is a constant.

The pair $(V,\ _q\|.\|)$ is called a quasi-normed space.

Definition 2.3 [7] : Let $(V,\ _q\|.\|)$ be a quasi-normed space, then

(a) A sequence $\{x_n\}$ in V is called convergent to the limit $x \in V$ if, for $\in > 0$, there exists a positive integer $N(\in)$ such that $_q\|x_n - x\| < \in \quad \forall\ n \geq N$ (or $_q\|x_n - x\| \to 0$ as $n \to \infty$).

(b) A sequence $\{x_n\}$ in V is called a Cauchy sequence if, for $\in > 0, \exists N(\in) > 0$ such that $_q\|x_m - x_n\| < \in \quad \forall n, m \geq N$ (or, $_q\|x_m - x_n\| \to 0$ as $n, m \to \infty$).

(c) V is called a complete quasi-normed space (or quasi-Banaed space) if every Cauchy sequence in V is convergent.

Definition 2.4 [7] : For $f \in L_p(\mu)$, $0 < p < 1$, let us define the quasi-norm of f which is denoted by $_q\|f\|$ as follows :- $_q\|f\| = \|f\|_p^p = \int |f|^p d\mu$, where $\|f\|_p$ as defined in Definition 2.1.

Theorem 2.1 [7] : The space $L_p(\mu)$ for $0 < p < 1$, with the quasi-norm given in Definition 2.4 is a quasi-Banach space.

Definition 2.5 [9] : Let T be a mapping of a quasi-normed space X into itself, then T is called a quasi-contraction mapping if there exists a constant k, $0 \leq k < 1$ such that $_q\|T(x) - T(y)\| \leq k\ _q\|x - y\| \quad \forall x, y \in X$.

Remark 2.1 : It is clear from the above definition that every quasi-contraction mapping is uniformly continuous.

Theorem 2.2 [6] : (Quasi-Banach contraction principle)

Every quasi-contraction mapping T defined on a quasi-Banach space X into itself has a unique fixed point $x^* \in X$. Moreover, if x_0 is any point in X and the sequence $\{x_n\}$ is defined by $x_1 = T(x_0)$, $x_2 = T(x_1)$, … , $x_n = T(x_{n-1})$ then $\lim_{n\to\infty} x_n = x^*$ and $_q\|x_n - x^*\| \leq (ck^n/1-k)\ _q\|x_1 - x_0\|$, where $c \geq 1$ is a constant.

Remark 2.2 [6] :

1. Theorem 2.2 is valid for complete quasi-metric space [6] (the proof is similar).

2. Since, every closed subset Y of a quasi-Banach space X is itself a complete quasi-metri space [6.Theorem2.2.7]. Therefor theorem 2.2 is valid for a quasi-contraction mapping defined on Y into itself.

3. By theorem 2.1, the space $L_p(\mu)$ for $0 < p < 1$ is a quasi-Banach space, thus the special case of theorem 2.2 is when we take $X = L_p(\mu)$ for $0 < p < 1$ [9].

Theorem 2.3 [6, theorem 2.2.6] : Let X and Y be a quasi-normed spaces and L be a linear transformation of X onto Y. Then the inverse L^{-1} exists and is continuous on its domain of definition if and only if there exists a constant $M > 0$, such that $M\ _q\|x\| \leq {}_q\|L(x)\|$, for all $x \in X$.

Definition 2.6 [9] : A family $T(t)$, $0 \leq t < \infty$ of bounded linear operators on a quasi-Banach space X is called a (one-parameter) semigroup on X if it satisfies the following conditions :

1. $T(0) = I$ *(I is the identity operator on X).*

2. $T(t+s) = T(t)T(s)$, for each $t, s \geq 0$.

Definition 2.7 [9] : The infinitesimal generator A of the semigroup $T(t)$ on a quasi-Banach space X is defined by $Ax = \lim_{t\to 0}(1/t)(T(t)x - x)$, where the limit exists and the domain of A is $D(A) = \{x \in X :\ \lim_{t\to 0^+}(1/t)(T(t)x - x)$ exists $\}$.

Definition 2.8 [9] : A semigroup $T(t)$, $0 \le t < \infty$ of bounded linear operator on a quasi-Banach space X is said to be strongly continuous semigroup (or C_0 - semigroup) if :

${}_q\|T(t)x - x\| \to 0$ as $t \to 0^+$ for all $x \in X$.

Theorem 2.4 [6] : Let X be a quasi-Banach space and $T(t)$ be a C_0-semigroup generated by A. Then the following hold :

(i): For each $x_0 \in D(A)$,

$T(t)x_0 \in D(A)$(domain of A) and $A\,T(t)x_0 = T(t)Ax_0$, $\forall\, t \ge 0$.

(ii): For each $x_0 \in D(A)$ and $T(t)x_0 \in D(A)$, $(d/dt)(T(t)x_0) = A\,T(t)x_0 = T(t)A\,x_0$.

(iii): For each $x_0 \in X$, $\lim_{h\to 0}(1/h)\int_t^{t+h} T(s)x_0 ds = T(t)x_0$.

(iv): For each $x_0 \in X$, $\int_0^t T(s)x_0 ds \in D(A)$ and $A\left(\int_0^t T(s)x_0 ds\right) = T(t)x_0 - x_0$.

(v): For each $x_0 \in D(A)$,

$T(t)x_0 - T(s)x_0 = \int_s^t T(\tau)Ax_0\, d\tau = \int_s^t A\,T(\tau)x_0 d\tau$

.

For more details about semigroup and C_0-semigroup on a Banach space see [2, 10].

3. Controllability of Nonlinear System in Quasi-Banach Spaces :

In this section we will study the existence theorem of the controllability of the mild solution to the nonlinear boundary-value control problem in appropriate quasi-Banach spaces, by using strongly continuous semigroup theory and quasi-Banach contraction principle theorem.

3.1. Preliminaries

Let (E, ${}_q\|\,.\,\|$) , $(U, {}_q\|\,.\,\|)$ be a real quasi-Banach spaces, and A be a linear closed bounded and densely operator with $D(A) \subseteq E$, ${}_q\|A\| \le C_1$, where C_1 is a constant and let τ be a linear operator with $D(\tau) \subseteq E$ and range $(\tau) \subseteq X$, where X is a quasi-Banach space.

Consider the boundary control nonlinear system of the form

$$\left.\begin{aligned} &\frac{d}{dt}\Big(x(t) + g\big(t, x(t)\big)\Big) = Ax(t) + Bu(t) + F\big(t, N(t, x(t))\big) \\ &\tau\Big(x(t) + g\big(t, x(t)\big)\Big) = B_1 u(t), t \in J = [0, b] \\ &x(0) = x_0 \end{aligned}\right\} \quad (1)$$

Where $B_1 : U \to X$ is a linear continuous operator, the control function $u(.) \in U$ a quasi-Banach space of admissible control functions. Let $A_1 : E \to E$ be the linear operator defined by : $A_1 x = Ax$, $x \in D(A_1)$, where $D(A_1) = \{x \in D(A) : \tau x = 0\}$.

Let $B_r = \{x \in E : {}_q\|x\| \le r$, for some $r > o\}$.

We shall make the following hypotheses:

(i): $D(A) \subset D(\tau)$ and the restriction of τ to $D(A)$ is continuous relative to graph norm of $D(A_1)$.

(ii): The operator A_1 is the infinitesimal generator of a C_0-semigroup $T(t)$ and there exist a constant $M > 0$, such that ${}_q\|T(t)\| \le M$.

(iii): There exist a linear continuous operator $B_2 : U \to E$, such that $AB_2 \in L_p(U, E)$ where $0 < p < 1$, $\tau(B_2 U) = B_1 u$ for all $u \in U$,also $B_2 u(t)$ is continuously differentiable.

(iv): For all $t \in (0, b]$ and $u \in U$, $T(t)B_2 u \in D(A_1)$, moreover there exist a positive function $v_0 \in L^1(0, b)$, such that ${}_q\|A_1 T(t) B_2\| \le v_0(t)$ almost everywhere $t \in (0, b)$.

(v): The nonlinear operator $N : J \times E \to E$ is continuous and satisfies Lipschitz condition on the second argument, such that for all $x_1, x_2 \in B_r$, and a positive constant M_3, we have:

${}_q\|N(t, x_1) - N(t, x_2)\| \le M_3\ {}_q\|x_1 - x_2\|$.

(vi): The nonlinear operator $F : J \times E \to E$ is continuous and satisfies Lipschitz condition on the second argument, for $x_1, x_2 \in B_r$ and the positive constants M_1 and M_2, we have:

${}_q\left\|F\big(t, N(t, x_1)\big) - F\big(t, N(t, x_2)\big)\right\| \le M_1\ {}_q\|x_1 - x_2\|$

and $M_2 = \max_{t\in J}\ {}_q\|F(t, N(t, 0))\|$.

(vii): $B : U \to E$ is bounded linear operator, ${}_q\|B\| \le C$, where C is a positive constant.

(viii): The nonlinear operator $g : J \times E \to E$, satisfy Lipschitz condition on the second argument, let $L_1, L_2 > 0$ be constants, such that for all $x_1, x_2 \in B_r$ we have: ${}_q\|g(t, x_1) - g(t, x_2)\| \le L_1\ {}_q\|x_1 - x_2\|$, and $L_2 = \max_{t\in J}\ {}_q\| g(t, 0)\|$.

The main aim of this section is to find the mild solution of (1).

Now let $x(t)$ be the solution of (1). Then we can define a function

$$Z(t) = x(t) + g(t, x(t)) - B_2\, u(t) \qquad (2)$$

From our assumption we want to show that $Z(t) \in D(A_1)$, by (2) :

$$\tau(Z(t)) = \tau\left(x(t) + g(t, x(t)) - B_2 u(t)\right)$$

$$= \tau\left(x(t) + g(t, x(t)) - \tau(B_2\, u(t))\right)$$

By condition (iii) and equation (1) we have $\tau(Z(t)) = B_1 u(t) - B_1 u(t) \equiv 0$

So, by definition of $D(A_1)$, we have $Z(t) \in D(A_1)$, and $A\, Z(t) = A_1 Z(t)$. Therefore, from (2), (1) can be written in term A_1 and B_2 as :

$$\frac{d}{dt}\left(x(t) + g(t, x(t))\right) = A\left(Z(t) - g(t, x(t)) + B_2 u(t)\right) + Bu(t) + F\left(t, N(t, x(t))\right)$$

since A is linear operator, then : $\frac{d}{dt}\left(x(t) + g(t, x(t))\right) = AZ(t) - Ag(t, x(t)) + AB_2 u(t) + Bu(t) + F\left(t, N(t, x(t))\right)$

$$= A_1 Z(t) - Ag(t, x(t)) + AB_2 u(t) + Bu(t) + F\left(t, N(t, x(t))\right)$$

, Thus we have :

$$\frac{d}{dt}\left(x(t) + g(t, x(t))\right) = A_1 Z(t) - Ag(t, x(t)) + AB_2 u(t) + Bu(t) + F\left(t, N(t, x(t))\right)$$

$$x(t) + g(t, x(t)) = Z(t) + B_2 u(t)$$

$$x(0) = x_0 \qquad (3)$$

By condition (iii), $B_2 u(t)$ is continuously differentiable, if x is continuously differentiable on [0,b], then $Z(t)$ can define as a mild solution to the Cauchy problem : $\frac{d}{dz} Z(t) = \frac{d}{dt}(x(t) + g(t, x(t))) - B_2 \frac{d}{dt} u(t)$, by equation (3), we get :

$$\frac{d}{dt} Z(t) = A_1 Z(t) - Ag(t, x(t)) + AB_2 u(t) + Bu(t) - B_2 \frac{d}{dt} u(t) + F\left(t, N(t, x(t))\right)$$

$$Z(0) = x_0 + g(0, x_0) - B_2\, u(0) \qquad (4)$$

Since in condition (ii), we have $T(t), \forall\, t \geq 0$ is a C_0-semigroup generated by the linear operator A_1 and $Z(t)$ is a solution of (4), then by theorem 2.4, the function $H(s) = T(t-s)Z(s)$ is differentiable for $0 < s < t$, and $\frac{d}{ds} H(s) = T(t-s)\frac{d}{ds} Z(s) + Z(s)\frac{d}{ds} T(t-s)$,

thus by (4) and theorem 2.4 (ii), we have :

$$\frac{d}{ds} H(s) = T(t-s)[A_1 Z(s) - Ag(s, x(s)) + AB_2 u(s) + Bu(s) - B_2 \tfrac{d}{ds} u(s) + F(s, N(s, x(s)))] + Z(s)[-A_1 T(t-s)]$$

$$= T(t-s)A_1 Z(s) - T(t-s)Ag(s, x(s)) + T(t-s)AB_2 u(s) + T(t-s)Bu(s) - T(t-s)B_2 \tfrac{d}{ds} u(s) + T(t-s)F\left(s, N(s, x(s))\right) - T(t-s)A_1 Z(s)$$

and by integration from *0* to *t*, yield.

$$H(t) - H(0) = -\int_0^t T(t-s)Ag(s, x(s))ds + \int_0^t T(t-s)AB_2 u(s)ds + \int_0^t T(t-s)Bu(s)ds - \int_0^t T(t-s)B_2 \tfrac{d}{ds} u(s)ds + \int_0^t T(t-s)F\left(s, N(s, x(s))\right)ds \qquad (5)$$

By integrate the part $\int_0^t T(t-s)B_2 \frac{d}{ds} u(s)ds$ in (5) by part, we get :

$$x(t) = T(t)x_0 + T(t)g(0, x_0) - g(t, x(t)) - \int_0^t T(t-s)Ag(s, x(s))ds + \int_0^t T(t-s)F\left(s, N(s, x(s))\right)ds + \int_0^t [(T(t-s)A - A_1 T(t-s))B_2 + T(t-s)B]u(s)ds \qquad (6)$$

Definition 3.1: A function $x : [0, b] \to E$ defined by the Integro equation (6) is called a mild solution of (1) if x is continuously differentiable on $(0, b)$, continuous on $[0, b]$ and $x(t) \in E$ for $0 < s < t$.

Definition 3.2 : The boundary value control problem (1) is said to be controllable on the interval $J = [0, b]$ if for every $x_0, x_1 \in E$, there exists a control $u \in U$ such that the mild solution (6) satisfied $x(b) = x_1$.

Here we further consider the following additional conditions :

(ix): There exists a constant $k > 0$ such that $\int_0^b v_0(t)dt \leq k_1$.

(x): Define the linear continuous operator w from U onto E as follows $w\,u = \int_0^b [(T(b-s)A - A_1T(b-s))B_2 + T(b-s)B]u(s)ds$, and suppose that for every $u(.) \in U$, there exists a constant $k > 0$ such that $k\ {}_q\|u\| \leq {}_q\|w\,u\|$.

From the above condition (x) and theorem 2.3, we see that the inverse operator of w *exists and is continuous (bounded). i.e., the operator* w^{-1} *: Rang* $w \to U$ *defined by* $w^{-1}(wu(t)) = u(t)$ *exists and there exists a positive constant* $k_2 > 0$*, such that* ${}_q\|w^{-1}\| \leq k_2$.

(xi): $C_2M\ {}_q\|x_0\| + C_2MC_3h_1 + C_2C_4h_2 + C_2bMC_1C_4h_2 + C_2bMC_5h_3 + (C_2bM\ {}_q\|AB_2\| + bMC + k_1)k_2[\ {}_q\|x_1\| + M\ {}_q\|x_0\| + MC_3h_1 + C_6(L_1\ {}_q\|x_1\| + L_2) + bMC_1C_4h_2 + bMC_5h_3] \leq r$,

where $C_i \geq 1$, $i = 2,3,4,5,6$, are constants and $h_1 = L_1\ {}_q\|x_0\| + L_2$, $h_2 = L_1r + L_2$, and $h_3 = M_1r + M_2$.

(xii): Let $q = C_7L_1 + C_7MbC_1L_1 + C_7bMM_1 + C_7(bk_2MC_1 + k_2k_1 + bk_2MC)(bMC_1L_1 + bMM_1)$, such that $0 \leq q < 1$, where $C_7 \geq 1$ be a constant.

3.2. Main Result

Theorem 3.1 : If the hypotheses (i)-(xii) are satisfied, then the boundary control nonlinear system (1) is controllable on J.

Proof. By definition 3.2, and condition (x) we have

$$x_1 = x(b) = T(b)x_0 + T(b)g(0,x_0) - g(b,x_1) - \int_0^b T(b-s)Ag(s,x(s))ds + \int_0^b T(b-s)F\left(s,N(s,x(s))\right)ds + w\,u(t).$$

Since $u(t) = w^{-1}(w\,u)$, then we get that

$$u(t) = w^{-1}[x_1 - T(b)x_0 - T(b)g(0,x_0) + g(b,x_1) + \int_0^b T(b-s)Ag(s,x(s))ds - \int_0^b T(b-s)F\left(s,N(s,x(s))\right)ds](t) \quad (7)$$

Let $Y = C(J, B_r)$. Using this control (7), we shall show that the operator $\emptyset$ defined by:

$$\emptyset x(t) = T(t)x_0 + T(t)g(0,x_0) - g(t,x(t)) - \int_0^t T(t-s)Ag(s,x(s))ds + \int_0^t T(t-s)F\left(s,N(s,x(s))\right)ds + \int_0^t [(T(t-s)A - A_1T(t-s))B_2 + T(t-s)B]w^{-1}[x_1 - T(b)x_0 - T(b)g(0,x_0) + g(b,x_1) + \int_0^b T(b-r_1)Ag(r_1,x(r_1))dr_1 - \int_0^b T(b-r_1)F\left(r_1,N(r_1,x(r_1))\right)dr_1](s)ds$$

, has a unique fixed point.

First we show that $\emptyset$ map Y into itself, for $x \in Y$ to show that ${}_q\|\emptyset x(t)\| \leq r$.

$${}_q\|\emptyset x(t)\| \leq C_2\{\ {}_q\|T(t)x_0\| + {}_q\|T(t)g(0,x_0)\| + {}_q\|g(t,x(t)\| + \int_0^b {}_q\|T(t-s)\|\ {}_q\|A\|\ {}_q\|g(s,x(x)\|ds + \int_0^t {}_q\|T(t-s)\|\ {}_q\left\|F\left(s,N(s,x(s))\right)\right\| ds + \int_0^t [\ {}_q\|T(t-s)AB_2\| + {}_q\|A_1T(t-s)B_2\| + {}_q\|T(t-s)B\|]\ {}_q\|w^{-1}\| [\ {}_q\|x_1\| + {}_q\|T(b)\|\ {}_q\|x_0\| + {}_q\|T(b)\|\ {}_q\|g(0,x_0)\| + {}_q\|g(b,x_1)\| + \int_0^b {}_q\|T(b-r_1)\|\ {}_q\|A\|\ {}_q\|g(r_1,x(r_1))\|dr_1 + \int_0^b {}_q\|T(b-r_1)\|\ {}_q\left\|F\left(r_1,N(r_1,x(r_1))\right)\right\| dr_1](s)ds\},$$

where $C_2 \geq 1$ be a constant. By (ii),(iv), and ${}_q\|A\| \leq C_1$, we have

$${}_q\|\emptyset x(t)\| \leq C_2M\ {}_q\|x_0\| + C_2M\ {}_q\|g(0,x_0) - g(0,0) + g(0,0)\| + C_2\ {}_q\|g(t,x(t)) - g(t,0) + g(t,0)\|$$

$$+\int_0^t C_2MC_1\ {}_q\|g(s,x(s))-g(s,0)+g(s,0)\|ds+$$

$$\int_0^t C_2M\ {}_q\left\|F\left(s,N(s,x(s))\right)-F(s,N(s,0))+F(s,N(s,0))\right\|ds+$$

$$\int_0^t C_2(M\ {}_q\|AB_2\|+v_0(s)+MC)k_2[\ {}_q\|x_1\|+$$

$$M\ {}_q\|x_0\|+M\ {}_q\|g(0,x_0)-g(0,0)+g(0,0)\|+$$

$${}_q\|g(b,x_1)-g(b,0)+g(b,0)\|+$$

$$\int_0^b MC_1\ {}_q\|g(r,x(r))-g(r,0)+g(r,0)\|dr+$$

$$\int_0^b M\ {}_q\left\|F\left(r_1,N(r_1,x(r_1))\right)-F(r_1,N(r_1,0))+F(r_1,N(r_1,0))\right\|dr_1](s)ds$$

Since $x\in B_r$, then ${}_q\|x\|\le r$. Thus by conditions (vi),(vii) and (viii) we get that

$${}_q\|\emptyset x(t)\|\le C_2M\ {}_q\|x_0\|+$$

$$C_2MC_3(L_1\ {}_q\|x_0\|+L_2)+C_2C_4(L_1r+L_2)+$$

$$C_2bMC_1C_4(L_1r+L_2)+C_2bMC_5(M_1r+M_2)+$$

$$(C_2bM\ {}_q\|AB_2\|+bMC+k_1)$$

$$k_2[\ {}_q\|x_1\|+M\ {}_q\|x_0\|+MC_3(L_1\ {}_q\|x_0\|+L_2)+$$

$$C_6(L_1\ {}_q\|x_1\|+L_2)+bMC_1C_4(L_1r+L_2)+$$

$bMC_5(M_1r+M_2)]$, where $C_i\ge 1, i=3,4,5.6,$ are constants. By condition (xi)we have

$${}_q\|\emptyset x(t)\|\le C_2M\ {}_q\|x_0\|+C_2MC_3h_1+$$

$$C_2C_4h_2+C_2bMC_1C_4h_2+C_2bMC_5h_3+$$

$$(C_2bM\ {}_q\|AB_2\|+bMC+k_1)k_2[\ {}_q\|x_1\|+$$

$$M\ {}_q\|x_0\|+MC_3h_1+C_6(L_1\ {}_q\|x_1\|+L_2)+$$

$$bMC_1C_4h_2+bMC_5h_3]\le r.$$

Thus $\emptyset$ map Y into itself. Now for $x_1,x_2\in Y$ we have

$${}_q\|\emptyset x_1(t)-\emptyset x_2(t)\|\le C_7\{\ {}_q\|g(t,x_1(t))-g(t,x_2(t))\|+$$

$$\int_0^t\ {}_q\|T(t-s)\|\ {}_q\|A\|$$

$${}_q\|g(s,x_1(s))-g(s,x_2(s))\|ds+$$

$$\int_0^t\ {}_q\|T(t-s)\|\ {}_q\|F(s,N(s,x_1(s)))-F(s,N(s,x_2(s)))\|ds$$

$$+\int_0^t\|T(t-s)A-A_1T(t-s)B_2+$$

$$T(t-s)B\|\ {}_q\|w^{-1}\|[\int_0^b\ {}_q\|T(b-r_1)\|\ {}_q\|A\|$$

$${}_q\|g(r_1,x_1(r_1))-g(r_1,x_2(r_1))\|dr_1+\int_0^b\ {}_q\|T(b-r_1)\|$$

$${}_q\left\|F(r_1,N(r_1,x_1(r_1))-F\left(r_1,N(r_1,x_2(r_1))\right)\right\|dr_1](s)ds\}$$

where $C_7\ge 1$ be a constant. By conditions (ii), (vii) and ${}_q\|A\|\le C_1$, we obtain that

$${}_q\|\emptyset x_1(t)-\emptyset x_2(t)\|\le C_7L_1\ {}_q\|x_1(t)-x_2(t)\|+$$

$$\int_0^t C_7MC_1L_1\ {}_q\|x_1(s)-x_2(s)\|ds+$$

$$\int_0^t C_7MM_1\ {}_q\|x_1(s)-x_2(s)\|ds+$$

$$\int_0^t C_7(MC_1+v_0(s)+MC)k_2[bMC_1L_1\ {}_q\|x_1(t)-x_2(t)\|+$$

$$bMM_1\ {}_q\|x_1(t)-x_2(t)\|\ ](s)ds,$$

and by condition (xii) we see that

$${}_q\|\emptyset x_1(t)-\emptyset x_2(t)\|\le[C_7L_1+C_7bMC_1L_1$$

$$+C_7bMM_1+C_7(bk_2MC_1+k_2k_1+$$

$$MCbk_2)(bMC_1L_1+bMM_1)]\ {}_q\|x_1(t)-x_2(t)\|$$

$$=q\ {}_q\|x_1(t)-x_2(t)\|$$

Thus $\emptyset$ is a contraction mapping. Hence by quasi-Banach contraction principle theorem (theorem 2.2) there exist a unique fixed point $x(t)\in Y$ such that $\emptyset x(t)=x(t)$. Any fixed point is a mild solution of (1) on J which satisfies $x(b)=x_1$. Thus the system (1) is controllable on.

3.3 Application

The Leslie model [7] is a powerful tool used the matrices to determine the growth of a population as well as the age distribution within a population over certain time interval.

Definition 3.3 [7] : An infinite matrix $(a_{ij})_1^{\infty}$ whose elements satisfy $a_{ij}=\begin{cases}F_i & i=1 \text{ and } j=1,2,\dots\\ P_i & i=2,3,\dots, \text{and } j=i-1\\ 0 & \text{otherwise}\end{cases}$ where $F_i\ge 0$ is the average reproduction of females in the i-th age class, and $0<P_i<1$ is the survival rate of a females in the i-th age class, is called an infinite dimensional Leslie matrix.

Let $(a_{ij})_1^{\infty}$ be Leslie matrix whose elements are function in the quasi-Banach space L_p for $0<p<1$. Then

Theorem 3.2 [7] : An infinite-dimensional Leslie matrix $(a_{ij})_1^{\infty}$ defines a bounded linear operator from L_p into L_p when $0<p<1$.

Theorem 3.3 [7] : An infinite- dimensional Leslie matrix $(a_{ij})_1^\infty$ defines a compact linear operator from L_p into L_p when $0 < p < 1$.

Now, let $E = U = X = L_p$ for $0 < p < 1$ be real quasi-Banach spaces, and consider the problem (1), where $A = (a_{ij})_1^\infty$ is an infinite-dimensional Leslie matrix, B is a matrix whose elements are functions in the quasi-Banach space $L_p, 0 < p < 1$, B_1 is the identity operator, and assume that the operators $g(.,.)$ and $F(.,.)$ in (1) are identical to zero operator.

Then by theorems 3.2 and 3.3, the matrix $A = (a_{ij})_1^\infty$ defines a bounded (compact) linear operator from L_p into L_p for $0 < p < 1$.

By the same way we see that the operator B is bounded.

Thus the operator A is the infinitesimal generator of a C_0-semigroup defined by $T(t) = e^{tA} = \sum_{k=0}^{\infty}(t^k A^k / k!), t \geq 0$, which is bounded [10].

Therefore it is not difficult to check that all assumptions of theorem 3.1, are satisfied for the above problem [9].

4. Conclusions:

In this paper we extend the study of controllability of control problem in any quasi-Banach spaces. Thus the concepts of a quasi-Banach space are introduced, such as a quasi-Banach contraction principle theorem, strongly continuous semigroup and used it to prove theorem deals with the controllability for nonlinear boundary-value control problem in the quasi-Banach spaces.

5. Future work:

The observability and optimality for the problem (1) may be considered.

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