On Right (σ,τ)- Derivation of Prime Rings

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Abstract

Let R be a prime ring and δ a right (σ,τ)-derivation on R. In the present paper we will prove the following results:

First, suppose that R is a prime ring and I a non-zero ideal of R if δ acts as a homomorphism on I then $\delta=0$ on R, and if δ acts an anti- homomorphism on I then either $\delta=0$ on R or R is commutative.

Second, suppose that R is 2-torsion-free prime ring and J a non-zero Jordan ideal and a subring of R, if δ acts as a homomorphism on J then $\delta=0$ on J, and if δ acts an anti- homomorphism on J then either $\delta=0$ on J or $J\subseteq Z(R)$.

Keywords: prime rings, (σ,τ)- derivations, right (σ,τ)-derivations.

مشتقات– (σ,τ) اليمنى على الحلقات الاولية

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الخلاصة:

لتكن R حلقة اولية و δ الاشتقاق– (σ,τ) الايمن على R . في بحثنا هذا سوف نبرهن النتائج الاتية :

اولا: افرض انه R حلقة اولية و I مثالي غير صفري على R ، إذا كانت δ هو تشاكل على I فأن $\delta=0$ وإذا كانت δ تشاكل ضد على I فانه اما $\delta=0$ على R او R حلقة ابدالية.

ثانيا: افرض انه R حلقة الالتواء–2 و J مثالي جوردان غير صفري وحلقة جزئية من R ، إذا كانت δ هو تشاكل او تشاكل ضد على J فانه اما $\delta=0$ على J او J مجموعة جزئية من $Z(R)$.

المفاتيح: الحلقات الاولية، مشتقات– (σ,τ)،مشتقات– (σ,τ) اليمنى.

1. Introduction

Throughout the present paper, R will denote an associative ring with center $Z(R)$. We will write for $x, y \in R$, $[x, y]=xy-yx$ and for $x\circ y=xy+yx$ for Lie product and Jordan product, respectively.

Recall that R is a prime if $aRb=\{0\}$ implies that $a=0$ or $b=0$. A ring said to be a 2-torsion-free if whenever $2a=0$, with a $\in R$ then $a=0$. An additive subgroup J of R is said to be a Jordan ideal of R if $u\circ r \in J$ for all $u \in J$ and $r \in R$. An additive mapping $d: R \rightarrow R$ is called derivation (resp., Jordan derivation) if $d(xy)= d(x)y+xd(y)$ (resp., $d(x^2)= d(x)x+ xd(x)$) holds, for all $x, y \in R$. Let σ, τ are two mappings of R. An additive mapping $d: R \rightarrow R$ is called a (σ, τ)-derivation (resp., Jordan (σ, τ)-derivation) on R if $d(xy)= d(x)\sigma(y)+\tau(x)d(y)$ (resp., $d(x^2)= d(x)\sigma(x)+\tau(x)d(x)$) holds, for all $x, y \in R$, of course every $(1,1)$-derivation (resp., Jordan $(1,1)$-derivation), where 1 is the identity mapping on R is derivation (resp., Jordan derivation) on R. An additive mapping $\delta: R \rightarrow R$ is called a left derivation (resp., Jordan left derivation) on R if $\delta(xy)=x\delta(y)+y\ \delta(x)$ (resp., $\delta(x^2)= 2x\ \delta(x)$) holds, for all $x, y \in R$. An additive mapping $\delta: R \rightarrow R$ is called a right

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derivation (resp., Jordan right derivation) on R if $\delta(xy)=\delta(y)x+\delta(x)y$ (resp., $\delta(x^2)= 2\delta(x)x$) holds, for all $x, y \in R$.

In view of the definition of a (σ, τ)-derivation the notation of left (σ, τ)-derivation and right (σ, τ)-derivation can be extended as follows: An additive mapping $\delta: R \rightarrow R$ is called a left (σ, τ)-derivation (resp., Jordan left (σ, τ)-derivation) on R if $\delta(xy)=\sigma(x)\delta(y)+\tau(y)\delta(x)$(resp., $\delta(x^2)= \sigma(x)\delta(x)+\tau(x)\delta(x)$) holds, for all $x, y \in R$. Clearly every left $(1, 1)$-derivation (resp., Jordan left $(1,1)$-derivation) on R.

An additive mapping $\delta: R \rightarrow R$ is called a right (σ, τ)-derivation (resp., Jordan right (σ, τ)-derivation) on R if $\delta(xy)= \delta(y)\sigma(x) + \delta(x)\tau(y)$ (resp., $\delta(x^2)= \delta(x)\sigma(x) + \delta(x)\tau(x)$) holds, for all $x, y \in R$. Clearly every right $(1, 1)$-derivation (resp., Jordan right $(1,1)$-derivation) on R.

Bell and Kappe [1] proved that if d is a derivation of a prime ring R which acts as a homomorphism or as an anti- homomorphism on a nonzero right ideal I of R, then $d=0$ on R, further Yenigul and Arac [2] obtained the above result for α-derivation in prime rings. Recently M. Ashraf [3] extended the result for (σ,τ)-derivation in prime and semiprime ring. In [4] the authors extended the above results for (σ, σ)-derivation which acts as a homomorphism or as an anti- homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion -free prime ring R, then they generalized the above extension for generalized (σ, σ)-derivation. Also they proved that if $d:R \rightarrow R$ is a (σ,τ)-derivation which acts as a homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring R, then either $d=0$ on R or $J \subseteq Z(R)$.

In [5] M. Ashraf proved that if R is a 2-torsion-free prime ring, J a nonzero Jordan ideal and a subring of R and d is a left (σ, σ)-derivation of R, which acts as a homomorphism or as an anti- homomorphism on R, then $d=0$ on R, the authors in [6] extended this result to a left (σ,τ)- derivation which acts as a homomorphism or as an anti- homomorphism on a nonzero Jordan ideal and a subring J of R, for more details and fundamental results used in this paper without mention we refer to [7-13]).

In the present paper our objective is to extend the above results for a right (σ,τ)-derivation which acts as a homomorphism or as an anti-homomorphism on a nonzero ideal I of prime ring R, and on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring R.

2. Right (σ,τ)-derivation as a homomorphism or as anti- homomorphism on ideals.

Let R be a ring and d a derivation of R. If $d(xy)=d(x)d(y)$ (resp., $d(x^2)=d(x)d(x)$) holds, for all $x, y \in R$, then we say that d acts as a homomorphism (resp., anti- homomorphism) on R.

Bell and Kappe [1] proved that if d is a derivation of a prime ring R which acts as homomorphism or anti- homomorphism on a nonzero right ideal I of R; then $d=0$ on R. This result was extended for (σ,τ) by M. Asheraf [3] as follows:

Theorem (2.1): [3]

Let R be a prime ring and I a nonzero ideal of R. Suppose σ, τ are automorphism of R and $d:R \rightarrow R$ is a (σ,τ)-derivation of R.

(i) If d acts as homomorphism on I, then $d=0$ on R.

(ii) If d acts as anti-homomorphism on I, then $d=0$ on R.

M. Ashraf in [5] was extended the above results for a left (σ,σ)-derivation and recently the authors in [6] extend this results to a left (σ,τ)-derivation which acts as a homomorphism or as an anti- homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring R as follows:

Theorem (2.2):[6]

Let R be a 2-torsion-free prime ring, J a nonzero Jordan ideal and a subring of R suppose that σ, τ are automorphism of R and $\delta:R \rightarrow R$ is a left (σ,τ)-derivation of R.

(i) If δ acts as homomorphism on J, then either $\delta =0$ on R or $J\subseteq Z(R)$.

(ii) If δ acts as anti-homomorphism on J, then either $\delta =0$ on R or $J\subseteq Z(R)$.

In the present paper first we will extend the above result to a right (σ,τ)-derivation which acts as a homomorphism or as an anti-homomorphism on a nonzero ideal I of a prime ring R. Secondly, we will extend the above result to a right (σ, τ)-derivation which acts as a homomorphism or as an anti- homomorphism on a nonzero ideal and Jordan ideal and a subring J of a 2-torsion-free prime ring R. To proof the first theorem we begin with the following Lemmas.

Lemma (2.3): [14]

Let R be a semiprime ring, I a right ideal of R, then $Z(I)\subset Z(R)$.

Lemma (2.4): [14]

Let R be a semiprime ring, I a nonzero right ideal of R. If I is a commutative as a ring, then $I \subset Z(R)$. In addition if R is a prime then R must be commutative.

Now, we will prove the first main theorem of this paper.

Theorem(2.5):

Let R be a prime ring, I a nonzero ideal of R. Suppose σ, τ are automorphisms of R and $\delta: R \rightarrow R$ is a right (σ,τ)- derivation of R.

(i) If δ acts as a homomorphism on I, then $\delta = 0$ on R.

(ii) If δ acts as an anti- homomorphism on I, then either $\delta = 0$ on R or R is commutative.

Proof:

(i) If δ acts as a homomorphism on I, then we have

$\delta(uv) = \delta(v)\sigma(u) + \delta(u)\tau(v) = \delta(u)\,\delta(v)$, for all $u,v \in I$ ... (2.1)

Replacing u by ut, $t \in I$ in (2.1), we get

$[\delta(v)\sigma(u) - \delta(u)\,\delta(v)]\,\sigma(t) = 0$, for all $u,v,t \in I$ ... (2.2)

In (2.2) Replacing t by rt, $r \in R$ we get $[\delta(v)\sigma(u) - \delta(u)\ \delta(v)]\ \sigma(rt) = 0$, for all $u,v,t \in I, r \in R$ i.e., $\sigma^{-1}([\delta(v)\sigma(u) - \delta(u)\delta(v)])rt = 0$ and hence $\sigma^{-1}([\delta(v)\sigma(u) - \delta(u)\delta(v)])RI = \{0\}$, for $u,v \in R$. Since R is a prime ring and I a nonzero ideal of R, we have $\delta(v)\sigma(u) - \delta(u)\ \delta(v) = 0$, for all $u,v \in I$. Since δ acts as a homomorphism on I, the last equation yields that $\delta(u)\tau(v) = 0$ for all $u,v \in I$. Replacing v by rv, $r \in R$, we get $\delta(u)\tau(rv) = 0$ for all $u,v \in I, r \in R$. Since τ is an automorphisms of R, we have $\delta(u)RI$, for all $u \in I$. Since R is prime ring and I a nonzero ideal of R, we get $\delta(u) = 0$, for all $u \in I$. Replacing u by ur, $r \in R$, in the last relation to get $0 = \delta(ur) = \delta(r)\sigma(u) + \delta(u)\tau(r) = \delta(r)\sigma(u)$, for all $u \in I, r \in R$. Since R is a prime ring and I is a nonzero ideal of R, we get $\delta = 0$ on R.

(ii) If δ acts as an anti- homomorphism on I, we have

$\delta(uv) = \delta(v)\sigma(u) + \delta(u)\tau(v) = \delta(v)\ \delta(u)$, for all $u,v \in I$ ... (2.3)

Replacing u by uv in (2.3), we get

$\delta(v)\sigma(u)\sigma(v) = \delta(v)\ \delta(v)\sigma(u)$, for all $u,v \in I$ . ...(2.4)

Replacing u by ut, $t \in I$ in (2.4), then we get

$\delta(v)\sigma(u)\sigma(t)\sigma(v) = \delta(v)\delta(v)\sigma(u)\sigma(t)$ ... (2.5)

In view of (2.4), the relation (2.5) yields that $\delta(v)\sigma(u)[\ \sigma(v),\ \sigma(t)] = 0$, for all $u,v,t \in I$ i.e., $\sigma^{-1}(\delta(v))\ u\ [v,t] = 0$, for all $u,v,t \in I$. Replacing u by ru, $r \in R$ in the last relation to get $\sigma^{-1}(\delta(v))\ ru\ [v,t] = 0$, for all $u,v,t \in I$, this implies that $\sigma^{-1}(\delta(v))\ RI\ [v,t] = \{0\}$, for all $v,t \in I$. Since R is a prime ring and I a nonzero ideal of R, we have either $\delta(v) = 0$ or $[v,t] = 0$, for all $v,t \in I$. If $\delta(v) = 0$, for all $v \in I$, replace v by vr, $r \in R$ to get $0 = \delta(vr) = \delta(r)\sigma(v) + \delta(v)\tau(r) = \delta(r)\sigma(v)$, for all $v \in I, r \in R$. Since I is ideal of R and σ is automorphisms of R, we get $\delta = 0$ on R. If $[v,t] = 0$, for all $v,t \in I$, this implies that I is commutative, i.e. $I = Z(I)$ by Lemma (2.1), we have $Z(I) \subset Z(R)$. Now since $I = Z(I)$ and $Z(I) \subset Z(R)$ we have $I \subset Z(R)$, thus by Lemma(2.2) R is commutative.

3. Right (σ,τ)-derivation as a homomorphism or as anti- homomorphism on Jordan ideals.

The following Lemmas which are essential to proof the second main theorem of our paper.

Lemma (3.1): [5]

Let R be a prime ring and J a nonzero Jordan ideal of R. If $a \in R$ and $aJ = \{0\}$ (or $Ja = \{0\}$), then $a = 0$.

Lemma (3.2): [5]

Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal of R. $aJb = \{0\}$ then $a = 0$ or $b = 0$.

Lemma (3.3): [5]

Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal of R. If J is a commutative Jordan ideal, then $J \subseteq Z(R)$.

Now, we obtained the following theorem which also includes the main results:

Theorem (3.4):

Let R be a 2-torsion-free prime ring and J a nonzero Jordan and subring of R. Suppose that σ, τ are automorphisms of R and $\delta: R \rightarrow R$ is a right (σ,τ)- derivation of R.

(i) If δ acts as a homomorphism on J, then either $\delta = 0$ on R or $J \subseteq Z(R)$.

(ii) If δ acts as an anti-homomorphism on J, then either $\delta = 0$ on R or $J \subseteq Z(R)$.

Proof: suppose that $J \not\subseteq Z(R)$

(i) If δ acts as a homomorphism on J, then we have $\delta(uv) = \delta(u)\delta(v) = \delta(v)\sigma(u) + \delta(u)\tau(v)$ holds, for all $u, v \in J$ …(3.1)

replacing u by ut, $t \in J$ in (3.1), we get

$\delta(v)\sigma(ut) + \delta(ut)\tau(v) = \delta(ut)\delta(v) = \delta(u)\delta(tv) = \delta(u)(\delta(v)\sigma(t) + \delta(t)\tau(v)) = \delta(u)\delta(v)\sigma(t) + \delta(u)\delta(t)\tau(v)$, for all $u,v,t \in J$

This implies that $[\delta(v)\sigma(u) - \delta(u)\delta(v)]\sigma(t) = \{0\}$, for all $u,v,t \in J$ and hence $[\delta(v)\sigma(u)-$

$\delta(u)\delta(v)]\sigma(J)=\{0\}$, for all $u,v\in J$. Since σ is an automorphisms of R and J a nonzero Jordan ideal of R, $\sigma(J)$ is also a nonzero Jordan ideal of R. Application of Lemma (3.1) yields that $0=\delta(v)\sigma(u)-\delta(u)\delta(v)$, for all $u,v\in J$. Since δ is a homomorphism right (σ, τ) derivation we have, $0=\delta(v)\sigma(u)-\delta(uv)=\delta(v)\sigma(u)-\delta(v)\sigma(u)$
$\delta(u)\tau(v)=\delta(u)\tau(v)$, for all $u,v\in J$. Since τ is an automorphism of R and by Lemma (3.1), we get $\delta(u)=0$, for all $u\in J$. Replacing u by $u\circ r$, $r\in R$, we have

$0=\delta(u\circ r)=\delta(ur+ru)=\delta(ur)+\delta(ru)$
$=\delta(r)\sigma(u)+\delta(u)\tau(r)+\delta(u)\sigma(r)+\delta(r)\tau(u)$
$=\delta(r)\sigma(u)+\delta(r)\tau(u)=\delta(r)[\sigma(u)+\tau(u)]$, for all $r\in R$, $u\in J$.

Hence we get $\delta(r)[\sigma(J)+\tau(J)]=\{0\}$, for all $r\in R$. Since σ, τ are automorphism of R and J is a nonzero Jordan ideal of R, we get $\sigma(J)$ and $\tau(J)$ are a nonzero Jordan ideals of R and hence we get $\sigma(J)+\tau(J)$ is a nonzero Jordan ideal of R, then by Lemma (3.1) we get $\delta(r)=0$ for all $r\in R$, this implies that $\delta=0$ on R.

(ii) If δ acts as an anti- homomorphism on J, then we have

$\delta(uv)=\delta(v)\delta(u)=\delta(v)\sigma(u)+\delta(u)\tau(v)$ holds, for all $u, v\in J$……… …(3.2)

replacing u by uv in (3.2), we get

$\delta(v)\delta(uv)=\delta(v)\delta(v)\sigma(u)+\delta(v)\delta(u)\tau(v)$
$=\delta(v)\sigma(u)\sigma(v)+\delta(v)\delta(u)\tau(v)$ for all $u, v\in J$

or equivalently,

$\delta(v)\delta(v)\sigma(u)=\delta(v)\sigma(u)\sigma(v)$, for all $u, v\in J$ (3.3)

replacing u by ut, $t\in J$ in (3.3), we get

$\delta(v)\delta(v)\sigma(u)\sigma(t)=\delta(v)\sigma(u)\sigma(t)\sigma(v)$, for all $u, v\in J$ ……………………………………………(3.4)

In view of (3.3), the relation (3.4) yields that $\delta(v)\sigma(u)[\sigma(v)\sigma(t)-\sigma(t)\sigma(v)]=0$, for all $u, v\in J$, this implies that

$\sigma^{-1}(\delta(v))\, J\, [vt-tv]=\{0\}$, for all $v, t\in J$ by Lemma (3.2), we get either $\delta(v)=0$ or $[v, t]=0$, for all $v, t\in J$.

Now, let $J_1=\{v\in J \mid [v, t]=0$, for all $t\in J\}$ and $J_2=\{v\in J \mid \delta(v)=0\}$. Clearly, J_1 and J_2 are additive proper subgroups of J whose union in J. Hence by Brauer's trick, either $J=J_1$ or $J=J_2$.

If $J=J_1$ then $[v, t]=0$, for all $v, t\in J$, it follows that J is commutative, hence by Lemma (3.3), we get $J\subseteq Z(R)$, a contradiction. Hence, we have remaining possibility that $\delta(v)=0$, for all $u\in J$. Replacing v by $v\circ r$, $r\in R$ in the above relation to get $\delta(r)[\sigma(v)+\tau(v)]=\{0\}$, for all $v\in J$, $r\in R$. By similar manner in part (i), we can get our result.

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