Minimal and Maximal Feebly Open Sets

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Abstract
In this paper we introduced many new concepts all of these concepts completely depended on the concept of feebly open set. The main concepts which introduced in this paper are minimal f-open and maximal f-open sets. Also new types of topological spaces introduced which called $T_{f\ min}$ and $T_{f\ max}$ spaces. Besides, we present a package of maps called: minimal f-continuous, maximal f-continuous, f-irresolute minimal, f-irresolute maximal, minimal f-irresolute and maximal f-irresolute. Additionally we investigated some fundamental properties of the concepts which presented in this paper.
Keywords: f-open set minimal f-open, maximal f-open, minimal f-continuous, maximal f-continuous, f-irresolute minimal, f-irresolute maximal, minimal f-irresolute and maximal f-irresolute

المجموعات الضئيلة ألاصغرية وألاكبرية

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الخلاصة
في هذا البحث قدمنا عدة مفاهيم جديدة وجميع هذه المفاهيم أعتمدت كليا على مفهوم المجموعة الضئيلة. المفاهيم الرئيسية التي قدمت في هذا البحث هي المجموعة الضئيلة الاصغرية و المجموعة الضئيلة الاكبرية. كذلك تم تقديم نوعين جديدين من الفضاءات التبولوجية سميت فضاءات $T_{f\ min}$ و $T_{f\ max}$. الى جانب ذلك قدمنا حزمة من الدوال أسميناها: الدالة ضئيلة الاستمرار الاصغرية، الدالة ضئيلة الاستمرار الاكبرية، الدالة الضئيلة المتحيرة الاصغرية، الدالة الضئيلة المتحيرة الاكبرية، الدالة الاصغرية الضئيلة المتحيرة و الدالة الاكبرية الضئيلة المتحيرة. بالاضافة الى ذلك بحثنا في بعض الخواص الاساسية للمفاهيم التي طرحت في هذا البحث.

1. Introduction

In [1] and [2] introduced the concepts of minimal open, maximal closed sets and their complement sets. The present paper introduce the concept of a new class of open sets called minimal f-open, maximal f-open, minimal f-closed and maximal f-closed.

1.1 Definition: Let X be a topological space then:

1- A proper nonempty open subset O of X is said to be minimal open set if any open set which is contained in O is ϕ or O. [1]

2- A proper nonempty open subset O of X is said to be maximal open set if any open set which is contains O is O or X. [2]

3- A proper nonempty closed subset F of X is said to be minimal closed set if any closed set which is contained in F is ϕ or F. [3]

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4- A proper nonempty closed subset F of X is said to be maximal closed set if any closed set which is contains F is F or X. [3]

1.2 Definition [4]: A set B in a space X is called semi-open (s.o.) if there exists an open subset O of X such that $O \subseteq S \subseteq \overline{O}$.The complement of a semi-open set is defined to be semi-closed.

1.3 Definition [5]: Let X be a space and $A \subseteq X$. Then the intersection of all semi-closed subsets of X which contains A is called the semi-closure of A and it is denoted by $\overline{A}^S$.

1.4 Definition [6, 9]: A subset A of a space X is called a feebly open (f-open) set if there exists an open subset U of X, such that $U \subseteq A \subseteq \overline{U}$.The complement of a feebly open set is defined to be a feebly closed (f-closed) set. An equivalent definition of being A is f-open is $A \subseteq \overline{A^o}^{o}$.

1.5 Definition [7]: Let X and Y be topological spaces and f:X→Y is a map then f is called an f-continuous function if f^{-1}(A) is an f-open set in X for every open set A in Y.

1.6 Definition [8] : Let X and Y be topological spaces and f:X→Y is a map then f is called f-irresolute if the inverse image of every f-open subset of Y is an f-open set in X.

2. Minimal and Maximal feebly open sets

2.1 Definition: Let X be a topological space then:

1- A proper nonempty f-open subset U of X is said to be a minimal f-open set if any f-open set which is contained in U is ϕ or U.

2- A proper nonempty f-open subset U of X is said to be a maximal f-open set if any f-open set which is contains U is X or U.

3- A proper non-empty f-closed subset F of X is said to be a minimal f-closed set if any f-closed set which is contained in F is ϕ or F.

4- A proper nonempty f-closed subset F of X is said to be a maximal f-closed set if any f-closed set which is contains F is X or F.

2.2 Remarks:

(1) The family of all minimal f-open (resp. minimal f-closed) set of a topological space X is denoted by $M_iFO(X)$ (resp. $M_iFC(X)$).

(2) The family of all maximal f-open (resp. maximal f-closed) set of a topological space X is denoted by $M_aFO(X)$ (resp. $M_aFC(X)$).

2.3 Remark: The concept of minimal f-open, maximal f-open, minimal f-closed and maximal f-closed are independent of each other as in the following example.

2.4 Example: let X={a, b, c} and $\tau=\{\phi, \{a\}, \{a,b\}\}$, so $FO(X)=\{\phi,\{a\},\{a,b\},\{a,c\},X\}$, $M_iFO(X)=\{\{a\}\}$, $M_iFC(X)=\{\{c\},\{b\}\}$, $M_aFO(X)=\{\{a,b\},\{a,c\}\}$, $M_aFC(X)=\{\{b,c\}\}$

Table 1-

	minimal f-open	maximal f-open	minimal f-closed	maximal f-closed
{a}	Yes	No	No	No
{a, b}	No	Yes	No	No
{b}	No	No	Yes	No
{b, c}	No	No	No	Yes

2.5 Theorem: let F be a subset of a topological space X, then F is a minimal f-closed if and only if X-F is maximal f-open set.

Proof: $\Rightarrow$ let F is a minimal f-closed, so X-F is f-open. We have to show that X-F is maximal f-open suppose not, so there is a f-open subset D of X such that $X-F \subset D$ hence $X-D \subset F$ and this contradict being F is minimal f-closed.

$\Leftarrow$ let F be an f-closed subset of X, suppose that there is an f-closed $K \neq \phi$ such that $K \subset F$ thus $X-F \subset X-K$ but X-K is proper f-open set. Contradiction to the assumption of being X-F is maximal f-open.■

2.6 Theorem: Let U and V be maximal f-open subsets of a Topological space X, then $U \cup V = X$ or U=V.

Proof: if $U \cup V = X$ then the proof is complete.

If not, i.e. $U \cup V \neq X$ so we have to show that U=V.

Since $U \cup V \neq X$ so $U \subset U \cup V$ and $V \subset U \cup V$.

But U is maximal f-open set, so $U \cup V = X$ or $U \cup V = U$

Thus $U \cup V = U$ and so $V \subset U$.

Now since $V \subset U \cup V$ and V is maximal f-open set, so $U \cup V = X$ or $U \cup V = V$,but $U \cup V \neq X$ so $U \cup V = V$ and hence $U \subset V$

Therefore U=V.■

2.7 Theorem: Let U be a maximal f-open and V be an f-open subsets of a Topological space X then $\mathbf{U} \bigcup \mathbf{V} = \mathbf{X}$ or $\mathbf{V} \subset \mathbf{U}$.

Proof: If $\mathbf{U} \bigcup \mathbf{V} = \mathbf{X}$ then the proof is complete.

If $\mathbf{U} \bigcup \mathbf{V} \neq \mathbf{X}$ so $\mathbf{U} \subset \mathbf{U} \bigcup \mathbf{V}$ and $\mathbf{V} \subset \mathbf{U} \bigcup \mathbf{V}$.

Since U is maximal f-open and $\mathbf{U} \subset \mathbf{U} \bigcup \mathbf{V}$ so by definition of Maximal f-open we have that $\mathbf{U} \bigcup \mathbf{V} = \mathbf{X}$ or $\mathbf{U} \bigcup \mathbf{V} = \mathbf{U}$ but $\mathbf{U} \bigcup \mathbf{V} \neq \mathbf{X}$ so $\mathbf{U} \bigcup \mathbf{V} = \mathbf{U}$ and hence $\mathbf{V} \subset \mathbf{U}$.■

2.8 Theorem: Let U be a maximal f-open subset of a Topological space X with $\mathbf{x} \in \mathbf{X}/\mathbf{U}$ then $\mathbf{X}/\mathbf{U} \subset \mathbf{V}$ for any f-open subset of X with $\mathbf{x} \in \mathbf{V}$.

Proof: Let $\mathbf{x} \in \mathbf{X}/\mathbf{U}$ and $\mathbf{x} \in \mathbf{V}$, so $\mathbf{V} \not\subset \mathbf{U}$, thus by (2.7) we have that $\mathbf{U} \bigcup \mathbf{V} = \mathbf{X} \Rightarrow (\mathbf{X} \setminus \mathbf{U}) \bigcap (\mathbf{X} \setminus \mathbf{V}) = \phi \Rightarrow \mathbf{X} \setminus \mathbf{U} \subset \mathbf{V}$.■

2.9 Theorem: let F be a minimal f-closed and K be an f-closed subsets of a Topological space X then $\mathbf{F} \bigcap \mathbf{K} = \phi$ or $\mathbf{F} \subset \mathbf{K}$.

Proof: If $\mathbf{F} \bigcap \mathbf{K} = \phi$ then the proof is complete.

If $\mathbf{F} \bigcap \mathbf{K} \neq \phi$ then we have to show that $\mathbf{F} \subset \mathbf{K}$.

Since $\mathbf{F} \bigcap \mathbf{K} \neq \phi$ then $\mathbf{F} \bigcap \mathbf{K} \subset \mathbf{F}$ and $\mathbf{F} \bigcap \mathbf{K} \subset \mathbf{K}$.

But F is minimal f-closed, so we have $\mathbf{F} \bigcap \mathbf{K} = \mathbf{F}$ or $\mathbf{F} \bigcap \mathbf{K} = \phi$.

Thus $\mathbf{F} \bigcap \mathbf{K} = \mathbf{F}$ which means that $\mathbf{F} \subset \mathbf{K}$.■

2.10 Theorem: let F and K be minimal f-closed subsets of a Topological space X then $\mathbf{F} \bigcap \mathbf{K} = \phi$ or $\mathbf{F} = \mathbf{K}$.

Proof: If $\mathbf{F} \bigcap \mathbf{K} = \phi$ then the proof is complete.

If $\mathbf{F} \bigcap \mathbf{K} \neq \phi$ then we have to show that $\mathbf{F} = \mathbf{K}$.

Since $\mathbf{F} \bigcap \mathbf{K} \neq \phi$ so $\mathbf{F} \bigcap \mathbf{K} \subset \mathbf{F}$ or $\mathbf{F} \bigcap \mathbf{K} \subset \mathbf{K}$.

Since F is minimal f-closed so we have $\mathbf{F} \bigcap \mathbf{K} = \mathbf{F}$ or $\mathbf{F} \bigcap \mathbf{K} = \phi$. But $\mathbf{F} \bigcap \mathbf{K} \neq \phi$ hence $\mathbf{F} \bigcap \mathbf{K} = \mathbf{F}$ which means $\mathbf{F} \subset \mathbf{K}$.

Now since K is minimal f-closed so we have $\mathbf{F} \bigcap \mathbf{K} = \mathbf{K}$ or $\mathbf{F} \bigcap \mathbf{K} = \phi$. But $\mathbf{F} \bigcap \mathbf{K} \neq \phi$ hence $\mathbf{F} \bigcap \mathbf{K} = \mathbf{K}$ which means $\mathbf{K} \subset \mathbf{F}$. Therefore F=K. ■

2.11 Theorem: Let U, V and W be maximal f-open subsets of a Topological space X such that $\mathbf{U} \neq \mathbf{V}$, if $\mathbf{U} \bigcap \mathbf{V} \subset \mathbf{W}$, then either U=W or V=W.

Proof: Suppose that $\mathbf{U} \bigcap \mathbf{V} \subset \mathbf{W}$, if U=W then the proof is complete.

If $\mathbf{U} \neq \mathbf{W}$ we have to show that V=W.

$$\mathbf{V} \cap \mathbf{W} = \mathbf{V} \cap (\mathbf{X} \cap \mathbf{W}) \quad \textbf{Set Theory}$$
$$= \mathbf{V} \cap [\mathbf{W} \cap (\mathbf{U} \cup \mathbf{V})] \quad \textbf{by (2.6)}$$

$$= \mathbf{V} \cap [(\mathbf{W} \cap \mathbf{U}) \cup (\mathbf{W} \cap \mathbf{V})] \quad \textbf{Set Theory}$$
$$= (\mathbf{V} \cap \mathbf{W} \cap \mathbf{U}) \cup (\mathbf{V} \cap \mathbf{W} \cap \mathbf{V}) \textbf{Set Theory}$$
$$= (\mathbf{U} \cap \mathbf{V}) \cup (\mathbf{V} \cap \mathbf{W}) \ \textbf{since}\ \mathbf{U} \cap \mathbf{V} \subset \mathbf{W}$$

$$= (\mathbf{U} \cup \mathbf{W}) \cap \mathbf{V} \quad \textbf{Set Theory}$$
$$= \mathbf{X} \cap \mathbf{V} \quad \textbf{since}\ \mathbf{U} \cup \mathbf{W} = \mathbf{X}$$
$$= \mathbf{V}$$

Thus $\mathbf{V} \bigcap \mathbf{W} = \mathbf{V}$ implies $\mathbf{V} \subset \mathbf{W}$ but V is maximal f-open therefore V=W or $\mathbf{V} \bigcup \mathbf{W} = \mathbf{X}$ but $\mathbf{V} \bigcup \mathbf{W} \neq \mathbf{X}$ so V=W. ■

2.12 Theorem: U, V and W be maximal f-open subsets of a Topological space X which are different from each other, then $\mathbf{U} \bigcap \mathbf{V} \not\subset \mathbf{U} \bigcap \mathbf{W}$

Proof:

Let $\mathbf{U} \bigcap \mathbf{V} \subset \mathbf{U} \bigcap \mathbf{W}$

$$\Rightarrow (\mathbf{U} \cap \mathbf{V}) \cup (\mathbf{W} \cap \mathbf{V}) \subset (\mathbf{U} \cap \mathbf{W}) \cup (\mathbf{W} \cap \mathbf{V})$$
$$\Rightarrow (\mathbf{U} \cap \mathbf{W}) \cup \mathbf{V} \subset (\mathbf{U} \cap \mathbf{V}) \cup \mathbf{W}$$
$$\Rightarrow \mathbf{X} \cup \mathbf{V} \subset \mathbf{X} \cup \mathbf{W}$$
$$\Rightarrow \mathbf{V} \subset \mathbf{W}$$

But V is maximal f-open and W is a proper subset of X so V=U, this result contradicts the fact that U, V and W are different from each other. Hence $\mathbf{U} \bigcap \mathbf{V} \not\subset \mathbf{U} \bigcap \mathbf{W}$ ■

2.13 Theorem: Let F be a minimal f-closed subset of a Topological space X , if $\mathbf{x} \in \mathbf{F}$ then $\mathbf{F} \subset \mathbf{K}$ for any f-closed subset K of X containing x.

Proof: Suppose $\mathbf{x} \in \mathbf{K}$ and $\mathbf{F} \not\subset \mathbf{K}$ so $\mathbf{F} \bigcap \mathbf{K} \subset \mathbf{F}$ and $\mathbf{F} \bigcap \mathbf{K} \neq \phi$ since $\mathbf{x} \in \mathbf{F} \bigcap \mathbf{K}$.

But F is minimal f-closed so $\mathbf{F} \bigcap \mathbf{K} = \mathbf{F}$ or $\mathbf{F} \bigcap \mathbf{K} = \phi$.

hence $\mathbf{F} \bigcap \mathbf{K} = \mathbf{F}$ which contract the relation $\mathbf{F} \bigcap \mathbf{K} \subset \mathbf{F}$. Therefore $\mathbf{F} \subset \mathbf{K}$.■

2.14 Theorem: Every minimal f-open subset of a topological space is open.

Proof: Let M be a minimal f-open subset of a topological space X.

Since M is f-open so there is an open subset U of X such that $\mathbf{U} \subset \mathbf{M} \subset \overline{\mathbf{U}}^{\mathbf{s}}$.

But U is open so it is f-open. Since M is minimal so either U=ϕ contradiction or M=U thus $\mathbf{U} = \mathbf{M^o}$ hence M is open. ■

3. $\mathbf{T_{f\,min}}$ and $\mathbf{T_{f\,max}}$ spaces

3.1 Definition: A topological space X is said to be $\mathbf{T_{f\,min}}$ space if every nonempty proper f-open subset of X is minimal f-open set.

3.2 Definition: A topological space X is said to be $\mathbf{T_{f\,max}}$ space if every nonempty proper f-open subset of X is maximal f-open set.

3.3 Example: Let X={a, b, c} and $\tau = \{\phi,\{a,b\},\{c\},X\}$ thus FO(X)=τ, it is clear that {a, b} and {c} are maximal and minimal f-open sets thus the space X is $\mathbf{T_{f\,min}}$ and $\mathbf{T_{f\,max}}$.

3.4 Remark: $\mathbf{T_{f\,min}}$ and $\mathbf{T_{f\,max}}$ spaces are identical.

3.5 Theorem: A space X is $\mathbf{T_{f\,min}}$ if and only if it is $\mathbf{T_{f\,max}}$.

Proof: $\Rightarrow$ Let X is $\mathbf{T_{f\,min}}$ space. Suppose that X is not $\mathbf{T_{f\,max}}$, so there is a proper f-open subset K of X which is not maximal, this mean there exist an f-open subset of X with $\mathbf{K} \subset \mathbf{H} \neq \phi$. Thus we get that H is not minimal which is contradict of being X is $\mathbf{T_{f\,max}}$.

$\Leftarrow$ Let X is $\mathbf{T_{f\,max}}$ space. Suppose that X is not $\mathbf{T_{f\,min}}$, so there is a proper f-open subset K of X which is not minimal, this mean there exist an f-open subset of X with $\phi \neq \mathbf{H} \subset \mathbf{K}$. Thus we get that H is not maximal which is contradict of being X is $\mathbf{T_{f\,max}}$.■

3.6 Theorem: A topological space X is T_{fmin} space if and only if every nonempty proper f-closed subset of X is maximal f-closed set in X.

Proof: $\Longrightarrow$ let F be a proper f-closed subset of X and suppose F is not maximal.

So there exists an f-closed subset K of X with $\mathbf{K} \neq \mathbf{X}$ such that $\mathbf{F} \subset \mathbf{K}$.

Thus $\mathbf{X-K} \subset \mathbf{X-F}$. Hence X-F is a proper f-open which is not minimal and this contradicts of being X is T_{fmin} space.

$\Longleftarrow$ Suppose U is a proper f-open subset of X. thus X-U is a proper f-closed subset of X, so X-U is maximal f-closed subset of X. and by (2.5) U is minimal f-open . thus X is T_{fmin} space. ■

3.7 Theorem: A topological space X is T_{fmax} space if and only if every nonempty proper f-closed subset of X is minimal f-closed set in X.

Proof:

$\Longrightarrow$ let F be a proper f-closed subset of X, suppose F is not minimal f-closed in X, so there is a proper f-closed subset of X such that $\mathbf{K} \subset \mathbf{F}$

Thus $\mathbf{X-F} \subset \mathbf{X-K}$ but X-K is proper f-open in X so X-F is not maximal in X. Contradiction to the fact X-F is maximal f-open.

$\Longleftarrow$ let U be a proper f-open subset of X, then X-U is a proper f-closed subset of X and so it is minimal f-closed set by (2.5) hence we get that U is maximal f-open. ■

3.8 Theorem: Every pair of different minimal f-open sets of $\mathbf{T_{f\,min}}$ are disjoint.

Proof: Let U and V be minimal f-open subsets of $\mathbf{T_{f\,min}}$ space X such that $\mathbf{U} \neq \mathbf{V}$ to show that $\mathbf{U} \cap \mathbf{V} = \phi$ suppose not i.e. $\mathbf{U} \cap \mathbf{V} \neq \phi$.

So $\mathbf{U} \cap \mathbf{V} \subset \mathbf{U}$ **and** $\mathbf{U} \cap \mathbf{V} \subset \mathbf{V}$. Since $\mathbf{U} \cap \mathbf{V} \subset \mathbf{U}$ and U is minimal f-open then $\mathbf{U} \cap \mathbf{V} = \mathbf{U}$ **or** $\mathbf{U} \cap \mathbf{V} = \phi$ thus $\mathbf{U} \cap \mathbf{V} = \mathbf{U}$ … (1).

Now since $\mathbf{U} \cap \mathbf{V} \subset \mathbf{V}$ and V is minimal f-open then $\mathbf{U} \cap \mathbf{V} = \mathbf{V}$ **or** $\mathbf{U} \cap \mathbf{V} = \phi$

thus $\mathbf{U} \cap \mathbf{V} = \mathbf{V}$ … (2)

Hence from (1) and (2) we get that U=V this result contradicts the fact that U and V are different. Therefore $\mathbf{U} \cap \mathbf{V} = \phi$.■

3.9 Theorem: Union of every pair of different maximal f-open sets in $\mathbf{T_{f\,max}}$ space X is X.

Proof: Let U and V be maximal f-open subsets of $\mathbf{T_{f\,max}}$ space X such that $\mathbf{U} \neq \mathbf{V}$ to show that $\mathbf{U} \cup \mathbf{V} = \mathbf{X}$ suppose not i.e. $\mathbf{U} \cup \mathbf{V} \neq \mathbf{X}$.

So $\mathbf{U} \subset \mathbf{U} \cup \mathbf{V}$ **and** $\mathbf{V} \subset \mathbf{U} \cup \mathbf{V}$.

Since $\mathbf{U} \subset \mathbf{U} \cup \mathbf{V}$ and U is maximal f-open then $\mathbf{U} \cup \mathbf{V} = \mathbf{U}$ **or** $\mathbf{U} \cup \mathbf{V} = \mathbf{X}$

thus $\mathbf{U} \cup \mathbf{V} = \mathbf{U}$ … (1).

Now since $\mathbf{V} \subset \mathbf{U} \cup \mathbf{V}$ and V is maximal f-open then $\mathbf{U} \cup \mathbf{V} = \mathbf{V}$ **or** $\mathbf{U} \cap \mathbf{V} = \mathbf{X}$

thus $\mathbf{U} \cup \mathbf{V} = \mathbf{V}$ … (2)

Hence from (1) and (2) we get that U=V this result contradicts the fact that U and V are different. Therefore $\mathbf{U} \cap \mathbf{V} = \mathbf{X}$.■

3.10 Theorem: Let X be a topological space, if X is $\mathbf{T_{f\,max}}$ then every proper f-open is open.

Proof: Let F be a proper f-open subset of X. so there exist an open subset U of X such that $U \subset F \subset \overline{U}^{s}$ but U is f-open subset of X, since X is $T_{f\,max}$ so U is maximal f-open and so $U = F$. Thus $U = F^{o}$ hence F is open. ■

4. Some Maps Via Minimal and Maximal f-open Sets

4.1 Definition: Let X and Y be topological spaces, a map $f: X \to Y$ is called minimal f-continuous if $f^{-1}(U)$ is minimal f-open in X for any open subset U of Y.

4.2 Example: Let X=Y={a, b, c}and $f:(X,\tau) \to (Y,\sigma)$ is the identity map, where τ={ϕ, {a}, {b}, {a, b}, X} and σ={ϕ, {b}, Y} then f is minimal f-continuous since the only proper open subset of Y is {b} and $f^{-1}(\{b\}) = \{b\}$ is minimal f-open in X.

4.3 Definition: Let X and Y be topological spaces, a map $f: X \to Y$ is called maximal f-continuous if $f^{-1}(U)$ is maximal f-open in X for any open subset U of Y.

4.4 Example: Let X=Y={a, b, c}and $f:(X,\tau) \to (Y,\sigma)$ is the identity map, where τ={ϕ, {a}, {c},{a, c}, X} and σ={ϕ, {a, c}, Y} then f is maximal f-continuous since the only proper open subset of Y is {a, c} and $f^{-1}(\{a, c\}) = \{a, c\}$ is maximal f-open in X.

4.5 Theorem: Every minimal f-continuous map is f-continuous.

Proof: Let $f: X \to Y$ be a minimal f-continuous map and U be open subset of Y. then $f^{-1}(U)$ is minimal f-open in X and so $f^{-1}(U)$ is f-open subset of X.■

4.6 Remark: The converse is not true in general as in the following example.

4.7 Example: Let X=Y={a, b, c} and $f:(X,\tau) \to (Y,\sigma)$ is the identity map, where τ={ϕ, {a}, {c}, {a, c}, X} and σ={ϕ, {a, c}, Y} then f is f-continuous but f is not minimal f-continuous since $f^{-1}(\{a,c\}) = \{a,c\}$ is not minimal f-open since $\phi \neq \{a\} \subset \{a,c\}$, .

4.8 Theorem: Let X and Y be topological spaces, if $f: X \to Y$ is an f-continuous onto map and X is $T_{f\,min}$ space then f is minimal f-continuous.

Proof: It is clear that the inverse image of ϕ and Y are f-open subsets of X. So let U be a proper open subset of Y. Since f is f-continuous so $f^{-1}(U)$ is proper f-open subset of X, but X is $T_{f\,min}$ so $f^{-1}(U)$ minimal f-open.■

4.9 Remark: the converse is not true in general as in the following example.

4.10 Example: In (4.2) f is minimal f-continuous but X is not $T_{f\,min}$.

4.11 Theorem: Let X and Y be topological spaces, if $f: X \to Y$ is an f-continuous onto map and X $T_{f\,max}$ space then f is maximal f-continuous.

Proof: It is clear that the inverse image of ϕ and Y are f-open subsets of X. So let U be a proper open subset of Y. Since f is f-continuous so $f^{-1}(U)$ is a proper f-open subset of X but X is $T_{f\,max}$ so $f^{-1}(U)$ is maximal f-open.■

4.12 Remark: the converse is not true in general as in the following example.

4.13 Example: In (4.4) f is maximal f-continuous but X is not $T_{f\,max}$ space.

4.14 Theorem: Every maximal f-continuous map is f-continuous.

Proof: Let $f: X \to Y$ be a maximal f-continuous map and U be open subset of Y. then $f^{-1}(U)$ is maximal f-open in X and so $f^{-1}(U)$ is f-open subset of X.■

4.15 Remark: The Converse is not true in general as in the following example.

4.16 Example : Let X=Y={a, b, c}and $f:(X,\tau) \to (Y,\sigma)$ is the identity map, then where τ={ϕ, {a}, {a, c}, X} and σ={ϕ, {a}, Y} then f is f-continuous but f is not maximal f-continuous since $f^{-1}(\{a\}) = \{a\}$ is not maximal f-open since $\phi \neq \{a,c\} \supset \{a\}$.

4.17 Remark: Minimal f-continuous and maximal f-continuous maps are independent of each other and the following examples show that.

4.18 Example: In (4.4) f is maximal f-continuous since $f^{-1}(\{a,c\}) = \{a,c\}$ is f-open but f is not minimal f-continuous.

4.19 Example: In (4.2) f is minimal f-continuous but it is not maximal f-continuous since $f^{-1}(\{b\}) = \{b\}$ is not maximal f-open in X.

4.20 Definition: Let X and Y be topological spaces, a map $f : X \to Y$ is called f-irresolute minimal if $f^{-1}(U)$ is f-open in X for any minimal f-open subset U of Y.

4.21 Example: Let X=Y={a, b, c}and $f : (X,\tau) \to (Y,\sigma)$ is the identity map and τ={ϕ, {a}, {b}, {a, b}, X}, σ ={ϕ, {a}, {a, b},Y}, then f is f-irresolute minimal since the only minimal f-open set in Y is {a} and $f^{-1}(\{a\}) = \{a\}$ is an f-open in X.

4.22 Definition: Let X and Y be topological spaces, a map $f : X \to Y$ is called f- irresolute maximal if $f^{-1}(U)$ is f-open in X for any maximal f-open subset U of Y.

4.23 Example: Let X=Y={a, b, c}and $f : (X,\tau) \to (Y,\sigma)$ is the identity map and τ={ϕ, {a}, {a, b},X}, σ ={ϕ, {a}, {b}, {a, b}, Y}, then f is f-irresolute maximal since the only maximal f-open set in Y is {a, b} and $f^{-1}(\{a,b\}) = \{a,b\}$ is an f-open in X.

4.24 Theorem: Every f-irresolute map is f-irresolute minimal.

Proof: Let $f : X \to Y$ be an f- irresolute map and U be a minimal f-open subset of Y so U is an f-open. Then $f^{-1}(U)$ is f-open in X and so $f^{-1}(U)$ is f-open subset of X.■

4.25 Remark: The converse is not true in general as in the following example.

4.26 Example: Let X=Y={a, b, c}and $f : (X,\tau) \to (Y,\sigma)$ is the identity map and τ={ϕ, {a}, {b, c}, X}, σ ={ϕ, {a},Y}, then f is f-irresolute minimal since the only minimal f-open set in Y is {a} and $f^{-1}(\{a\}) = \{a\}$ is an f-open in X. But f is not f-irresolute since {a, b} is f-open in Y but $f^{-1}(\{a,b\}) = \{a,b\}$ is not f-open in X.

4.27 Theorem: Every minimal f-irresolute map is f-irresolute minimal.

Proof: Let X and Y be topological spaces and the map $f : X \to Y$ is minimal f-irresolute, to show that f is f-irresolute minimal. Let U be a minimal f-open subset of Y, so U is f-open, thus $f^{-1}(U)$ is minimal f-open subset of X, therefore $f^{-1}(U)$ is f-open. ■

4.28 Remark: The converse is not true in general as in the following example.

4.29 Example: In (4.21) f is f-irresolute minimal but f is not minimal f-irresolute since $f^{-1}(\{a,b\}) = \{a,b\}$ is not minimal f-open in X.

4.30 Theorem: Every maximal f-irresolute map is f-irresolute maximal.

Proof: Let X and Y be topological spaces and the map $f : X \to Y$ is maximal f-irresolute, to show that f is f-irresolute maximal. Let U be a maximal f-open subset of Y, so U is f-open, thus $f^{-1}(U)$ is maximal f-open subset of X, therefore $f^{-1}(U)$ is f-open . ■

4.31 Remark: The converse is not true in general as in the following example.

4.32 Example: In (4.23) f is f-irresolute maximal but f is not maximal f-irresolute since $f^{-1}(\{a\}) = \{a\}$ is not maximal f-open in X.

4.33 Theorem: Every minimal f-irresolute map is minimal f-continuous.

Proof: Let X and Y be topological spaces and the map $f : X \to Y$ is minimal f-irresolute, to show that f is minimal f-continuous. Let U be an open subset of Y, so U is f-open, thus $f^{-1}(U)$ is minimal f-open subset of X. ■

4.34 Remark: The converse is not true in general as in the following example.

4.35 Example: Let X=Y={a, b, c} and $f : (X,\tau) \to (Y,\sigma)$ is the identity map, then where τ={ϕ, {a}, {b, c}, X} σ={ϕ, {a}, Y} then f is minimal f-continuous since the only proper open subset of Y is {a} and $f^{-1}(\{a\}) = \{a\}$ which is minimal f-open subset in X. Now f is not minimal f-irresolute since {a, c} is f-open in Y but $f^{-1}(\{a,c\}) = \{a,c\}$ which is not f-open in X so it is not minimal f-open.

4.36 Theorem: Every maximal f-irresolute map is maximal f-continuous.

Proof: Let X and Y be topological spaces and the map $f : X \to Y$ is maximal f-irresolute, to show that f is maximal f-continuous. Let U be an open subset of Y, so U is f-open, thus $f^{-1}(U)$ is maximal f-open subset of X. ■

4.37 Remark: The converse is not true in general as in the following example.

4.38 Example: Let X=Y={a, b, c} and $\mathbf{f:(X,\tau)\to(Y,\sigma)}$ is the identity map, then where τ={ ϕ, {a}, {b, c}, X} σ={ ϕ, {a}, Y} then f is maximal f-continuous since the only proper open subset of Y is {a} and $\mathbf{f^{-1}(\{a\})=\{a\}}$ which is maximal f-open subset in X. Now f is not maximal f-irresolute since {a, b} is f-open in Y but $\mathbf{f^{-1}(\{a,b\})=\{a,b\}}$ which is not f-open in X so it is not maximal f-open.

4.39 Theorem: Every f-irresolute map is f-irresolute maximal.

Proof: Let X and Y be topological spaces and the map $\mathbf{f:X\to Y}$ is f-irresolute, to show that f is f-irresolute maximal. Let U be a maximal f-open subset of Y, so U is f-open, thus $\mathbf{f^{-1}(U)}$ is f-open subset of X. ■

4.40 Remark: The converse is not true in general as in the following example.

4.41 Example: In (4.23) f is f-irresolute maximal but it is not f-irresolute since {b} f-open in Y but $\mathbf{f^{-1}(\{b\})=\{b\}}$ is not f-open in X.

4.42 Definition: Let X and Y be topological spaces, a map $\mathbf{f:X\to Y}$ is called minimal f-irresolute if $\mathbf{f^{-1}(U)}$ is minimal f-open in X for any f-open subset U of Y.

4.43 Example: Let X=Y={ a, b }and $\mathbf{f:(X,\tau)\to(Y,\sigma)}$ is the identity map and τ={ ϕ, {a}, {b}, X}, σ={ ϕ, {a}, Y} then f is minimal f-irresolute.

4.44 Definition: Let X and Y be topological spaces, a map $\mathbf{f:X\to Y}$ is called maximal f-irresolute if $\mathbf{f^{-1}(U)}$ is maximal f-open in X for any f-open subset U of Y.

4.45 Example: Let X = Y= {a, b} and $\mathbf{f:(X,\tau)\to(Y,\sigma)}$ is the identity map and τ={ ϕ, {a}, {b}, X}, σ ={ ϕ, {b}, Y} then f is minimal f-irresolute.

446 Theorem: Every maximal f-irresolute is f-irresolute minimal.

Proof: Let X and Y be topological spaces and the map $\mathbf{f:X\to Y}$ is maximal f-irresolute, to show that f is f-irresolute minimal. Let U be a minimal f-open subset of Y, U is f-open, thus $\mathbf{f^{-1}(U)}$ is maximal f-open subset of X and so it is f-open. ■

4.47 Remark: The converse is not true in general as in the following example.

4.48 Example: In (4.21)f is f-irresolute minimal but it is not maximal f-irresolute since {a} is an f-open set in Y but $\mathbf{f^{-1}(\{a\})=\{a\}}$ is not maximal f-open in X.

4.49 Theorem: Every minimal f-irresolute is f-irresolute maximal

Proof: Let $\mathbf{f:X\to Y}$ be a minimal f-irresolute map and U be a maximal f-open subset of Y so it is f-open. Then $\mathbf{f^{-1}(U)}$ is minimal f-open in X and so $\mathbf{f^{-1}(U)}$ is f-open subset of X.■

4.50 Remark: The converse is not true in general as in the following example.

4.51 Example: In (4.23) f is f-irresolute maximal but it is not minimal f-irresolute since {a, b} is an f-open set in Y but $\mathbf{f^{-1}(\{a,b\})=\{a,b\}}$ is not minimal f-open in X.

4.52 Remark: The concepts of minimal f-continuous and f-irresolute maximal are independent for each other as in the following examples.

4.53 Example: In (4.2) f is minimal f-continuous but it is not f-irresolute maximal since {b, c} is an f-open in Y but $\mathbf{f^{-1}(\{b,c\})=\{b,c\}}$ is not f-open in X so it is not maximal f-open.

4.54 Example: In (4.7) f is f-irresolute maximal but it is not minimal f-continuous since $\mathbf{f^{-1}(\{a,c\})=\{a,c\}}$ is not minimal f-open.

4.55 Theorem: Every maximal f-continuous is f-irresolute minimal.

Proof: Let X and Y be topological spaces and the map $\mathbf{f:X\to Y}$ maximal f-continuous , to show that f is f-irresolute minimal. Let U be a minimal f-open subset of Y, so by (2.14) U is open, thus $\mathbf{f^{-1}(U)}$ is maximal f-open subset of X hence $\mathbf{f^{-1}(U)}$ is f-open . ■

4.56 Remark: The converse is not true in general as in the following example.

4.57 Example: In (4.2) f is f-irresolute minimal but it is not maximal f-continuous since $\mathbf{f^{-1}(\{b\})=\{b\}}$ is not maximal f-open in X.

4.58 Remark: The concepts of maximal f-continuous and f-irresolute maximal are independent for each other as in the following example;

4.59 Example: Let X=Y={a, b, c}and $\mathbf{f:(X,\tau)\to(Y,\sigma)}$ is the identity map, where τ={ ϕ, {a}, {a, b}, {a, c}, X} and σ={ ϕ, {a}, Y} then f is f-irresolute maximal but f is not maximal f-continuous since $\mathbf{f^{-1}(\{a\})=\{a\}}$ is not maximal f-open since $\phi \neq \{a,c\} \supset \{a\}$.

4.60 Example: in (4.35) f is maximal f-continuous but it is not f-irresolute maximal since {a, b} is maximal f-open in Y but $\mathbf{f^{-1}(\{a,b\})=\{a,b\}}$ is not f-open in X.

4.61 Theorem: Every minimal f-continuous is f-irresolute minimal.

Proof: Let X and Y be topological spaces and the map $\mathbf{f:X \to Y}$ minimal f-continuous, to show that f is f-irresolute minimal. Let U be a minimal f-open subset of Y, so by (2.14) U is open, thus $\mathbf{f^{-1}(U)}$ is minimal f-open subset of X hence $\mathbf{f^{-1}(U)}$ is f-open. ■

4.62 Remark: The converse is not true in general as in the following example.

4.63 Example: In (4.21) f is f-irresolute minimal but it is not minimal f-continuous since $\mathbf{f^{-1}(\{a,b\})=\{a,b\}}$ is not minimal f-open in X.

4.64 Remark: The concepts of f-irresolute minimal and f-irresolute maximal are independent for each other as in the following examples.

4.65 Example: Let X=Y={a, b, c}and $\mathbf{f:(X,\tau) \to (Y,\sigma)}$ is the identity map and τ={ ϕ, {a}, {b, c}, X}, σ ={ ϕ, {a}, {a, b},Y}, then f is f-irresolute minimal since the only minimal f-open set in Y is {a} and $\mathbf{f^{-1}(\{a\})=\{a\}}$ is an f-open in X. But it is not f-irresolute maximal since {a, b} is maximal f-open but $\mathbf{f^{-1}(\{a,b\})=\{a,b\}}$ is not f-open in X.

4.66 Example: Let X=Y={a, b, c}and $\mathbf{f:(X,\tau) \to (Y,\sigma)}$ is the identity map and τ={ ϕ, {a, b},X}, σ ={ ϕ, {a}, {b}, {a, b}, Y}, then f is f-irresolute maximal since the only maximal f-open set in Y is {a, b} and $\mathbf{f^{-1}(\{a,b\})=\{a,b\}}$ is an f-open in X but {a} is minimal f-open in X and $\mathbf{f^{-1}(\{a\})=\{a\}}$ is not f-open in X so f is not f-irresolute minimal.

4.67 Theorem: Every minimal f- irresolute map is f-irresolute.

Proof: Let X and Y be topological spaces and the map $\mathbf{f:X \to Y}$ is minimal f-irresolute, to show that f is f-irresolute. Let U be an f-open subset of Y, thus $\mathbf{f^{-1}(U)}$ is minimal f-open subset of X, therefore $\mathbf{f^{-1}(U)}$ is f-open. ■

4.68 Remark: The converse is not true in general as in the following example.

4.69 Example: Let X=Y={a, b, c}and $\mathbf{f:(X,\tau) \to (Y,\sigma)}$ is the identity map, where τ={ ϕ, {a}, {a, b}, {a, c}, X} and σ={ ϕ, {a}, {a, c}, Y} then f is f- irresolute but it is not minimal f-irresolute since $\mathbf{f^{-1}(\{a,c\})=\{a,c\}}$ is not minimal f-open

4.70 Theorem: Every maximal f- irresolute map is f-irresolute.

Proof: Let X and Y be topological spaces and the map $\mathbf{f:X \to Y}$ is maximal f-irresolute, to show that f is f-irresolute. Let U be an f-open subset of Y, thus $\mathbf{f^{-1}(U)}$ is maximal f-open subset of X, therefore $\mathbf{f^{-1}(U)}$ is f-open . ■

4.71 Remark: The converse is not true in general as in the following example.

4.72 In (4.69) f is an f-irresolute but it is not maximal f- irresolute since $\mathbf{f^{-1}(\{a\})=\{a\}}$ is not maximal f-open in X.

4.73 Remark: The following diagram shows the implications of various relations among concepts which introduced in this paper.

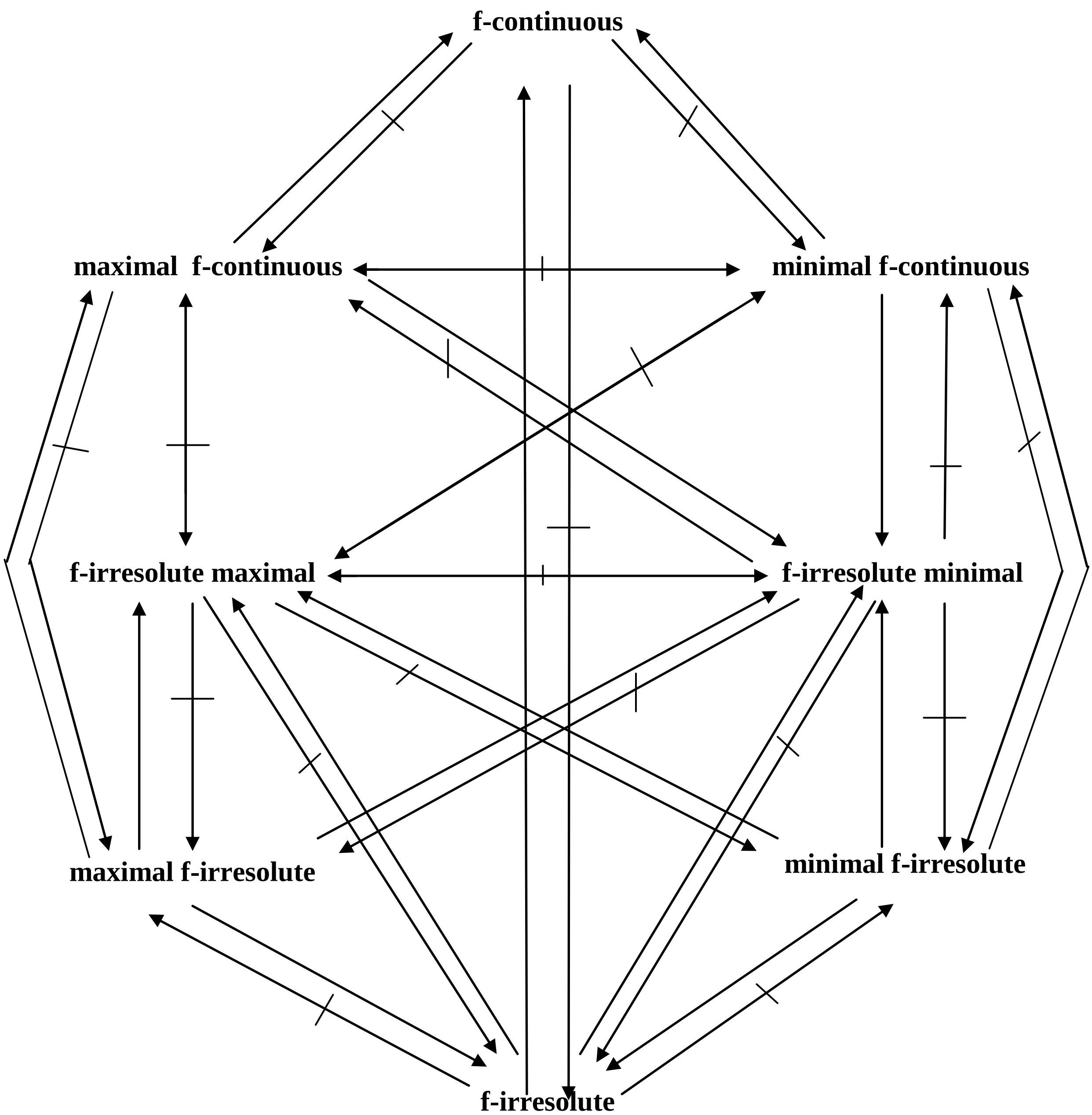

Diagram 1-

5. Conclusion

New types of minimal and maximal sets introduced here defined by using one of generalized open set (feebly open set) and we investigated many important properties of these sets. Also these sets used to define new type of topological spaces. Some maps also defined by term of minimal and maximal feebly open sets. And finally we presented an interesting diagram relevant to the maps which discussed in this paper.

There is a possible problem to study and this can accomplish by use the procedure which adopted in this paper on generalized topology [10]

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