On the Size of Complete Arcs in Projective Space of Order 17

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Abstract
The main goal of this paper is to show that a q-arc in $PG(3,q)$ and $q=17$ is subset of a twisted cubic, that is, a normal rational curve. The maximum size of an arc in a projective space or equivalently the maximum length of a maximum distance separable linear code are classified. It is then shown that this maximum is $q+1$ for all dimensions up to q.
Keyword: Arc, MDS code, normal rational curve, main conjecture.

حول حجم الأقواس التامة في الفضاء ألإسقاطي من الرتبة 17

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الخلاصة:
الهدف الرئيسي من هذا البحث هو توضيح بان القوس في فضاء الإسقاط ثلاثي الإبعاد من الرتبة السابعة عشر هو مجموعة جزئية من المنحني النسبي الطبيعي. أن حجم النهاية العظمى للقوس في فضاء إسقاطي أو مكافئا لطول النهاية العظمى ذات الشفرة الخطية مفصولة المسافة تم تصنيفها. كما تم توضيح بان هذه النهاية العظمى هي $q+1$ لكل الإبعاد حتى q.

1. Introduction

The subject of this paper is suggested by Prof. J.W.P Hirschfeld in 2008. The main conjecture MC_k for codes, always taking $q>k$, is the following:

$$m(k-1,q)=\begin{cases} q+2, & q \text{ even for } k=q-1 \\ q+1, & \text{other wise} \end{cases} \qquad (1)$$

such that the value $q+2$ for $k=q-1$ both with q even, and $q+1$, in all other cases. Also

$$m(k,q)=k+1, \text{ for } q \le k \qquad (2)$$

Establish this for 6-dimensional code and small values of q. In projective space of $k-1$ dimensions over the finite field of $q=p^h$ elements, p prime $\mathbb{F}_q$, the following three notions are equivalent for $n \ge k$:

1) An n-arc in $PG(k-1,q)$, that is, a set of n points with at most $k-1$ in any hyperplane;
2) A set of n vectors in $V(k,q)$ with any k linearly independent;
3) A maximum distance separable linear code of length n, dimension k, and hence minimum distance $d=n-k+1$, that is, an $[n,k,n-k+1]$ code.

For more details see [1-6].

2. Previous Results

Definition (1)[3]: For any given q, the set $\mathbb{F}_q$ satisfies the following properties:

i. The set $\mathbb{F}_q$, where $q=p^h$, is a *field of characteristic* p;

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ii. The elements x of $\mathbb{F}_q$ satisfy $x^q - x = 0$;

iii. There exists δ in $\mathbb{F}_q$ such that $\delta^{q-1} = 1$ and $\mathbb{F}_q = \{0,1,\delta,\dots\delta^{q-2}\}$; such an δ is called a *primitive* .

Definition (2)[3]: Let $V = V(n+1,K)$ be $(n+1)$-dimensional vector space over the field K with origin O. Then consider the equivalence relation on the points of $V\backslash\{O\}$ whose equivalence classes are one-dimensional subspaces of V with the origin deleted; that is, if $X,Y \in V\backslash\{O\}$ and for some basis $X = (x_0,\dots,x_n), Y = (y_0,\dots,y_n)$, X is equivalent to Y if, for some t in K_0, $y_i = tx_i$. For all i.
Then the set of equivalence classes is the n-dimensional projective space over K and is denoted by $PG(n,K)$ or; if $K = GF(q)$, by $PG(n,q)$. The elements of $PG(n,q)$ are called points.

Definition (3)[3]: A subspace of dimension r of $PG(n,q)$ will be called an *r-space* and is denoted by π_r. It is a set of points represented by vectors which form (with the origin) a subspace of $V(n+1,q)$ of dimension $r+1$. When $r = 0$ then π_0 is exactly a point of $PG(n,q)$. When $r = 1$, π_1 is called a *line* and, when $r = 2$, π_2 is called a *plane*. π_{n-1} is called a *prime* or a *hyperplane*.

Definition (4)[3]:

i. A *linear code* C is a subspace of $V(n,q)$.

ii. If dim $(C) = k$, then C is an $[n,k]$-code.

Theorem (5)[3]: The space $PG(r-1,q)$ contains

i. $(q^r-1)/(q-1)$ points,

ii. $\frac{(q^r-1)(q^{r-1}-1)}{(q^2-1)(q-1)}$ lines,

iii. $q+1$ points on a line,

iv. $(q^{r-1}-1)/(q-1)$ lines through a point.

Definition (6)[3]: An n-arc is *complete* if it is maximal with respect to inclusion; that is, it is not contained in an $(n+1)$-arc.

Definition (7)[4]:A *normal rational curve* in $PG(r,q)$ is any subset of $PG(r,q)$ which is projectively equivalent to $\{(t^r,t^{r-1},\cdots,t,1) \in PG(r,q) | t \in \mathbb{F}_q \cup \{\infty\}\}$.
For $r = 2$, it is a *conic*; for $r = 3$, it is a *twisted cubic.*

Definition (8)[4]: Let $m(r,q)$ be the maximum size of an arc in $PG(r,q)$; also, let $m'(r,q)$ denote the size of the second largest complete arc in $PG(r,q)$. Then an n-arc with $n > m'(r,q)$ is contained in an $m(r,q)$-arc.

Theorem (9)[5]:

i. The dual code of a linear MDS code is also MDS.

ii. An n-arc exists in $PG(k-1,q)$ if and only if an n-arc exists in $PG(n-k-1,q)$.

Corollary (10)[4]:

i. A $(q+1)$-arc exists in $PG(k-1,q)$ if and only if a $(q+1)$-arc exists in $PG(q-k,q)$.

ii. A $(q+2)$-arc exists in $PG(k-1,q)$ if and only if a $(q+2)$-arc exists in $PG(q-k+1,q)$; hence if $m(k-1,q) = q+1$ so also $m(q-k+1,q) = q+1$.

iii. A $(q+3)$-arc exists in $PG(k-1,q)$ if and only if a $(q+3)$-arc exists in $PG(q-k+2,q)$.

Theorem (11)[3]: In $PG(2,q)$, q odd, a q-arc lies on a conic.

Theorem (12)[4]: In $PG(3,q)$, q odd, a $(q+1)$-arc is a twisted cubic.

Theorem (13)[4]: Let K be a k-arc in $PG(r,q)$ with $q+1 \geq k \geq r+3 \geq 6$ and suppose there exist $P_0, P_1 \in K$ and a hyperplane π containing neither P_0 nor P_1 such that, for $i = 0,1$, each projection K_i of K into π is rational in π. Then the arc K is contained in one and only one normal rational curve in $PG(r,q)$.

Theorem (14)[4]: Let K be a $(q+2)$-arc in $PG(r,q)$ with $q+1 \geq r+3 \geq 6$. If a hyperplane π of $PG(r,q)$ contains neither of the points P_0, P_1 of K, then it cannot happen that both projections K_i of K from P_i, $i = 0,1$, onto π are rational in π. In particular, if every $(q+1)$-arc in $PG(r-1,q)$ is rational, then $m(r,q) = q+1$.

Rationality of $(q-3)$-arcs in $PG(3,q)$ for $q=17$

Throughout this section let $q=17$ and $\delta=3$, a primitive element of $\mathbb{F}_{17}$. Arcs in $PG(2,17)$ have been classified [1]. In particular, there exists a complete 14-arc, unique up to projectivity. A construction will now be given of such a complete 14-arc. The following formulas are useful for addition in $\mathbb{F}_{17}$:

$$1-\delta=\delta^6,\quad 1-\delta^2=\delta^2,\quad 1-\delta^3=\delta^{10},$$
$$1-\delta^4=\delta^5,\quad 1-\delta^7=\delta^{11},\quad 1-\delta^8=\delta^{14},$$
$$1-\delta^9=\delta^{12},\quad 1-\delta^{13}=\delta^{15}.$$

Let

$$\begin{array}{ll} P_1=(1,0,0), & P_2=(0,1,0), \\ P_3=(0,0,1), & P_4=(1,1,1), \\ P_5=(\delta,\delta^{10},1), & P_6=\delta^{12},\delta^{15},1), \\ P_6=\delta^{12},\delta^{15},1), & P_7=(\delta^{10},\delta^{12},1), \\ P_8=(\delta^2,\delta^7,1), & P_9=(\delta^3,\delta^4,1), \\ P_{10}=(\delta^7,\delta^{11},1), & P_{11}=(\delta^{13},\delta^8,1), \\ P_{12}=(\delta^9,\delta^6,1), & P_{13}=(\delta^6,\delta^5,1), \\ P_{14}=(\delta^8,\delta^{14},1) & \end{array} \tag{3}$$

Lemma(1)[1]: Let P_j be as in (3). Then $K=\{P_j|1\le j\le q-3\}$ is a complete 14-arc in $PG(2,17)$.

The stabilizer group of K is denoted by $G(K)$ is generated by g_1, g_2 where

$$g_1=\begin{pmatrix} 0 & \delta^{15} & 0 \\ \delta & 0 & 0 \\ \delta^{14} & \delta^{13} & \delta^8 \end{pmatrix},$$
$$g_2=\begin{pmatrix} \delta^7 & 1 & \delta^6 \\ \delta^{13} & \delta^{13} & \delta^{13} \\ 0 & 0 & \delta^{14} \end{pmatrix}.$$

Then $G(K)$ has the following orbits on K: one orbit $O_4=\{P_1,P_2,P_4,P_5,P_7,P_8,P_{11},P_{14}\}$, one orbit $O_5=\{P_6,P_9,P_{10},P_{12}\}$ and one orbit $O_1=\{P_3,P_{13}\}$. The group $G(K)$ stabilizes a line $\ell=v(x-3y)$ containing O_1 on a conic $\boldsymbol{C}=v(xy-7xz+6yz)$, and partitions the line ℓ into three orbits are as following:

$$\{(\delta^4,\delta^3,1),(\delta^3,\delta^2,1),(\delta^{14},\delta^{13},1),(\delta^5,\delta^4,1)\}$$

$$\{(\delta^{11},\delta^{10},1),(\delta^{10},\delta^9,1),(\delta^9,\delta^8,1),(1,\delta^{15},1)\}$$

$$\{(\delta^2,\delta,1),(\delta^{15},\delta^{14},1),(\delta^7,\delta^6,1),(\delta^{12},\delta^{11},1)\}$$

Also three orbits $O_1=\{P_3,P_{13}\}$, $O_2=\{(\delta,1,1),\ (\delta^{13},\delta^{12},1)\}$, $O_3=\{(\delta,1,0),\ (\delta^8,\delta^7,1)\}$. Then K consists of ten points $P_1,P_2,P_3,P_4,P_5,P_7,P_8,P_{11},P_{13},P_{14}$ on a conic $\boldsymbol{C}$, two of them P_2, P_{13} on ℓ, and eight points in O_4 on $\boldsymbol{C}$. The points in O_5 not on $\boldsymbol{C}$. The points in O_2 and O_3 on ℓ, where

$$(\delta^{13},\delta^{12},1)=P_6P_9\cap\ell=P_{10}P_{12}\cap\ell$$

$$(\delta,1,1)=P_6P_{10}\cap\ell=P_9P_{12}\cap\ell$$

$$(\delta,1,0)=P_6P_{12}\cap\ell,\ (\delta^8,\delta^7,1)=P_9P_{10}\cap\ell.$$

The tangents at P_3 and P_{13} to $\boldsymbol{C}$ meet at $\boldsymbol{R}$. The lines

$$P_1R,\ P_2R, P_3R, P_4R, P_5R, P_7R, P_8R, P_{11}R, P_{13}R, P_{14}R;$$

Are part of a pencil. However $O_4{}'=\boldsymbol{C}-\{O_2\cup O_4\}$ is inequivalent to O_4. The other eight lines of the pencil meet $\boldsymbol{C}$ in O_4.

Theorem (2): Let G be the projective automorphism group of the complete $(q-3)$-arc K in Lemma 1. Then

i. G acts transitively on K;

ii. The stabilizer G_3 of P_3 in G acts 3-transitively and faithfully on the set of five unisecants of K through P_3;

iii. $|G_3|=4$ and $|G|=8$.

Proof: Let $(t_i,1,0)$ be the projection of P_i from P_3 to the line $v(z)$ for $i\in\{1,2,\dots,q-3\}-\{3\}$. Then t_i takes the respective values $\infty,0,1,\delta^7,\delta^{13},\delta^{14},\delta^{11},\delta^{15},\delta^{12},\delta^5,\delta^3,\delta,\delta^{10}$. Therefore, the unisecant ℓ_i of K through P_3 takes the form
$v(X+3Y), v(X+2Y), v(X+4Y), v(X+8Y), v(X+Y)$.

We identify a line ℓ through P_3 with $x\in\mathbb{F}_q\cup\{\infty\}$ satisfying $\ell\cap v(Z)=(x,1,0)$. Now, a projectivity $g\in G_3$ induced by a 3×3 matrix $[a_{ij}]$ maps a line x to

$$(a_{11}x+a_{21})/(a_{12}x+a_{22}).$$

Let $\sigma\in\boldsymbol{Z}_4$. We shall show that there exists a unique projectivity $g\in G_3$ such that $g(\ell_i)=\ell_{\sigma(i)}$.
Suppose a matrix $A=[a_{ij}]$ induces such a projectivity g. Since the linear fractional transformation

$$(a_{11}x + a_{21})/(a_{12}x + a_{22})$$

Maps $\delta^2, \delta^4, \delta^6, \delta^8, \delta^9$ to $\delta^6, \delta^4, \delta^8, \delta^9, \delta^2$ respectively, A takes the form

$$A = \begin{pmatrix} \delta^7 & 1 & \delta^6 \\ \delta^{13} & \delta^{13} & \delta^{13} \\ \alpha & \beta & \gamma \end{pmatrix}.$$

Since $g(K) = K$, so $gP_1 = P_5$ and $gP_2 = P_4$; hence $\alpha = \beta = 0$. Similarly, $gP_4 = P_8$ yields $\gamma = \delta^{14}$. Note that $A^4 = I_3$. It can be verified that g sends $P_1, \dots, P_{14}$ to $P_5, P_4, P_3, P_8, P_{14}, P_9, P_2, P_7, P_{12}, P_6, P_1, P_{10}, P_{13}, P_{11}$, respectively.

Let ϑ be a group homomorphism from G_3 to $\boldsymbol{Z_4}$ such that $g(\ell_i) = \ell_{\vartheta(g)(i)}$. Since g generate $\boldsymbol{Z_4}$, so ϑ is surjective. Finally $|\vartheta^{-1}(\sigma)| = 1$, which implies that ϑ is injective. ■

Lemma(3)[1]: An 13-arc in $PG(2,17)$ is projectively isomorphic to either $\{(t^2, t, 1) | t \in \mathbb{F}_{17} - \{0,1\}\}$ or $K - \{P_{14}\}$.

Theorem (4): Let Q_i, $1 \le i \le 14$, be points in $PG(2,17)$ with the following coordinates:

$Q_1 = (1,0,a_1)$, $Q_2 = (0,1,a_2)$,
$Q_3 = (0,0,1)$, $Q_4 = (1,1,a_4)$,
$Q_5 = (\delta, \delta^{10}, a_5)$, $Q_6 = (\delta^{12}, \delta^{15}, a_6)$,
$Q_7 = (\delta^{10}, \delta^{12}, a_7)$, $Q_8 = (\delta^2, \delta^7, a_8)$,
$Q_9 = (\delta^3, \delta^4, a_9)$, $Q_{10} = (\delta^7, \delta^{11}, a_{10})$,
$Q_{11} = (\delta^{13}, \delta^8, a_{11})$, $Q_{12} = (\delta^9, \delta^6, a_{12})$,
$Q_{13} = (\delta^6, \delta^5, a_{13})$, $Q_{14} = (x, 1, a_{14})$.

Then

$$K' = \{Q_i | 1 \le i \le 13\}$$

Is an 13-arc isomorphic to $K - \{P_{14}\}$ under a projectivity g with $Q_{14} = gP_{14}$ if and only if there exist constants $\alpha, \beta, \gamma \in \mathbb{F}_{17}$ with $\gamma \neq 0$ such that one of the following six conditions holds:

1. $a_i = 1, 1 \le i \le 14$, and $x = \delta$.
2. $a_1 = (\alpha + \beta + \gamma)\,\delta^2$,
$a_2 = \alpha\delta^{11} + \beta\delta^2 + \gamma$,
$x = \delta^2$,
$a_4 = \alpha\delta^6 + \beta + \gamma$,
$a_5 = \alpha\delta^{13} + \beta\delta^9 + \gamma\delta^3$,
$a_6 = \alpha\delta^{15} + \beta\delta^5 + \gamma\delta^{15}$,
$a_7 = \alpha\delta^{11} + \beta\delta^{15} + \gamma\delta^{12}$,
$a_8 = \alpha\delta^4 + \beta\delta^3 + \gamma\delta^7$,
$a_9 = \alpha + \beta\delta + \gamma\delta^4$,
$a_{10} = \alpha + \beta\delta^2 + \gamma\delta^{11}$,
$a_{11} = \alpha\delta^{12} + \beta\delta^9 + \gamma\delta^8$,
$a_{12} = \alpha\delta^4 + \beta\delta^{15} + \gamma\delta^6$,
$a_{13} = \alpha\delta^{14} + \beta\delta^3 + \gamma\delta^5$,
$a_{14} = \alpha\delta^{15} + \beta\delta^2 + \gamma\delta^{14}$.
3. $a_1 = \alpha + \beta\delta^5 + \gamma\,\delta^2$,
$a_2 = \beta\delta$,
$x = \delta^4$,
$a_4 = \alpha + \beta\delta^{12} + \gamma\delta^{13}$,
$a_5 = \alpha\delta + \beta\delta^{11} + \gamma\delta^3$,
$a_6 = \alpha\delta^{12} + \beta\delta^9 + \gamma\delta^8$,
$a_7 = \alpha\delta^{10} + \beta\delta^{14} + \gamma\delta^{11}$,
$a_8 = \alpha\delta^2$,
$a_9 = \alpha\delta^3 + \beta\delta + \gamma\delta^{12}$,
$a_{10} = \alpha\delta^7 + \beta\delta^9 + \gamma\delta^2$,
$a_{11} = \alpha\delta^{13} + \beta\delta^3 + \gamma\delta^{13}$,
$a_{12} = \alpha\delta^9 + \beta\delta^{12} + \gamma\delta^8$,
$a_{13} = \alpha\delta^6 + \beta\delta^{14} + \gamma$,
$a_{14} = \alpha\delta^8 + \beta\delta^3 + \gamma\delta^{10}$.
4. $a_1 = \beta$,
$a_2 = \alpha\delta^6$,
$x = \delta^6$,
$a_4 = \alpha\delta^6 + \beta + \gamma$,
$a_5 = \alpha + \beta\delta + \gamma\delta^4$,
$a_6 = \alpha\delta^5 + \beta\delta^{12} + \gamma\delta^{10}$,
$a_7 = \alpha\delta^2 + \beta\delta^{10} + \gamma$,
$a_8 = \alpha\delta^{13} + \beta\delta^2 + \gamma\delta^4$,
$a_9 = \alpha\delta^{10} + \beta\delta^3 + \gamma$,
$a_{10} = \alpha\delta + \beta\delta^7 + \gamma\delta$,
$a_{11} = \alpha\delta^{14} + \beta\delta^{13} + \gamma\delta$,
$a_{12} = \alpha\delta^{12} + \beta\delta^9 + \gamma$,
$a_{13} = \alpha\delta^{11} + \beta\delta^6 + \gamma\delta^{13}$,
$a_{14} = \alpha\delta^4 + \beta\delta^8 + \gamma\delta^5$.
5. $a_1 = \alpha\delta^3 + \beta\delta^5 + \gamma\,\delta^{14}$,
$a_2 = \alpha\delta^3 + \beta\delta^6 + \gamma\delta^2$,
$x = \delta^8$,
$a_4 = (\alpha + \beta + \gamma)\,\delta$,
$a_5 = \alpha\delta^{10} + \beta\delta^8 + \gamma\delta^3$,
$a_6 = \alpha\delta^2 + \beta\delta^{10} + \gamma\delta^7$,
$a_7 = \alpha + \beta\delta^6 + \gamma$,
$a_8 = \alpha\delta^4 + \beta\delta^{15} + \gamma\delta^6$,
$a_9 = \alpha\delta^2 + \beta\delta^{11} + \gamma$,
$a_{10} = \alpha\delta^3 + \beta\delta^{11} + \gamma$,
$a_{11} = \alpha\delta^{10} + \beta\delta^7 + \gamma\delta^6$,
$a_{12} = \alpha + \beta\delta^{15} + \gamma\delta^3$,
$a_{13} = \alpha\delta^4 + \beta\delta^9 + \gamma\delta^{11}$,
$a_{14} = \alpha\delta^3 + \beta\delta^{10} + \gamma$.

6. $a_1 = \alpha\delta^{11} + \beta + \gamma\,\delta^2$,
$a_2 = \alpha\delta^{14}$,
$x = \delta^9$,
$a_4 = \alpha\delta^2 + \beta + \gamma\delta^{11}$,
$a_5 = (\alpha + \beta + \gamma)\,\delta$,
$a_6 = \alpha\delta^{15} + \beta\delta^{12} + \gamma\delta^{11}$,
$a_7 = \alpha\delta^4 + \beta\delta^{10} + \gamma\delta^4$,
$a_8 = \beta\delta^2$,
$a_9 = \alpha\delta^7 + \beta\delta^3 + \gamma\delta^4$,
$a_{10} = \alpha\delta^{15} + \beta\delta^7 + \gamma$,
$a_{11} = \alpha\delta^9 + \beta\delta^{13} + \gamma\delta^{10}$,
$a_{12} = \alpha\delta^2 + \beta\delta^9 + \gamma\delta^7$,
$a_{13} = \alpha\delta^4 + \beta\delta^6 + \gamma\delta^{15}$,

$a_{14} = \alpha\delta^9 + \beta\delta^8 + \gamma\delta^{12}$.

Proof: Suppose K' is an 13-arc. Then there exists a point $Q_{14} = (x, 1, a_{14})$ such that $K' \cup \{Q_{14}\}$ is isomorphic to the 14-arc K in Theorem (2). We identify a line ℓ through $P_3 = Q_3$ with $t \in \mathbb{F}_{17} \cup \{\infty\}$ such that

$$\ell \cap v(Z) = (t, 1, 0).$$

Now the lines P_3P_i and Q_3Q_i coincide and they are

$$\infty, 0, 1, \delta^7, \delta^{13}, \delta^{14}, \delta^{11}, \delta^{15}, \delta^{12}, \delta^5, \delta^3, \delta, \delta^{10}$$

For $i = ,2,4,5, \dots, 13$, respectively. Therefore, the set of five unisecants of K through P_3 is

$$\{\delta^2, \delta^4, \delta^6, \delta^8, \delta^9\},$$

While the set of five unisecants of $K' \cup \{Q_{14}\}$ through Q_3 is

$$\{\delta^2, \delta^4, \delta^6, \delta^8, \delta^9\}, \{\delta^8, \delta, \delta^6, \delta^2, \delta^9\}, \{\delta, \delta^8, \delta^9, \delta^4, \delta^6\},$$

$$\{\delta^8, \delta^6, \delta^4, \delta^2, \delta\}, \{\delta^2, \delta^9, \delta^4, \delta^8, \delta\}, \{\delta^9, \delta^2, \delta, \delta^6, \delta^4\}$$

According as x is $\delta, \delta^2, \delta^4, \delta^6, \delta^8, \delta^9$.

1. Let $x = \delta$. By Theorem (2) there exists a unique projectivity g from K to $K' \cup \{Q_{14}\}$ fixing P_3 and sending each unisecant to itself. A 3×3-matrix $A = [a_{ij}]$ inducing g takes the form

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & \gamma \end{pmatrix}.$$

Since $gK = K' \cup \{Q_{14}\}$, it follows that $\alpha = \beta = 0$ and $\gamma = 1$. Thus condition (1) holds.

2. Let $x = \delta^2$ and let g be a projectivity from K to $K' \cup \{Q_{14}\}$ fixing P_3 and sending the unisecants $\delta^2, \delta^4, \delta^6, \delta^8, \delta^9$ of K through P_3 to $\delta^8, \delta, \delta^6, \delta^2, \delta^9$, respectively. Then a matrix A inducing g takes the form

$$A = \begin{pmatrix} 1 & \delta & 0 \\ \delta^8 & \delta^8 & 0 \\ \alpha & \beta & \gamma \end{pmatrix}.$$

This g maps the line P_3P_i to

$$\delta^{15}, 1, 0, \delta^{13}, \delta^{11}, \delta^{11}, \delta^{14}, \infty, \delta^3, \delta^5, \delta^4, \delta^{12}$$

For $i = 1,2,4,5, \dots, 14$, respectively. Now it can be verified that condition (2) holds.

3. Let $x = \delta^4$ and let g be a projectivity from K to $K' \cup \{Q_{14}\}$ fixing P_3 and sending the five unisecants $\delta^2, \delta^4, \delta^6, \delta^8, \delta^9$ of K through P_3 to $\delta, \delta^8, \delta^9, \delta^4, \delta^6$, respectively. A matrix A inducing g is of the form

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \delta^5 & \delta^8 & 0 \\ \alpha & \beta & \gamma \end{pmatrix}.$$

Now it can be seen that condition (3) holds.

4. Let $x = \delta^6$ and let g be a projectivity from K to $K' \cup \{Q_{14}\}$ fixing P_3 and sending the five unisecants $\delta^2, \delta^4, \delta^6, \delta^8, \delta^9$ of K through P_3 to $\delta^8, \delta^6, \delta^4, \delta^2, \delta$, respectively. A matrix A inducing g is of the form

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \delta^{10} & 0 & 0 \\ \alpha & \beta & \gamma \end{pmatrix}.$$

Now it can be seen that condition (4) holds.

5. Let $x = \delta^8$ and let g be a projectivity from K to $K' \cup \{Q_{14}\}$ fixing P_3 and sending the five unisecants $\delta^2, \delta^4, \delta^6, \delta^8, \delta^9$ of K through P_3 to $\delta^2, \delta^9, \delta^4, \delta^8, \delta$, respectively. A matrix A inducing g is of the form

$$A = \begin{pmatrix} 1 & \delta^5 & 0 \\ \delta^7 & \delta^{13} & 0 \\ \alpha & \beta & \gamma \end{pmatrix}.$$

Now it can be seen that condition (5) holds.

6. Let $x=\delta^9$and let g be a projectivity from K to $K'\cup\{Q_{14}\}$ fixing P_3 and sending the five unisecants $\delta^2,\delta^4,\delta^6,\delta^8,\delta^9$ of K through P_3 to $\delta^9,\delta^2,\delta,\delta^6,\delta^4$, respectively. A matrix A inducing g is of the form

$$A=\begin{pmatrix}0&1&0\\ \delta^2&\delta^{13}&0\\ \alpha&\beta&\gamma\end{pmatrix}.$$

As in the preceding cases we can see that condition (6) holds. Conversely, if one of the conditions (1)-(6) holds, the corresponding matrix A induces a projectivity g such that $g(K)=K'\cup\{Q_{14}\}$. Hence, K' is 13-arc. ■

Theorem (5): Let $q=17$.
1. A k-arc in $PG(3,q)$ with $k=q-3,q+1$ is rational.
2. $m(4,q)=m(8,q)=m(9,q)=q+1$.

Proof: Assume that there exists a nonrational 14-arc

$$L=\{Q_i|1\le i\le q-3\}$$

In $PG(3,17)$. Among the fourteen projections L from one of its points to a plane, at most one of these plane 13-arc can be rational, by Theorem (13). So assume that the projections from Q_1,Q_3,Q_4 are not rational. This assumption leads us to a contradiction.

The projection L_j of L from Q_j to a hyperplane π_j not through Q_j is isomorphic to the 13-arc $K-\{P_{14}\}$ in Lemma (3) for $j=1,3,4$. Without loss of generality we further assume that Q_j has the following coordinates

$Q_1=(1,0,0,b_1)$, $Q_2=(0,1,0,b_2)$,
$Q_3=(0,0,1,b_3)$, $Q_4=(0,0,0,1)$,
$Q_5=(1,1,1,b_5)$, $Q_6=(\delta,\delta^{10},1,b_6)$,
$Q_7=(\delta^{12},\delta^{15},1,b_7)$, $Q_8=(\delta^{10},\delta^{12},1,b_8)$,
$Q_9=(\delta^2,\delta^7,1,b_9)$, $Q_{10}=\delta^3,\delta^4,1,b_{10})$,
$Q_{11}=(\delta^7,\delta^{11},1,b_{11})$, $Q_{12}=(\delta^{13},\delta^8,1,b_{12})$,
$Q_{13}=(\delta^9,\delta^6,1,b_{13})$, $Q_{14}=(\delta^6,\delta^5,1,a_{14})$.

Thus the projection L_4 of L from Q_4 to the hyperplane $v(X_4)$, a point $(x,y,z,0)$ of which is identified with $(x,y,z)\in PG(2,17)$, consists of the following points:

$R_1=(1,0,b_1)$,
$R_2=(0,1,b_2)$,
$R_3=(0,0,1)$,
$R_4=(1,1,b_5-b_3)$,
$R_5=(\delta,\delta^{10},b_6-b_3)$,
$R_6=(\delta^{12},\delta^{15},b_7-b_3)$,
$R_7=(\delta^{10},\delta^{12},b_8-b_3)$,
$R_8=(\delta^2,\delta^7,b_9-b_3)$,
$R_9=(\delta^3,\delta^4,b_{10}-b_3)$,
$R_{10}=(\delta^7,\delta^{11},b_{11}-b_3)$,
$R_{11}=(\delta^{13},\delta^8,b_{12}-b_3)$,
$R_{12}=(\delta^9,\delta^6,b_{13}-b_3)$,
$R_{13}=(\delta^6,\delta^5,b_{14}-b_3)$.

The projection L_1 of L from Q_1 onto the plane $v(X_1)$, a point $(0,x,y,z)$ of which is identified with $(x,y,z)\in PG(2,17)$, consists of the following points:

$S_1=(1,0,b_2)$,
$S_2=(0,1,b_3)$,
$S_3=(0,0,1)$,
$S_4=(1,1,b_5-b_1)$,
$S_5=(\delta^{10},1,b_6-b_1\delta)$,
$S_6=(\delta^{15},1,b_7-b_1\delta^{12})$,
$S_7=(\delta^{12},1,b_8-b_1\delta^{10})$,
$S_8=(\delta^7,1,b_9-b_1\delta^2)$,
$S_9=(\delta^4,1,b_{10}-b_1\delta^3)$,
$S_{10}=(\delta^{11},1,b_{11}-b_1\delta^7)$,
$S_{11}=(\delta^8,1,b_{12}-b_1\delta^{13})$,
$S_{12}=(\delta^6,1,b_{13}-b_1\delta^9)$,
$S_{13}=(\delta^5,1,b_{14}-b_1\delta^6)$.

Then L_1 is precisely the 13-arc $\{Q_i|1\le i\le 13\}$ of Theorem (4). Reorder the S_i as follows:

$T_1=(1,0,b_2)$, $T_2=(0,1,b_3)$,
$T_3=(0,0,1)$, $T_4=(1,1,b_5-b_1)$,
$T_5=(\delta,\delta^{10},b_6\delta^{10}-b_1)$,
$T_6=(\delta^{12},\delta^{15},b_7\delta^{15}-b_1)$,
$T_7=(\delta^{10},\delta^{12},b_8\delta^{12}-b_1)$,
$T_8=(\delta^2,\delta^7,b_9\delta^7-b_1)$,
$T_9=(\delta^3,\delta^4,b_{10}\delta^4-b_1)$,
$T_{10}=(\delta^7,\delta^{11},b_{11}\delta^{11}-b_1)$,
$T_{11}=(\delta^{13},\delta^8,b_{12}\delta^8-b_1)$,

$T_{12} = (\delta^9, \delta^6, b_{13}\delta^6 - b_1)$,
$T_{13} = \delta^6, \delta^5, b_{14}\delta^5 - b_1)$.

First assume that case(1) of Theorem(4) holds for the arc $L_3 = \{R_i | 1 \leq i \leq 13\}$. Then

$$b_5 = b_6 = b_7 = b_8 = b_3 + 1 \text{ and}$$
$$Q_5 = Q_6 = Q_7 = Q_8$$

Lie in the plane $v((b_3 + 1)X_3 + X_4)$, a contradiction. Similarly, case(1) does not hold for the arc L_1.

Suppose now that case(2) of Theorem(4) holds for L_3. Then there are constants $\alpha, \beta, \gamma \in \mathbb{F}_{17}$ with $\gamma \neq 0$ such that, putting $b = b_3$ gives the following:

$b_1 = (\alpha + \beta + \gamma)\,\delta^2$,
$b_2 = \alpha\delta^{11} + \beta\delta^2 + \gamma$,
$b_5 = \alpha\delta^6 + \beta + \gamma + b$,
$b_6 = \alpha\delta^{13} + \beta\delta^9 + \gamma\delta^3 + b$,
$b_7 = \alpha\delta^{15} + \beta\delta^5 + \gamma\delta^{15} + b$,
$b_8 = \alpha\delta^{11} + \beta\delta^{15} + \gamma\delta^{12} + b$,
$b_9 = \alpha\delta^4 + \beta\delta^3 + \gamma\delta^7 + b$,
$b_{10} = \alpha + \beta\delta + \gamma\delta^4 + b$,
$b_{11} = \alpha + \beta\delta^2 + \gamma\delta^{11} + b$,
$b_{12} = \alpha\delta^{12} + \beta\delta^9 + \gamma\delta^8 + b$,
$b_{13} = \alpha\delta^4 + \beta\delta^{15} + \gamma\delta^6$+b,
$b_{14} = \alpha\delta^{14} + \beta\delta^3 + \gamma\delta^5 + b$.

Therefore the third component of T_i takes the form of the following:

$(\alpha + \beta\delta^7)\delta^{11} + \gamma$,
b,
1,
$(\alpha + \beta\delta^5)\delta^{13} + \gamma\delta^2 + b$,
$(\alpha + \beta\delta^2)\delta^{14} + \gamma\delta + b\delta^{10}$,
$(\alpha + \beta\delta^9)\delta^3 + \gamma\delta^{12} + \delta^{15}$,
$(\alpha + \beta\delta^8)\delta^{14} + \gamma\delta^{11} + b\delta^{15}$,
$(\alpha + \beta\delta^2)\delta^6 + \gamma\delta^3 + b\delta^7$,
$(\alpha + \beta\delta^8)\delta^{12} + \gamma\delta^{15} + b\delta^4$,
$(\alpha + \beta\delta^{11})\delta^6 + \gamma\delta^{15} + b\delta^{11}$,
$(\alpha + \beta\delta^{11})\delta^{12} + \gamma\delta^2 + b\delta^2$,
$(\alpha + \beta\delta^{12})\delta^8 + \gamma\delta^{13} + b\delta^6$,
$(\alpha + \beta\delta^{11}) + \gamma\delta^8 + b\delta^5$.

If case(2) of Theorem(4) holds for L_1, there are constants α', β', and $\gamma' \neq 0$ in $\mathbb{F}_{17}$ in terms of which the third component of T_i can be expressed. This gives twelve homogeneous linear equations in $\alpha, \beta, \gamma, b, \alpha', \beta', \gamma'$. It is not difficult to show that $\gamma' = 0$ for any solution of this system of equations, a contradiction.
The same situation also prevails in cases (3), (4), (5), (6) for L_1. Since the cases (3), (4), (5), (6) for L_3 can be dealt with similarly, only formulas expressing b_i and the third component of T_i in terms of $\alpha, \beta, \gamma, b = b_3$ are necessary. In each case, a contradiction is obtained.
since a 18-arc is a twisted cubic by Theorem (12), so $m(4,17) = 18$ by Theorem (14). Now since an 14-arc in $PG(3,17)$ is rational, For, if every 14-arc is rational, then a 15-arc in $PG(4,17)$ is rational, by Theorem (13). Hence, again by Theorem (13), a 16-arc in $PG(5,17)$ is a normal rational curve, also by Theorem (13), a 17-arc in $PG(6,17)$ is rational, Hence, again by Theorem (13), a 18-arc in $PG(7,17)$ is a normal rational curve. So, by Theorem(14), $m(8,17) = 18$. Now, Corollary (10) (ii) gives that), $m(9,17) = 18$. ■

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