Projectivity on y-closed Submodules

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Abstract:
In this notion we consider a generalization of the notion of a projective modules , defined using y-closed submodules . We show that for a module $M = M_1 \oplus M_2$. If M_2 is M_1 – y-closed projective , then for every y-closed submodule N of M with $M = M_1 + N$, there exists a submodule M\`of N such that $M = M_1 \oplus M$\`.
Keywords : projective moduls , y-closed submodules .

الاسقاطية على المقاسات الجزئية المغلقة من النمط y

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الخلاصة :
تـم في هذا البحث إعطاء تعميم للمقاسات الاسقاطية باستخدام المقاسات الجزئية المغلقة من النمط y . برهنا إن للمقــاس M = M1 ⊕ M2 إذا كان M2 مقاسا اسقاطيا على المقاس M1 بالنسبة للمقاسات الجزئية المغلقة من النمط y فأن لكل مقاس جزئي N مغلق من النمط y في M بحيث إن M = M1+N يوجد مقاس جزئي \`M من N بحيث إن \`M = M1⊕ M .

1. Introduction :

Throughout R will be an associative ring with identity and all modules will be unital left R – modules .

Let N be a module , a module M is said to be N – projective if for every submodule X of N, any homomorphism φ from M to $\frac{N}{X}$ can be lifted to a homomorphism ψ from M to N .

It is known that a module P is projective if P is M – projective , for every module M . A module M is called quasi – projective if M is M – projective , see [1] , [2] , [3] .

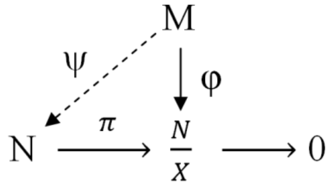
A submodule N of an R – module is said to be an y-closed submodule of M provided $\frac{M}{N}$ is nonsingular , see [4] .

Clearly that for a singular R – module M , M is the only y-closed submodule of M .

In this paper we define projectivity on y-closed submodules.

1- Projectivity on y-closed submodules

Definition 1.1 : Let N be a module . A module M is said to be N – y-closed projective if for every y-closed submodule X of N , any homomorphism $\varphi : M \to \frac{N}{X}$ can be lifted to a homomorphism $\psi : M \to N$ .i.e. , if $\pi : N \to \frac{N}{X}$ is the natural epimorphism , then there exists on R – homomorphism $\psi : M \to N$ such that $\pi\psi = \varphi$.



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A module M is called an y-closed projective module if M is N-y-closed projective , for every module N .

Remark 1.1: every singular module is y-closed projective .

Proof : Let M be a singular module . Let N be any R-module and let X be an y-closed submodule of N , then $\frac{N}{X}$ is nonsingular .

Let f : M $\to \frac{N}{X}$ be any R – homomorphism . Since M is singular , then by [4] f = 0 . So f can be lifted to a homomorphism 0 = φ : M → N . Thus M is y-closed projective .

Example 1.2 : It is clear that every projective is y-closed projective .

The converse is not true . For example , consider the module .

Z_n as a Z-module , for n ≥ 2 . since Z_n is singular, then Z_n is y-closed projective , by remark 1.1 . But it is known than Z_n is not projective .

Remark 1.3 : Let N be an R-module and Let M be a singular R-module . Then N is M-y-closed projective .

Proof : since M is singular , then M is the only y-closed submodule of M . Thus N is M-y-closed projective .

The following two remarks are in coodearl, we sketch their proofs .

Remark 1.5 : Let A and B be submodules of an R-module M such that A ⊆ B if A is y-closed in B and B is y-closed in M , then A is y-closed in M .

Proof : consider the following short exact sequence

$$0 \to \frac{B}{A} \xrightarrow{i} \frac{M}{A} \xrightarrow{\pi} \frac{\frac{M}{A}}{\frac{B}{A}} \to 0$$

Where i is the inclusion map and π is the natural epimorphism . since $\frac{\frac{M}{A}}{\frac{B}{A}} \simeq \frac{M}{B}$ is nonsingular and $\frac{B}{A}$ is non singular , then $\frac{M}{A}$ is non singular , see [4] .

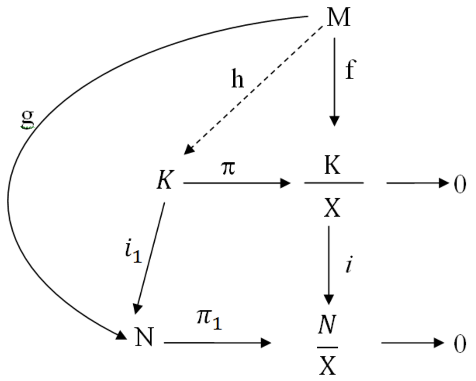
Remark 1.6 : Let $\{M_\alpha / \alpha\in\Lambda\}$ be a family of R-modules and Let A_α be a submodule of M_α , for each α∈Λ . Then $\oplus_{\alpha\in\Lambda} A_\alpha$ is y-closed in $\oplus_{\alpha\in\Lambda} M_\alpha$ if and only if $A_\alpha$is y-closed in M_α , for each α∈Λ

Proof : One can easily show that $\frac{\oplus_{\alpha\in\Lambda} M\alpha}{\oplus_{\alpha\in\Lambda} A\alpha} \simeq \oplus_{\alpha\in\Lambda}(\frac{M\alpha}{A\alpha})$

But $\oplus_{\alpha\in\Lambda}(\frac{M\alpha}{A\alpha})$ is non singular if and only if $\frac{M\alpha}{A\alpha}$ is non singular , for each α∈Λ . Therefore $\oplus_{\alpha\in\Lambda} A_\alpha$is y-closed in $\oplus_{\alpha\in\Lambda} M_\alpha$ if and only if A_α is y-closed in M_α , for each α∈Λ.

Proposition : Let N be a module and Let M be N-y-closed projective . Then M is K-y-closed projective , for every y-closed submodule K of N .

Proof : Let X be a y-closed submodule of K and f : M $\to \frac{K}{X}$ be any homomorphism . Since K is y-closed in N , then X is y-closed in N . consider the following diagram



Where i, i_1 are the inclusion maps and π , π_1 are the natural epimorphisms . Since M is N-y-closed projective , then there exists a homomorphism g : M → N such that $\pi_1 g$ = i f . claim that g(M) ⊆ K , to show that Let m ∈ M . π_1g(m) = i f(m) = f(m) Let f(m) = k + X . So g(m) + X = k + X . Thusg(m) – k∈ X ⊆ K and hence g(m) ∈ K .

Now define h : M → K by h(m) = g(m) , ∀m ∈ M .

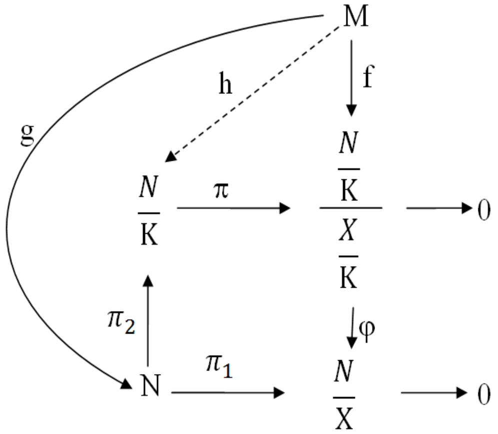
Let m ∈ M , then

π h(m) = π g(m) = π_1g(m) = i f(m) = f(m) , Thus π h = f .

Proposition 1.8 : Let N be an R-module and M be N-y-closed projective . if K is a submodule of N , then M is $\frac{N}{K}$ -y-closed projective .

Proof : Let $\frac{X}{K}$ be a y-closed sub module of $\frac{N}{K}$ and f : M $\to \frac{\frac{N}{K}}{\frac{X}{K}}$ be any R – homomorphism . Now $\frac{\frac{N}{K}}{\frac{X}{K}} \simeq \frac{N}{X}$. So Let φ : $\frac{\frac{N}{K}}{\frac{X}{K}} \to \frac{N}{X}$ be the isomorphism defined by $\varphi((n+K)+\frac{X}{K}) = n + X$, $\forall n \in N$. Since $\frac{N/K}{X/K}$ is nonsingular , then X is a y-closed submodule of N .

Now consider the following diagram :



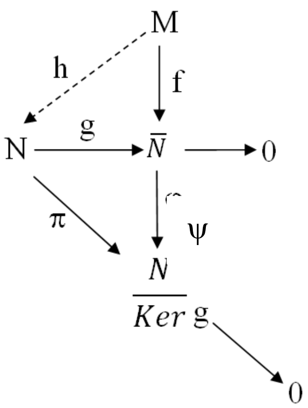
Where π , π_1 , π_2 are the natural epimorphisms since M is N-y-closed projective , then there exists a homomorphism g : M → N such that $\pi_1 g = \varphi f$.

Let h = $\pi_2 g$. claim that $\pi h = f$. To show that $\varphi f = \pi_1 g = \varphi\pi\pi_2 g = \varphi\pi h$. since φ is an isomorphism, then $\pi h = f$.

Proposition 1.9 : Let M and N be R- modules . Then M is N –y-closed projective if and only if for every epimorphismg: N →N\` with kerg is y-closed in N , where N\`is any R-module and any homomorphism f : M →N\`, there exists a homomorphism

h : M → N such that gh = f

Proof : By the first isomorphism theorem $\frac{N}{kerg} \simeq$N\`. So there exists an isomorphism $\psi$ : N\`→$\frac{N}{kerg}$ defined by ψ(n\`) = x + kerg , where g(x) = n\`. Now consider the following diagram



Where π is the natural epimorphism , then exists a homomorphism h : M → N such that π h = ψf.

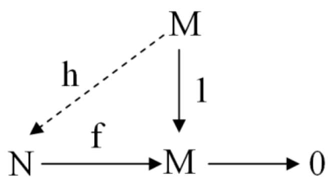
Now Let m ∈ M , then h(m) + kerg = x + kerg , where g(x) = f(m) .

So h(m) – x ∈ kerg . Thus g(h(m) –x) = 0 and hence gh(m) = g(x) = f(m) . Thus gh = f .

The converse is clear .

Proposition 1.10 : Let M and N be R – modules. if M is N-y-closed projective , then any epimorphism f : N → M with kerf is y-closed in N split . In addition , if N is indecomposable , then f is an isomorphism .

Proof : consider the following diagram



Where 1 is the identity . By prop 1.9 , there exists a homomorphism h : M → N such that f h = 1 . Thus f has a right inverse and hence split . Thus ker f is a direct summand of N .

Let M be an R – module , Recall that M is called a CLS – module if every y-closed submodule of M is a direct summand see [5] .

Proposition 1.11 : Let M be an R – module such that $\frac{M}{K}$ is M – y-closed projective , for every y-closed submodule K of M , then M is a CLS – module .

Proof : Let K be a y-closed submodule of M

Let π : M →$\frac{M}{K}$ be the natural epimorphism , then kerπ = K is a direct summand of M , by proposition 1.10 .

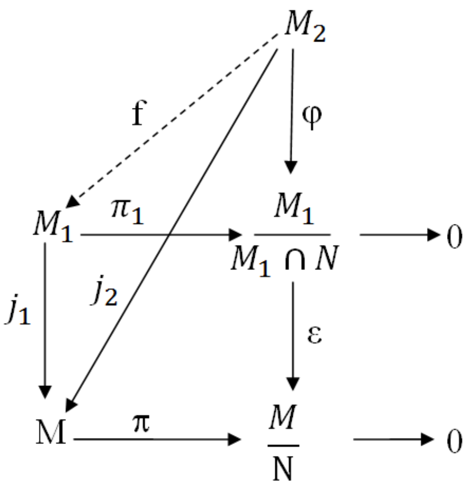
2- Direct sums and y-closed projectivity

Proposition 2.1 : Let M = $M_1 \oplus M_2$ be an R – module .If $M_2$is M_1– y-closed projective, then for every y-closed submoduleN of M with M = M_1 + N , there exists a submodule M\`of N such that M = $M_1 \oplus$M\`.

Proof : Let φ : $M_2 \to \frac{M_1}{M_1 \cap N}$ be a map defined by $\varphi(m_2) = \varphi(x + y) = x + M_1 \cap N$, where $m_2 \in M_2$, x ∈M_1 and y ∈ N .

It is easy to show that φ is well defined .

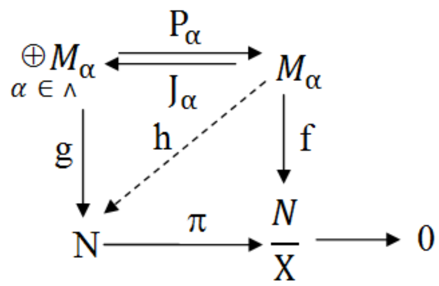
Let $\xi : \frac{M_1}{M_1 \cap N} \to \frac{M}{N}$ be the isomorphism defined as follows :
$\xi(m_1 + M_1 \cap N) = m_1 + N$. Now consider the following diagram



where π_1 and π are the natural epimorphisms and j_1 and j_2 are the inclusion maps . Since N is y-closed in M and $\frac{M}{N} = \frac{M_1 + N}{N} \simeq \frac{M_1}{M_1 \cap N}$, then $M_1 \cap N$ is y-closed in M_1 .
But M_2 is M_1 –y-closed projective , therefore there exists
$f : M_2 \to M_1$ such that $\pi_1 f = \varphi$. We can easily show that $M = M_1 + M_2 = M_1 + (j_1 f + j_2)(M_2)$. Now
Let $m_1 \in M_1 \cap (j_1 f + j_2)(M_2)$, $m_1 = j_1 f(m_2) - j_2(m_2)$.
Therefore $m_1 = f(m_2) - m_2$. Thus $f(m_2) - m_1 = m_2$. Therefore $m_2 \in M_1 \cap M_2 = 0$ and $f(m_2) - m_1 = 0$. Thus $m_1 = 0$. Hence $M = M_1 \oplus (j_1 f - j_2)(M_2)$.
Now , we only need to show that $(j_1 f - j_2)(M_2) \subseteq N$
Let $x \in M_2$, therefore $x = x_1 + y_1$, where $x_1 \in M_1$ and $y_1 \in N$.
Now we have
$f(x) - x + N = \pi((j_1 f - j_2)(x))$
$= \pi j_1 f(x) - \pi j_2(x)$
$= \xi \pi_1 f(x) - \pi j_2(x)$
$= \xi \varphi(x) - \pi j_2(x)$
$= (x_1 + N) - (x + N)$
$= x_1 - x + N$
$= -y_1 + N$
$= N$
Therefore , $f(x) - x \in N$, for every $x \in M_2$.
Thus $(j_1 f - j_2)(M_2) \subseteq N$.

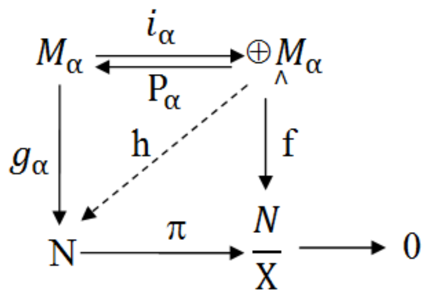
Proposition 2.2 : A direct sum $\oplus_{\alpha \in \Lambda} M_\alpha$ is N-y-closed projective if and only if M_α is N-y-closed projective , for every $\alpha \in \Lambda$.

Proof : Suppose $\oplus_{\alpha \in \Lambda} M_\alpha$ is N-y-closed projective . Let X be a y-closed submodule of N. Consider the following diagram



Where f is any homomorphism , P_α is the projection maps,J_α is the inclusion map and π is the natural epimorphism .
Then there exists a homomorphism $g : \oplus_{\alpha \in \Lambda} M_\alpha \to N$ such that $\pi g = f P_\alpha$. Let $h = g J_\alpha$.
Now $\pi h = \pi g J_\alpha = f P_\alpha J_\alpha = f I = f$.
Thus M_α is N-y-closed projective .

The converse , Let X be a y-closed sub module of N and
$f : \oplus_{\alpha \in \Lambda} M_\alpha \to \frac{N}{X}$ be any R – homomorphism . Now consider the following diagram



Where i_α is the inclusion , P_α is the projection and π is the natural epimorphism .
For every $\alpha \in \Lambda$, since M_α is N-y-closed projective , then have exists a homomorphism $g_\alpha : M_\alpha \to N$ such that $f i_\alpha = \pi g_\alpha$. Define $h : \oplus_{\alpha \in \Lambda} M_\alpha \to N$ as follows :
$h((m_\alpha)_{\alpha \in \Lambda}) = \sum_{\alpha \in \Lambda} g_\alpha(m_\alpha)$
h is well defined , since $m_\alpha \neq 0$ for at most a finite number of $\alpha \in \Lambda$. To show that $\pi h = f$. Let $(m_\alpha)_{\alpha \in \Lambda} \in \oplus_{\alpha \in \Lambda} M_\alpha$
$\pi h((m_\alpha))_{\alpha \in \Lambda} = \pi \sum_{\alpha \in \Lambda} g_\alpha(m_\alpha)$
$= \sum_{\alpha \in \Lambda} \pi g_\alpha(m_\alpha)$

$= \sum_{\alpha \in \Lambda} f\, i_\alpha(m_\alpha)$

$= f\,(\sum_{\alpha \in \Lambda} i_\alpha(m_\alpha))$

$= f\,((m_\alpha)_{\alpha \in \Lambda})$.

Recall that an R – module M has D_2 if for any submodule N of M which $\frac{M}{N}$ is isomorphic to a direct summand of M , then N is a direct summand of M , see [1] . It is known that if a module M is M – projective , then M has D_2 , see [1, prop 4.38] .

We have the following result when a module M is M –y-closed projective.

Proposition 2.3 : Let M be M-y-closed projective module and N be a y-closed submodule of M . If $\frac{M}{N}$ is isomorphic to a direct summand K of M , then N is a direct summand of M .

Proof : Let π : M $\to \frac{M}{N}$ be the natural epimorphism and

$\psi : \frac{M}{N} \to K$ be an isomorphism . Let f = ψπ : M → K

Clearly ψ is an epimorphism and ker f = N

By prop 2.2 , K is M-y-closed projective and hence f is split , by Prop. 1.10 . Thus ker f = N is a direct summand of N .

Let M be an R-module and N be a submodule of M .

N is called a fully invariant submodule of M if f (N) ⊆ N , for every R-homomorphism f : M → M , see [6] .

Proposition 2.4 : Let M be an R-module and N= $\oplus_{i=1}^{n} N_i$ be an R-module such that every y-closed submodule of N is fully invariant . If M is N_i-y-closed projective , for every i = 1,…,n , then M is N-y-closed projective . The converse is true if N is nonsingular .

Proof : Suppose M is Ni-y-closed projective, for every i = 1,…,n .

Let X be a y-closed submodule of $\oplus_{i=1}^{n} N_i$.

Let P_j : $\oplus_{i=1}^{n} N_i \to N_j$ be the projection, ∀ j=1,...,n since X is fully invariant .

then $P_i(X) \subseteq X \cap N_i$.

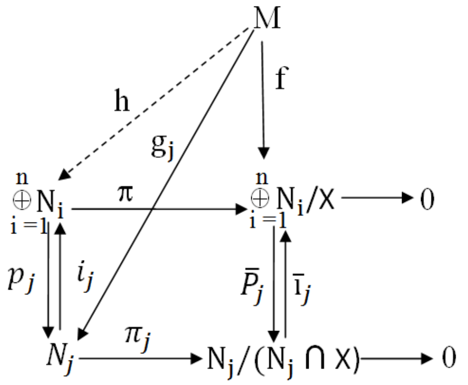
Let x ∈ X , then x = $\sum_{j=1}^{n} x_i = \sum_{i=1}^{n} P_i(x)$.

Thus $x_i \in X \cap N_i$. Thus X = $\oplus_{i=1}^{n}(X \cap N_i)$.

Since X is y-closed in N , then $\frac{N}{X}$ is nonsingular and hence ,

$\frac{X+N_i}{X} \subseteq \frac{N}{X}$ is nonsingular .

But $\frac{X+N_i}{X} \simeq \frac{N_i}{X \cap N_i}$, therefore $X \cap N_i$ is y-closed in N_i , for every i = 1,….,n . Now consider the following diagram



Where $\bar{p}_j\,(\sum_{i=1}^{n} x_i + X) = x_j + N_j \cap X$, $\bar{i}_j\,(x_j + N_j \cap X) = x_j + X$, i_j is the inclusion maps and π , π_j are the natural apimoprphisms .

Since M is N_j-y-closed projective , then there exists a homomorphism g_j : M $\to N_j$ such that $\pi_j q_j = \bar{p}_j$ f

Now define h : M $\to \oplus_{i=1}^{n} N_i$ as follows :

$h(m) = \sum_{i=1}^{n} g_i(m)$

clearly $\pi = \sum_{j=1}^{n} \bar{i}_j \pi_j p_j$

to show that π h = f , Let m ∈ M

$\pi\, h(m) = \pi\,(\sum_{i=1}^{n} g_i(m))$

$= \sum_{j=1}^{n} \bar{i}_j \pi_j p_j\,(\sum_{i=1}^{n} g_i(m))$

$= \sum_{j=1}^{n} \bar{i}_j \pi_j g_j(m)$

$= \sum_{j=1}^{n} \bar{i}_j \bar{p}_j f(m)$

= f(m)

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