On Reverse – Centralizers Of Semiprime Rings

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Abstract
In this paper we study necessary and sufficient conditions for a reverse-centralizer of a semiprime ring R to be orthogonal. We also prove that a reverse-centralizer T of a semiprime ring R having a commuting generalized inverse is orthogonal.
Keywords: Semiprime ring, left (right) reverse-centralizer, reverse-centralizer, orthogonal map, generalized inverse.

حَـول الدوال التمركزيـة العكسيـة على الحلقات شُبـه ألاولـية

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الخلاصة:
في هذا البحث ندرس الشروط الضرورية والكافية للدوال التمركزية العكسية على الحلقة شبه الاولية R لتكون متعامدة. ونبرهن ايضاً ان الدالة التمركزية العكسية على الحلقة شبه الاولية R والتي تمتلك تعميم معكوس ابدالي تكون متعامدة.
كلمات مفتاحية: الحلقة شبه الاولية ، تمركز عكسي ايسر (ايمن) ،التمركز العكسي، الدالة المتعامدة ، التعميم المعكوس.

1. Introduction:

The purpose of this paper is to investigate some further properties of reverse-centralizers of semiprime rings and we give necessary and sufficient conditions for a reverse-centralizer of a semiprime ring to to be orthogonal map. Recall that R is semiprime if aRa=0 implies a =0 and is prime if aRb=0 implies that a=0 or b=0. Recently, some authors have studied left (right) centralizers and centralizers in the general framework of semiprime rings (see [1-4]). An additive mapping $T:R\rightarrow R$ is called a left (resp. right) centralizer of R if $T(xy)=T(x)y$ ($T(xy)=xT(y)$) holds, for all $x,y\in R$. If T is both left as well right centralizer, then T is a centralizer. An additive mapping $T:R\rightarrow R$ is called a left (resp. right) reverse-centralizer of R if $T(xy)=T(y)x$ ($T(xy)=yT(x)$) holds, for all $x,y\in R$, if T is both left as well right reverse-centralizer, then T is a reverse-centralizer, see [5]. Let M be a subset of a ring R, following [6], we recall the orthogonal complement of M is the set $M^{\perp}=\{x\in R: xy=yx = 0 \text{ for all } y \in M\}$. Also we recall the mapping T of a ring R is called orthogonal if $R = R(T) \oplus R(T)^{\perp}$. We prove that a reverse-centralizer T of a semiprime ring R is orthogonal if and only if $R(T) = R(T^2)$, where R(T) denotes the range of T. [Theorem 2.10]. Also we prove that if T is a reverse-centralizer of a semiprime ring R then the following are equivalent:

* $R(T) = R(T^2)$
* $R = Ker(T) \oplus R(T)$. [Theorem 2.4].

Following [6], A mapping $T: R \rightarrow R$ of a ring R into itself is said to have a generalized inverse if there is a mapping $S: R \rightarrow R$ such that STS = S

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and TST = T. In this case S is said to be a generalized inverse of T or S is a g-inverse of T. Generalized inverses are useful in various fields of mathematics, statistics and engineering (see [6-9]. Finally we prove that if T is a reverse-centralizer of a semiprime ring having a commuting generalized inverse then T is orthogonal [Proposition 2.15].

2. Results:

Remark 2.1. It is easy to verify that if T is a reverse-centralizer of a ring R, then Ker(T) and R(T) are ideals of R.

Remark 2.2. Let S, T ∈V(R), the set of all reverse-centralizers of a ring R is denoted by V(R). We Define (S+T)(x)=S(x)+T(x) and (ST)(x)=S(T(x)) , x∈R. then V(R) with these two operations is a ring with identity.

It is easy to verify that if R is a semiprime ring then V(R) is commutative.

Proof: we want prove that V(R) is commutative.

Let S, T∈ V(R), we want prove that (ST)(x)= (TS)(x) , for all x∈ R .

Consider ((ST)(x)-(TS)(x))z = ST(x)z-TS(x)z = ST(x)z-T(S(x))z = ST(x)z-T(zS(x)) = ST(x)z-S(x)T(z) = ST(x)z-S(T(z)x) = ST(x)z-xS(T(z)) = ST(x)z-xST(z) = ST(x)z-ST(zx) = ST(x)z-ST(x)z = 0 , for all x,z∈R . By semiprimeness of R we obtain (ST)(x)-(TS)(x)=0, and hence (ST)(x) = (TS)(x) for all x∈ R. Thus V(R) is a commutative ring.

Proposition 2.3. Let T be a reverse-centralizer of a semiprime ring R, then Ker (T) = Ker (T^2).

Proof: Clearly, Ker (T) ⊆Ker (T^2). Now, let x∈ Ker (T^2), this implies that T^2(x)=0 , Now, since T is a reverse-centralizer then we have,T(x)rT(x)=(T(x)r)T(x)=T(rx)T(x)=T(T(x)r x)=rxT(T(x))=rxT^2(x)=0. By semiprimeness of R we get, T(x)=0, hence x∈ Ker (T), and hence Ker(T^2) ⊆Ker (T). Thus, Ker (T) = Ker (T^2).

Theorem 2.4. Let T be a reverse-centralizer of a semiprime ring R. Then R(T) = R(T^2) if and only if R = Ker (T) ⊕ R(T).

Proof:

Suppose that R(T) = R(T^2) , since T is a reverse-centralizer then by (Remark 2.1.) Ker(T) and R(T) are ideals of R, now we show that Ker(T) ∩ R(T)=0. Let w∈ Ker(T) ∩ R(T). Then w∈ Ker(T) and w∈R(T) and hence T(w)=0 and T(y)=w for some y∈R. Hence T(w)=T^2(y)=0. Thus y ∈Ker(T^2). But by (proposition 2.3.), we have Ker (T) = Ker (T^2). Thus y∈ Ker(T) which implies that T(y)=0 and hence w=0. Which implies that Ker(T) ∩ R(T)=0.

Now, suppose that z∈ R, then T(z) ∈R(T), by assumption we have T(z) ∈ R(T^2). Thus T(z)=T^2(v) for some v∈ R, and soT(z)-T^2(v)= T(z-T(v))=0. This implies that z-T(v) ∈Ker(T) , now we can write z=(z-T(v))+T(v). This implies that R = Ker (T) ⊕ R(T).

Conversely, Suppose that R = Ker (T) ⊕ R(T), we prove that R(T) = R(T^2). Clearly that R(T^2) ⊆ R(T). Now let y∈R(T), then y=T(x) for some x∈ R. Thus by assumption, x=x_1+x_2, where x_1∈ Ker(T) and x_2 ∈ R(T). Thus

y= T(x)=T(x_1+x_2)=T(x_1)+T(x_2)=T(x_2). Since x_2 ∈R(T), therefore x_2=T(w) for some w∈ R, hence

y = T(x_2) = T(T(w)) = T^2(w), which implies that

y ∈R(T^2). Thus R(T) ⊆ R(T^2), hence R(T) = R(T^2).

Definition 2.5. [6]. Let M be a subset of a ring R. We define the orthogonal complement of M to be the set $M^{\perp}$={ x∈R: xy = yx = 0 for all y ∈ M }.

Remark 2.6. [6]. It is easy to verify that if M is an ideal of R then $M^{\perp}$ is also an ideal of R.

Example.2.7 Let M_2(R) denote the ring of all 2×2 matrices over the set of all real numbers R and let T: M_2(R) → M_2(R) be an additive mapping defined by $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix})=\begin{bmatrix} 0 & 0 \\ b & d \end{bmatrix}$, for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ∈ M_2(R). One can easily show that Ker(T) ⊈ $R(T)^{\perp}$ and R(T) ⊈ $Ker(T)^{\perp}$. Farther, it is clear that T is not reverse-centralizer. Moreover, if T is reverse-centralizer then it is easy to verify that Ker (T) ⊆ $R(T)^{\perp}$ and R(T) ⊆ $Ker(T)^{\perp}$.

Proposition 2.8. Let R be a semiprime ring and T: R→R a reverse-centralizer of R. Then Ker(T) = $R(T)^{\perp}$.

Proof:

First, we want prove that Ker (T) ⊆ $R(T)^{\perp}$. Let x∈ Ker(T), then T(x)=0, now replace x with xy, we get T(xy)=0, for all x,y ∈ R. But T is reverse-centralizer of R, therefore

T(xy)=yT(x)=T(y)x=0 for all x,y∈ R . Thus x ∈ $R(T)^{\perp}$. and hence Ker (T) ⊆ $R(T)^{\perp}$. Now let z∈$R(T)^{\perp}$,then zT(x)=T(z)x=0 for all x ∈ R.

By semiprimness of R we get T(z)=0, that is z∈ Ker (T). Therefore $R(T)^{\perp}$ ⊆ Ker(T), and hence Ker(T) = $R(T)^{\perp}$.

Definition 2.9. [6]. A mapping T of a ring R is said to be orthogonal if R=R(T) ⊕ $R(T)^{\perp}$.

The following Theorem gives the necessary and sufficient conditions for a reverse-centralizer T of a semiprime ring to be orthogonal.

Theorem 2.10. A reverse - centralizer T of a semiprime ring R is orthogonal if and only if R(T)= $R(T^2)$.

Proof: Let T be a reverse-centralizer of semiprime ring R. Suppose that T is orthogonal then R=R(T) ⊕ $R(T)^{\perp}$. By proposition (2.8.), we have Ker(T) = $R(T)^{\perp}$, this implies that R= R(T) ⊕ Ker(T). Thus by Theorem (2.4.) we get R(T) = $R(T^2)$.Conversely, suppose that R(T)= $R(T^2)$. Thus by proposition (2.4.) we have

R = Ker (T) ⊕ R(T), hence by proposition (2.8.), we obtain R=R(T) ⊕ $R(T)^{\perp}$, and hence, T is orthogonal.

Corollary 2.11. An idempotent reverse-centralizer of a semiprime ring is orthogonal.

Definition 2.12. [6] A mapping T: R → R of a ring R into itself is said to have a generalized inverse if there is a mapping S: R → R such that STS = S and TST = T. In this case S is said to be a generalized inverse of T or S is a g-inverse of T.

Before we give our main result, we need the following proposition which can be found in [3].

Proposition 2.13. Let T: R → R be an additive mapping with S as a g-inverse.

Then the following hold:

(a) TS and ST are idempotents.

(b) R(TS) = R(T), and Ker(ST) = Ker(T).

Remark 2.14. [6] It is well known that a g-inverse S of a mapping T : R → R is not unique. But there is at most one g-inverse which commutes with T. If S and S′ are g-inverses of T, both commuting with T, then TS′ = TSTS′ = STS′T = ST, and hence S′ = S′TS′ = S′ST = S′TS = TS′S = STS = S.

The following proposition gives a condition under which a reverse-centralizer of a ring is orthogonal.

Proposition 2.15. Let T be a reverse-centralizer of a semiprime ring R. if T has a commuting g-inverse S∈V(R), then T is orthogonal map.

Proof:

Let T be a reverse-centralizer of a semiprime ring R. Suppose that T have a commuting g-inverse S∈V(R). Thus by Remark (2.2.) we get TS = ST. Also TS and ST are reverse-centralizers and are idempotent by Proposition (2.13). Thus by Corollary (2.11.) TS and ST are orthogonal. Thus R = Ker (TS) ⊕ R(TS) = Ker (ST) ⊕ R(TS). Now by Proposition (2.13.), we have Ker (ST)= Ker (T) and R(TS)= R(T). Thus R = Ker (T) ⊕ R(T). But by Proposition (2.8.), Ker (T)= $R(T)^{\perp}$, therefore R = $R(T)^{\perp}$ ⊕ R(T). Hence T is orthogonal.

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