Unsteady Flow of Electrically Conducting Fluid of Second Grade Over A Stretching Sheet Subject to A Transverse Magnetic Field

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Abstract

In this paper, the series solution for unsteady flow for an incompressible viscous electrically conducting fluid of second grad over a stretching sheet subject to a transverse magnetic field is presented by using homotopy analysis method (HAM). Also we examines the effects of viscoelastic parameter, magnetic parameter and time which they control the equation of motion.

Keywords: Homotopy analysis method (HAM), Unsteady, viscoelasticity, magnetic parameter.

الجريان اللامستقر لمائع من الرتبة الثانية قابل للتوصيل الكهربائي على صفيحة مستعرضة مطاطية في حقل مغناطيسي

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الخلاصة

في هذا البحث نتناول دراسة معادلة الحركة اللامستقرة لمائع من الرتبة الثانية غير قابل للانضغاط , ثابت اللزوجة , قابل للتوصيل الكهربائي على مقطع عرضي لصفيحة مطاطية في مجال مغناطيسي. وتم التعبير عن الحل باستخدام طريقة هوموتوبي التحليلية (HAM). كما درست تأثير الإعداد اللابعدية التي تحكم معادلة الحركة وهي الزمن (ξ)، اللزوجة (K) والحقل المغناطيسي (M_n).

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Introduction

There are many mathematical models which describing some properties, but not all, of non-Newtonian fluids, as ketchup, custard, paint, liquid detergent, liquid polymers and a variety of other liquids. Among these models, the fluids of differential type, for example, fluids of second grade and tired grade, have been received much attention in the past due to their elegance and simplicity [1].

The studies of boundary layer flow of non-Newtonian fluids over a stretching became more important because of the development in the industry Fox et al. [2]

Examine the boundary layer flow of a viscoelastic fluid characterized using both exact and approximate methods with a power law model. Vajravelu and Rollins [3] investigated the heat transfer of the boundary layer flow of a second grade fluid whose constitutive equation is given by

$$\mathrm{T} = -p\mathrm{I} + \mu \mathrm{A}_1 + \alpha_1 \mathrm{A}_2 + \alpha_2 \mathrm{A}_1^2 \quad (1)$$

Here T is the Cauchy stress tensor, p is the indeterminate pressure constrained by the incompressibility, μ is the viscosity, α_1 and α_2 are the moduli of the viscoelastic fluid, A_1 and A_2 are the first two Rivilin-Ericksen tensors defined by [4]

$$\mathrm{A}_1 = L + L^{\mathrm{T}}$$

$$\mathrm{A}_2 = \frac{d\mathrm{A}_1}{dt} + \mathrm{A}_1 L + L^{\mathrm{T}} \mathrm{A}_1, \quad (2)$$

Where d/dt is the material derivative and $L = \nabla V$. If the fluid of second grade is to satisfy the Clausius-Dehum inequality for all motions and the assumption that the specific Helmholtz free energy of the fluid is a minimum when it is locally at rest, Then the requirements for the moduli of the second grad fluid are

$$\mu \geq 0,\ \alpha_1 > 0, \text{and} \ \ \alpha_1 + \alpha_2 = 0 \quad (3)$$

Though the sign of α_1 has been a subject of much controversy.

If the second grade fluid is electrically conducting, the Lorentz force $J \times B$ where J is the electrical current and B is the magnetic field, must be included in the momentum equation when a transverse uniform magnetic field $B = (0, B_0, 0)$ is applied to the fluid layer. The terms due to Lorentz force can be simplified if the following assumption are made: (i) all physical quantities are constant; (ii) the magnetic field B is perpendicular to the velocity V and the induced magnetic field is small compared with the applied magnetic field; (iii) the electrical field is assumed to be zero. These assumptions are valid when the magnetic Reynolds number is small and there is no displacement current [5]. Thus, in the boundary layer approximation the Lorentz force is simply the term $-\sigma B_0^2 u$, where σ is the electrical conductivity, B_0 is the uniform magnetic field in the y-direction , and u is the x-component of the velocity V .

The flow problem of non-Newtonian fluids, characterized by Bingham plastic and the power law models, in a magnetic field has been investigated by Sarpkaya [6]. Sarpkaya also pointed out that some non-Newtonian fluids such as nuclear fuel slurries, liquid metals, mercury amalgams, biological fluids, plastic extrusions, paper coating, lubrication oils and greases, have applications in many areas in the absence as well as in the presence of magnetic field.

In this paper, using homotopy analysis method (HAM), one of the most effective methods [7, 8]. We present a general solution for unsteady velocity equation of a laminar boundary layer flow of an electrically conducting second grad fluid subject to a transverse uniform magnetic field over a stretching sheet with prescribed power-law surface temperature and prescribed power-law surface heat flux, the viscoelastic modulus α_1 of the second grad fluid is taken to be positive to satisfy thermodynamic restriction equation. 3

Mathematical Description

Consider the unsteady, two dimensional laminar flow of electrically conducting fluid caused by an impulsive stretching flat surface in two lateral direction in an otherwise quiescent fluid in the presence of transverse magnetic field. It is assumed that the contribution due to the normal stress is of the same order of magnetic as that due to shear stress.

The basic boundary layer equations for the unsteady flow are:

$$\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}=0 \qquad (4)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}u$$
$$+\frac{\alpha_1}{\rho}\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial y}\frac{\partial^2 \upsilon}{\partial y^2}+\upsilon\frac{\partial^3 u}{\partial y^3}\right] \qquad (5)$$

Where u and υ are velocity component in the x and y direction respectivelly, ρ is the density, $\nu=\mu/\rho$ is kinematic viscosity. The last term in (1) is the Lorentz Force. The gravity force assumed to be neglected and the modified pressure gradient is absent since the flow is driven by the stretching sheet.

The boundary conditions for velocity field are

$$\left.\begin{array}{l} u=Bx, \upsilon=0 \text{ at } y=0,\ B>0 \\ u\to 0,\ \partial u/\partial y\to 0 \text{ at } y\to\infty \end{array}\right\} \qquad (6)$$

Non-Dimensional form of Velocity Equation

We can write down the Velocity Equation in non-dimensional form through using the transformations:

$$\left.\begin{array}{l} u=Bxf'(\eta)\,, \upsilon=-(B\nu)^{1/2}f(\eta)\xi^{1/2} \\ \eta=-(B/\nu)^{1/2}y\xi^{-1/2}\,, \tau=Bt \\ \xi=1-e^{-\tau} \end{array}\right\} \qquad (7)$$

The substitution of these quantities into (5) gives:

$$\frac{1}{2}B^2xf''\eta(1-\xi^{-1})+B^2x(f')^2-B^2xf''f=$$
$$B^2x\xi^{-1}f'''-\frac{\sigma B_0^2}{\rho}Bxf'+\frac{\alpha_1 B^2x}{\rho}\left[2\frac{B}{\nu}\xi^{-1}f'f'''\right.$$
$$\left.-B\frac{1}{\nu}\xi^{-1}(f'')^2-B\frac{1}{\nu}\xi^{-1}f''''f\right] \qquad (8)$$

Dividing the above by B^2x, the above equation became

$$\frac{1}{2}f''\eta(1-\xi^{-1})+(f')^2-f''\ f=$$
$$\xi^{-1}f'''-M_nf'+$$
$$K\xi^{-1}\left[2f'\ f'''-(f'')^2-f^{IV}f\right] \qquad (9)$$

Where a prime denotes the differentiation with respect to η, $K=\alpha_1B/\mu$ is the viscoelastic parameter and M_n is the magnetic parameter.

The corresponding boundary conditions 6 became:

$$\left.\begin{array}{l} f(\eta,\xi)=0,\quad f'(\eta,\xi)=1 \text{ at } \eta=0 \\ f(\eta,\xi)=0 \text{ at } \eta\to\infty \end{array}\right\} \qquad (10)$$

Basic ideas of HAM

Consider a non-linear equation governed by

$$A(u)+f(r)=0 \qquad (11)$$

where A is a non-linear operator, $f(r)$ is a known function and u is a unknown function. By means of homotopy analysis method, one first constructs a family of equations

$$(1-p)L[\upsilon(r,p)-u_0(r)]=$$
$$ph\{A[\upsilon(r,p)-f(r)]\} \qquad (12)$$

where L is an auxiliary linear operator, $u_0(r)$ is an initial guess, h is an auxiliary parameter, $p\in[0,1]$ is an embedding parameter, $\upsilon(r,p)$ is an unknown function of r and p. Liao [9,10] expanded $\upsilon(r,p)$ in Taylor series about the embedding parameter

$$\upsilon(r,p)=u_0(r)+\sum_{m=1}^{\infty}u_m(r)p^m \qquad (13)$$

where

$$u_m(r)=\frac{1}{m!}\frac{\partial^m\upsilon(r,p)}{\partial p^m}\bigg|_{p=0} \qquad (14)$$

The convergence of the series (13) depends upon the auxiliary parameter h. If it is convergent at p=1, one has

$$u(r)=u_0(r)+\sum_{m=1}^{\infty}u_m(r) \qquad (15)$$

Differentiating the zeroth order deformation equation (12) m-times with respect to p and then dividing them by m! and finally setting p=0 we obtain the following mth-order deformation problem:

$$L[u_m(r)-\chi_m u_{m-1}(r)]=hR_m(r) \qquad (16)$$

in which

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases} \qquad (17)$$

$$R_m(r)=\frac{1}{(m-1)!}\left\{\frac{d^{k-1}}{d\ p^{k-1}}A[u_0(r)+\sum_{m=1}^{\infty}u_m(r)p^m]\right\}\bigg|_{p=0} \qquad (18)$$

There are many different ways to get the higher order deformation equations. However, according to the fundamental theorems in calculus [11], the term $u_m(r)$ in the series (13) is unique. Note that the HAM contains an auxiliary parameter h, which provides us with a simple way to control and adjust the convergence of the series solution (15).

Homotopy Analysis Solution

In order to solve equation 9 we select

$$f_0(\eta,\xi)=(1-e^{-\gamma\eta})/\gamma \qquad (19)$$

as initial approximation of f, where $\gamma=\sqrt{(1+M_n)/(1+K)}$ is a combined parameter relating the effects of viscoelasticity of the second grad fluid and the magnetic field. Besides to choose

$$L[\phi(\xi,\eta,q)]=\frac{\partial^3\phi}{\partial\eta^3}-\gamma^2\frac{\partial\phi}{\partial\eta} \qquad (20)$$

as the auxiliary linear operator with the property

$$L[C_1+C_2\exp(-\gamma\eta)+C_3\exp(\gamma\eta)]=0 \qquad (21)$$

$f_0(\eta)$ Satisfy the linear operator and the corresponding boundary conditions.

Zero-order deformation equation

Based on equation 9 we define the nonlinear operator

$$\mathrm{N}[\phi(\eta,\xi,q)]=\frac{1}{2}\phi''\eta(1-\xi^{-1})+\phi'^2$$
$$-\phi\phi''-\xi^{-1}\phi'''+M_n\phi'$$
$$-K\xi^{-1}\left[2\phi'\phi'''-\phi''^2-\phi^{IV}\phi\right] \qquad (22)$$

Let h denote the non-zero auxiliary parameter. We construct the zero-order deformation equation,

$$(1-q)\,L[\phi(\eta,\xi,q)-f_0(\eta,\xi)]=$$
$$q\,h\,H(r,t)\,\mathrm{N}[\phi(\eta,\xi,q)] \qquad (23)$$

Subject to the boundary conditions

$$\phi(\eta,\xi)\Big|_{\eta=0}=0\ ,\ \frac{\partial\phi(\eta,\xi)}{\partial\eta}\Bigg|_{\eta=0}=1$$
$$\frac{\partial\phi(\eta,\xi)}{\partial\eta}\Bigg|_{\eta\to\infty}=\frac{\partial^2\phi(\eta,\xi)}{\partial\eta^2}\Bigg|_{\eta\to\infty}=0 \qquad (24)$$

Where $\phi(\eta,\xi,q)$ is the solution which depends not only upon $f_0(\eta,\xi), L, H(\eta,\xi)$ and h but on the embedding parameter $q\in[0,1]$. When $q=0$ and $q=1$ the zero-order deformation equation have the solutions $\phi(\eta,\xi,0)=f_0(\eta,\xi)$ and $\phi(\eta,\xi,1)=f(\eta,\xi)$ respectively. Thus q increases from 0 to 1, $\phi(\eta,\xi,q)$ vary from the initial guesses $f_0(\eta,\xi)$ to the solution $f(\eta,\xi)$ of the considered unsteady problem. So expanding $\phi(\eta,\xi,p)$ by Taylor's series with respect to the embedding parameter q, we have

$$\phi(\eta,\xi,q)=\phi(\eta,\xi,0)+\sum_{m=1}^{\infty}f_m(\eta,\xi)q^m \qquad (25)$$

Where $f_m(\eta,\xi)=\frac{1}{m!}\frac{\partial^m\phi(\eta,\xi,q)}{\partial \mathrm{q}^{\mathrm{m}}}\Bigg|_{\mathrm{q}=0}$ (26)

Assuming that h properly chosen so that the series (25) convergent at $q=1$, we have, using (24), the solution series

$$\mathrm{f}(\eta,\xi)=f_0(\eta,\xi)+\sum_{m=1}^{\infty}f_m(\eta,\xi) \qquad (27)$$

High-order deformation equation

For simplicity, we define the vector

$$\overrightarrow{f_m}=\{f_0,f_1,f_2,\dots,f_m\} \qquad (28)$$

Differentiating the zero-order deformation equation m times with respect to q, then setting $q=0$, and finally divided it by $m!$, we obtain the m-th order deformation equation

$$L[f_m(\eta,\xi)-\chi_m f_{m-1}(\eta,\xi)]=hR_m(\vec{f}_{m-1}) \qquad (29)$$

Subject to the boundary conditions

$$f_m(\eta,\xi)\Big|_{\eta=0}=0\ ,\ \frac{\partial f_m(\eta,\xi)}{\partial\eta}\Bigg|_{\eta=0}=0$$
$$\frac{\partial f_m(\eta,\xi)}{\partial\eta}\Bigg|_{\eta\to\infty}=\frac{\partial^2 f_m(\eta,\xi)}{\partial\eta^2}\Bigg|_{\eta\to\infty}=0 \qquad (30)$$

Where

$$R_m(\vec{f}_{m-1})=\frac{1}{2}\eta(1-\xi^{-1})f''_{m-1}-\xi^{-1}f'''_{m-1}$$
$$+M_n f'_{m-1}+\sum_{i=0}^{m-1}\Big[(1+K\xi^{-1})f'_i f'_{m-i-1}$$
$$-f_i\,f''_{m-i-1}-2K\xi^{-1}f'_i f'''_{m-i-1}$$
$$+K\xi^{-1}f_i\,f_{m-i-1}{}^{IV}\Big] \qquad (31)$$

And $$\chi_m=\begin{cases}0, & m=1\\ 1, & m>1\end{cases} \qquad (32)$$

In this way, it is easy to solve the linear equation 29 one after the other in the order $\mathrm{m}=1,2,3,\dots$ by means of the symbolic computation software such as Matlab, Mathematica, and Maple.

General solution

The first step in the HAM is to find a set of base functions to express the sought solution of the problem under investigation. As mentioned by Liao [9, 12], a solution may be expressed with different base functions, among which some converge to the exact solution of the problem faster than others. Here, due to many boundary- layer flows decay exponentially at infinity, we assume that $f(\eta,\xi)$ can be expressed by a set of functions $\left\{\xi^k \eta^j \exp(-n\gamma\eta) \middle| k \ge 0, n \ge 0, j \ge 0)\right\}$ in the form

$$\varphi_m(\eta,\xi) = \sum_{k=0}^{m+1}\sum_{r=0}^{m} \psi_{m,r,k}(\eta)\ \ \xi^{-r}\exp(-k\gamma\eta)$$
$$m \ge 1 \tag{33}$$

where

$$\left.\begin{array}{l} \psi_{m,r,0} = b^0_{m,r,0},\ m \ge 1,\ k = 0 \\ \psi_{m,0,k} = b^0_{m,0,k},\ m \ge 1,\ k \ge 1,\ r = 0 \\ \psi_{m,0,0} = b^0_{m,0,0},\ m \ge 1,\ k = r = 0 \\ \psi_{m,r,k}(\eta) = \displaystyle\sum_{i=0}^{2m-(k+r)} b^i_{m,r,k}\ \ \eta^i \\ m \ge 1,\ 1 \le k \le m+1,\ 1 \le r \le m \end{array}\right\} \tag{34}$$

And from the initial guess $f_0(\eta,\xi)$ we have

$$b^0_{0,0,0} = \frac{1}{\gamma},\quad b^0_{0,0,1} = \frac{-1}{\gamma} \tag{35}$$

For simplicity we will define $\lambda^i_{m,r,k}$ as the following

$$\psi_{m,r,k}(\eta) = \sum_{i=0}^{2m-(k+r)} \lambda^i_{m,r,k}\ \ b^i_{m,r,k}\ \ \eta^i \tag{36}$$

where

$$\lambda^i_{m,r,k} = \left\{\begin{array}{l} 0,\ \ if\ \ m = k = r = 0,\ \ i > 0 \\ 0,\ \ if\ \ m > 0,\ k = 0,\ \ i \ge 1 \\ 0,\ \ if\ \ m > 0,\ k \ge 1, r = 0,\ \ i \ge 1 \\ 0,\ \ if\ \ m > 0,\ k = r = 0,\ \ i \ge 1 \\ 0,\ \ if\ \ k > m+1 \\ 0,\ \ if\ \ r > m \\ 0,\ \ if\ \ \mathrm{i > 2m - (r + k)} \\ 1,\ \ \mathrm{otherwies} \end{array}\right\} \tag{37}$$

From equation 33, 36 and 37 we can get

$$\varphi'_m(\eta,\xi) = \sum_{k=0}^{m+1}\sum_{r=0}^{m}\sum_{i=0}^{2m-(k+r)} a^i_{m,r,k}\eta^i\ \ \xi^{-r}\exp(-k\gamma\eta) \tag{38}$$

$$\varphi''_m(\eta,\xi) = \sum_{k=0}^{m+1}\sum_{r=0}^{m}\sum_{i=0}^{2m-(k+r)} c^i_{m,r,k}\eta^i\ \ \xi^{-r}\exp(-k\gamma\eta) \tag{39}$$

$$\varphi'''_m(\eta,\xi) = \sum_{k=0}^{m+1}\sum_{r=0}^{m}\sum_{i=0}^{2m-(k+r)} d^i_{m,r,k}\eta^i\ \ \xi^{-r}\exp(-k\gamma\eta) \tag{40}$$

$$\varphi^{(\mathrm{IV})}_m(\eta,\xi) = \sum_{k=0}^{m+1}\sum_{r=0}^{m}\sum_{i=0}^{2m-(k+r)} w^i_{m,r,k}\eta^i\ \ \xi^{-r}\exp(-k\gamma\eta) \tag{41}$$

where

$$a^i_{m,r,k} = (i+1)\lambda^{i+1}_{m,r,k}b^{i+1}_{m,r,k} - (k\gamma)\lambda^i_{m,r,k}b^i_{m,r,k} \tag{42}$$

$$c^i_{m,r,k} = (i+1)(i+2)\lambda^{i+2}_{m,r,k}b^{i+2}_{m,r,k}$$
$$-2(k\gamma)(i+1)\lambda^{i+1}_{m,r,k}b^{i+1}_{m,r,k} + (k\gamma)^2\lambda^i_{m,r,k}b^i_{m,r,k} \tag{43}$$

$$d^i_{m,r,k} = (i+1)\lambda^{i+1}_{m,r,k}c^{i+1}_{m,r,k} - (k\gamma)\lambda^i_{m,r,k}c^i_{m,r,k} \tag{44}$$

$$w^i_{m,r,k} = (i+1)\lambda^{i+1}_{m,r,k}d^{i+1}_{m,r,k} - (k\gamma)\lambda^i_{m,r,k}c^i_{m,r,k} \tag{45}$$

Now

$$G_m = hR_m \tag{46}$$

$$R_m = \frac{1}{2}\eta(1-\xi^{-1})\varphi''_{m-1} - \xi^{-1}\varphi'''_{m-1} + M_n\varphi'_{m-1}$$
$$+\sum_{s=0}^{m-1}\Big[(1+K\xi^{-1})\varphi'_s\varphi'_{m-s-1} - \varphi_s\varphi''_{m-s-1}$$
$$-2K\xi^{-1}\varphi'_s\varphi'''_{m-s-1} + K\xi^{-1}\varphi_s\varphi^{IV}_{m-s-1}\Big] \tag{47}$$

From equation (30) we can have:

$$\varphi'_s\varphi'_{m-s-1} = \left[\sum_{k1=0}^{s+1}\sum_{r1=0}^{s}\sum_{i1=0}^{2s-(k1+r1)} a^{i1}_{s,r1,k1}\eta^{i1}\xi^{-r1}\exp(-k1\eta\gamma)\right]$$
$$\times\left[\sum_{k2=0}^{m-s}\sum_{r2=0}^{m-s-1}\sum_{i2=0}^{2(m-s-1)-(k2+r2)} a^{i2}_{m-s-1,r2,k2}\eta^{i2}\xi^{-r2}\exp(-k2\,\eta\gamma)\right] \tag{48}$$

Which can be rewritten as

$$= \left[\sum_{k1=0}^{s+1}\sum_{k2=0}^{m-s}\exp(-(k1+k2)\,\eta\gamma)\right]$$
$$\times\left[\sum_{r1=0}^{s}\sum_{r2=0}^{m-s-1}\xi^{-(r1+r2)}\right]$$
$$\times\left[\sum_{i1=0}^{2s+(k1+r1)}\ \sum_{i2=0}^{2(m-s-1)-(k2+r2)} a^{i1}_{s,r1,k1}a^{i2}_{m-s-1,r2,k2}\eta^{i1+i2}\right]$$
$$= \left[\sum_{k=0}^{m+1}\exp(-k\eta\gamma)\sum_{k1=\max\{0,k-m+s\}}^{\min\{m+1,k\}}\right]\times\left[\sum_{r=0}^{m-1}\xi^{-r}\sum_{r1=\max\{0,r-m+s+1\}}^{\min\{m,r\}}\right]$$
$$\times\left[\sum_{i=0}^{2(m-1)+2k1-k+2r1-r}\eta^i\sum_{i1=\max\{0,i-2(m-s-1)+(k-k1+r-r1)\}}^{\min\{2s+k1+r1,i\}} a^{i1}_{s,r1,k1}a^{i-i1}_{m-s-1,r-r1,k-k1}\right]$$

This finally gives

$$\varphi'_s\varphi'_{m-s-1}=\left[\sum_{k=0}^{m+1}\exp(-k\eta\gamma)\sum_{r=0}^{m-1}\xi^{-r}\sum_{i=0}^{2(m-1)+2k1-k+2r1-r}\eta^i\right]\times\left[\sum_{k1=\max\{0,k-m+s\}}^{\min\{m+1,k\}}\sum_{r1=\max\{0,r-m+s+1\}}^{\min\{m,r\}}\sum_{i1=\max\{0,i-2(m-s-1)+(k-k1+r-r1)\}}^{\min\{2s+k1+r1,i\}}a^{i1}_{s,r1,k1}a^{i-i1}_{m-s-1,r-r1,k-k1}\right] \quad (49)$$

By the similar way, we can have

$$\varphi_s\varphi''_{m-s-1}=\left[\sum_{k=0}^{2m+1-s}\exp(-k\eta\gamma)\sum_{r=0}^{2m-s-1}\xi^{-r}\sum_{i=0}^{4m-2(s+1)-(k+r)}\eta^i\right]\times\left[\sum_{k1=\max\{0,k-m+s\}}^{\min\{m+1,k\}}\sum_{r1=\max\{0,r-m\}}^{\min\{m,r\}}\sum_{i1=\max\{0,i-2(m-s-1)+(k-k1+r-r1)\}}^{\min\{2(m-s-1)-(k-k1+r-r1),i\}}\lambda^{i1}_{m,r1,k1}b^{i1}_{m,r1,k1}c^{i-i1}_{m-s-1,r-r1,k-k1}\right] \quad (50)$$

$$\varphi'_s\varphi'''_{m-s-1}=\left[\sum_{k=0}^{m+1}\exp(-k\eta\gamma)\sum_{r=0}^{m-1}\xi^{-r}\sum_{i=0}^{2(m-1)-(k+r)}\eta^i\right]\times\left[\sum_{k1=\max\{0,k-m+s\}}^{\min\{s+1,k\}}\sum_{r1=\max\{0,r-m+s+1\}}^{\min\{s,r\}}\sum_{i1=\max\{0,i-2(m-s-1)+(k-k1+r-r1)\}}^{\min\{2s-(k1+r1),i\}}a^{i1}_{s,r1,k1}d^{i-i1}_{m-s-1,r-r1,k-k1}\right] \quad (51)$$

$$\varphi_s\varphi^{IV}_{m-s-1}=\left[\sum_{k=0}^{2m+1-s}\exp(-k\eta\gamma)\sum_{r=0}^{2m-s-1}\xi^{-r}\sum_{i=0}^{4m-2(s+1)-(k+r)}\eta^i\right]\times\left[\sum_{k1=\max\{0,k-m+s\}}^{\min\{m+1,k\}}\sum_{r1=\max\{0,r-m\}}^{\min\{m,r\}}\sum_{i1=\max\{0,i-2(m-s-1)+(k-k1+r-r1)\}}^{\min\{2(m-s-1)-(k-k1+r-r1),i\}}\lambda^{i1}_{m,r1,k1}b^{i1}_{m,r1,k1}w^{i-i1}_{m-s-1,r-r1,k-k1}\right] \quad (52)$$

Substitution equations 38-40 and 48-52 into (46-47), we obtain

$$G_m=h\left[\frac{1}{2}\eta(1-\xi^{-1})\varphi''_{m-1}-\xi^{-1}\varphi'''_{m-1}+M_n\varphi'_{m-1}+\sum_{s=0}^{m-1}\left[(1+K\xi^{-1})\varphi'_s\varphi'_{m-s-1}-\varphi_s\varphi''_{m-s-1}-2K\xi^{-1}\varphi'_s\varphi'''_{m-s-1}+K\xi^{-1}\varphi_s\varphi^{IV}_{m-s-1}\right]\right] \quad (53)$$

$$\begin{aligned}G_m=&\sum_{k=0}^{m}\exp(-k\gamma\eta)\sum_{i=0}^{2(m-1)-(k+r)}\eta^{i+1}A^i_{m-1,r,k}-\sum_{k=0}^{m}\exp(-k\gamma\eta)\sum_{i=0}^{2(m-1)-(k+r)}\eta^{i+1}A^i_{m-1,r-1,k}\\&+\sum_{k=0}^{m+1}\exp(-k\gamma\eta)\sum_{i=0}^{2m-(k+r)}\eta^iB^i_{m,r,k}+\sum_{k=0}^{m+1}\exp(-k\gamma\eta)\sum_{i=0}^{2(m-1)-(k+r)}\eta^iD^i_{m-1,r,k}\\&+\sum_{s=0}^{m-1}\sum_{k=0}^{m+1}\exp(-k\gamma\eta)\sum_{i=0}^{2(m-1)+2k1-k+2r1-r}\eta^i\Delta^i_{m,r,k}+\sum_{s=0}^{m-1}\sum_{k=0}^{m+1}\exp(-k\gamma\eta)\sum_{i=0}^{2(m-1)+2k1-k+2r1-r}\eta^iK\,\Delta^i_{m,r-1,k}\\&+\sum_{s=0}^{m-1}\sum_{k=0}^{m+1}\exp(-k\gamma\eta)\sum_{i=0}^{4m-2(s+1)-(k+r)}\eta^i\,\Gamma^i_{m,r,k}+\sum_{s=0}^{m-1}\sum_{k=0}^{m+1}\exp(-k\gamma\eta)\sum_{i=0}^{2(m-1)-(k+r)}\eta^i\,\Lambda^i_{m,r,k}\\&+\sum_{s=0}^{m-1}\sum_{k=0}^{m+1}\exp(-k\gamma\eta)\sum_{i=0}^{4m-2(s+1)-(k+r)}\eta^i\,\Omega^i_{m,r,k}\end{aligned} \quad (54)$$

Where

$$A^i_{m-1,r,k}=\frac{h}{2}\sum_{r=0}^{m-1}\xi^{-r}C^i_{m-1,r,k} \quad (55)$$

$$B^i_{m,r,k}=-h\sum_{r=0}^{m}d^i_{m,r,k}\xi^{-r} \quad (56)$$

$$D^i_{m-1,r,k}=-h\,M_n\sum_{r=0}^{m-1}a^i_{m-1,r,k}\xi^{-r} \quad (57)$$

$$\Delta^i_{m,r,k}=h\sum_{\mathrm{r=0}}^{\mathrm{m-1}}\sum_{k1=\max\{0,k-m+s\}}^{\min\{m+1,k\}}\sum_{\mathrm{r1=\max\{0,r-m+s+1\}}}^{\min\{m,r\}}\times\sum_{i1=\max\{0,i-2(m-s-1)+(k+k1+r-r1)\}}^{\min\{2s+k1+r1,i\}}\xi^{-r}\,a^{i1}_{s,r1,k1}\,a^{i-i1}_{m-s-1,r-r1,k-k1} \quad (58)$$

$$\Gamma^i_{m,r,k}=-h\sum_{\mathrm{r=0}}^{\mathrm{2m-s-1}}\xi^{-r}\sum_{k1=\max\{0,k-m+s\}}^{\min\{m+1,k\}}\sum_{\mathrm{r1=\max\{0,r-m\}}}^{\min\{m,r\}}\times\sum_{i1=\max\{0,2(m-s-1)+(k-k1+r-r1)\}}^{\min\{2(m-s-1)-(k-k1+r-r1),i\}}\lambda^{i1}_{m,r1,k1}\,b^{i1}_{m,r1,k1}\,c^{i-i1}_{m-s-1,r-r1,k-k1} \quad (59)$$

$$\Lambda^i_{m,r,k}=-2Kh\sum_{\mathrm{r=0}}^{\mathrm{m-1}}\xi^{-(r+1)}\sum_{k1=\max\{0,k-m+s\}}^{\min\{s+1,k\}}\sum_{\mathrm{r1=\max\{0,r-m+s+1\}}}^{\min\{s,r\}}\times\sum_{i1=\max\{0,i-2(m-s-1)+(k-k1+r-r1)\}}^{\min\{2s-(k1+r1),i\}}a^{i1}_{s,r1,k1}\,d^{i-i1}_{m-s-1,r-r1,k-k1} \quad (60)$$

$$\Omega^i_{m,r,k}=Kh\sum_{\mathrm{r=0}}^{\mathrm{2m-s-1}}\xi^{-(r+1)}\sum_{k1=\max\{0,k-m+s\}}^{\min\{m+1,k\}}\sum_{\mathrm{r1=\max\{0,r-m\}}}^{\min\{m,r\}}\times\sum_{i1=\max\{0,i-2(m-s-1)+(k-k1+r-r1)\}}^{\min\{2(m-s-1)-(k-k1+r-r1),i\}}\lambda^{i1}_{m,r1,k1}\,b^{i1}_{m,r1,k1}\,w^{i-i1}_{m-s-1,r-r1,k-k1} \quad (61)$$

In order to solve equation 54, we should first give solutions of the equation

$$y'''(\eta)-\gamma^2 y(\eta)=\eta^i \xi^{-r}\exp(-k\gamma\eta) \qquad (62)$$

The particular solution of this equation is

$$y_p=\exp(-k\gamma\eta)\sum_{j=0}^{i}\mu_{k,j,r}^{i}\,\eta^i \qquad (63)$$

Where $\mu_{k,j,r}^{i}=\dfrac{\xi^{-r}\ i!}{\gamma^2\ j!\,(k\gamma)^{i-j+1}}$

Using the solution (63) on the differential equation (54), we obtain the following general solution

$$\varphi_m-\chi_m\varphi_{m-1}=$$

$$\sum_{k=0}^{m}\sum_{i=0}^{2(m-1)-(k+r)}\exp(-k\gamma\eta)A^{i}_{m-1,r,k}\sum_{j=0}^{i+1}\mu_{k,j,r}^{i+1}\eta^j$$

$$-\sum_{k=0}^{m}\sum_{i=0}^{2(m-1)-(k+r)}\exp(-k\gamma\eta)A^{i}_{m-1,r-1,k}\sum_{j=0}^{i+1}\mu_{k,j,r}^{i+1}\eta^j$$

$$+\sum_{k=0}^{m}\sum_{i=0}^{2m-(k+r)}\exp(-k\gamma\eta)B^{i}_{m,r,k}\sum_{j=0}^{i}\mu_{k,j,r}^{i}\eta^j$$

$$+\sum_{k=0}^{m}\sum_{i=0}^{2(m-1)-(k+r)}\exp(-k\gamma\eta)D^{i}_{m-1,r,k}\sum_{j=0}^{i}\mu_{k,j,r}^{i}\eta^j$$

$$+\sum_{s=0}^{m-1}\sum_{k=0}^{m+1}\sum_{i=0}^{2(m-1)+2k1-k+2r1-r}\exp(-k\gamma\eta)\Delta^{i}_{m,r,k}\sum_{j=0}^{i}\mu_{k,j,r}^{i}\eta^j$$

$$+\sum_{s=0}^{m-1}\sum_{k=0}^{m+1}\sum_{i=0}^{2(m-1)+2k1-k+2r1-r}\exp(-k\gamma\eta)K\Delta^{i}_{m,r-1,k}\sum_{j=0}^{i}\mu_{k,j,r}^{i}\eta^j$$

$$+\sum_{s=0}^{m-1}\sum_{k=0}^{m+1}\sum_{i=0}^{4m-2(s+1)-(k+r)}\exp(-k\gamma\eta)\Gamma^{i}_{m,r,k}\sum_{j=0}^{i}\mu_{k,j,r}^{i}\eta^j$$

$$+\sum_{s=0}^{m-1}\sum_{k=0}^{m+1}\sum_{i=0}^{2(m-1)-(k+r)}\exp(-k\gamma\eta)\Lambda^{i}_{m,r,k}\sum_{j=0}^{i}\mu_{k,j,r}^{i}\eta^j$$

$$+\sum_{s=0}^{m-1}\sum_{k=0}^{m+1}\sum_{i=0}^{4m-2(s+1))-(k+r)}\exp(-k\gamma\eta)\Omega^{i}_{m,r,k}\sum_{j=0}^{i}\mu_{k,j,r}^{i}\eta^j$$

$$+C_1^m+C_2^m\exp(-\gamma\eta)+C_3^m\exp(k\gamma\eta) \qquad (64)$$

Convergence of the solution

It is noticed that the explicit analytical solution expression contain auxiliary parameter h. As pointed out by Liao [9,12], the convergence region and rate of approximations given by homotopy analysis method are strongly dependent upon h, so we have a family of solution expressions in the auxiliary parameter h, and the physical quantities also depend upon h. So, regarding h as an independent variable, it is easy to plot curves of these kinds of quantities versus h. So, if the solution is unique, all of them converge to the same value and therefore there exists a horizontal line segment in the h-curve, and if we set h any value in the horizontal line segment we quite sure that the corresponding solution series converge.

Figure 1 portrays the h-curve of the velocity profile. The range for admissible value of h for the velocity is $-0.4\le h\le 0.5$ we see that series converges in the whole region of η when $h=-0.3$, this value of h lie in the admissible range of h.

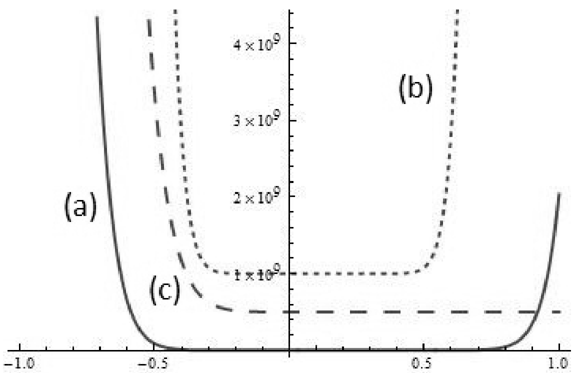

Figure 1- The h-curves of $f_{\eta\eta\eta}(0,\xi)$ obtained by the 14th-order approximation of the HAM, when(a) $M_n=1,\ \xi=1/2, K=1$
(b) $M_n=2,\ \xi=1/2, K=2$,
(c) $M_n=3,\ \xi=1/2, K=0$

Result and conclusion

The solution form of the velocity components shows that the combined parameter γ depends on the viscoelastic parameter K of the second grad fluid as well as the magnetic parameter M_n .

Effects of time ξ :

To study the effects of time ξ on the velocity distribution, we keep magnetic parameter M_n and the viscoelastic parameter K fixed at 1, 1 respectively, and we give time ξ for values $\pi/3$, $\pi/2$, π the following result is made:

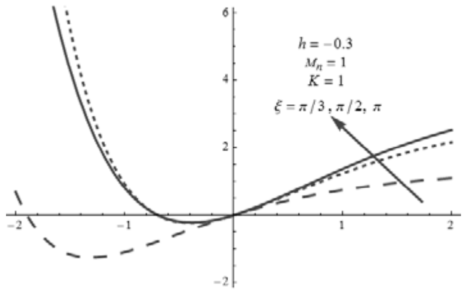

Figure 2- The approximation solution given by the 6th-order approximate solution

As time ξ increases, there is small decreasing in the velocity range. See figure. 2.

Effects of viscoelastic parameter K :

To study the effects of the viscoelastic parameter K on the velocity distribution, we keep magnetic parameter M_n and the time ξ fixed at 0.5, $\pi/4$ respectively, and we give the viscoelastic parameter K for values 0, 0.5, 1 the following result is made:

As viscoelastic parameter K increases, there is small decreasing in the velocity range. See figure 3.

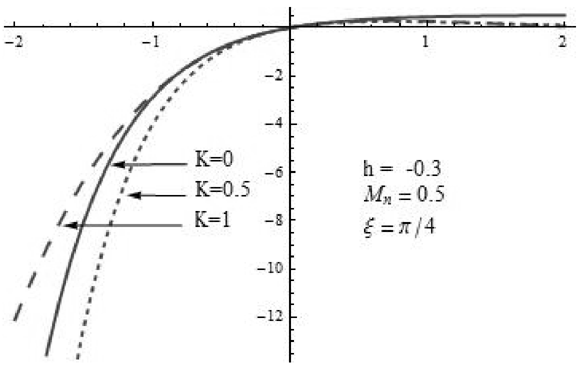

Figure 3- The approximation solution given by the 6th-order approximate solution

Effects of magnetic parameter M_n :

To study the effects of magnetic parameter M_n on the velocity distribution, we keep time ξ and the viscoelastic parameter K fixed at $\pi/4$, 0.5 respectively, and we give of magnetic parameter M_n for values 0, 0.5, 1, 2, 5 the following result is made:

As magnetic parameter M_n increases, there is small decreasing in the velocity range. See Figure. 4

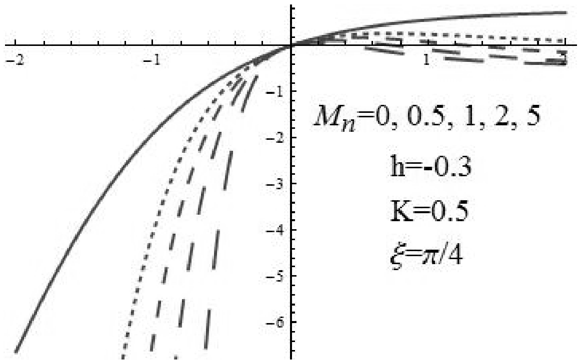

Figure 4- The approximation solution given by the 6th-order approximate solution

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