Effect of Mhd on Accelerated Flows of A Viscoelastic Fluid with The Fractional Burgers' Model

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Abstract

In this paper, we studied the effect of magnetic hydrodynamic (MHD) on accelerated flows of a viscoelastic fluid with the fractional Burgers' model. The velocity field of the flow is described by a fractional partial differential equation of fractional order by using Fourier sine transform and Laplace transform, an exact solutions for the velocity distribution are obtained for the following two problems: flow induced by constantly accelerating plate, and flow induced by variable accelerated plate. These solutions, presented under integral and series forms in terms of the generalized Mittag-Leffler function, are presented as the sum of two terms. The first term, represent the velocity field corresponding to a Newtonian fluid, and the second term gives the non-Newtonian contributions to the general solutions. The similar solutions for second grad, Maxwell and Oldroyd-B fluids with fractional derivatives, as well as, those for the ordinary models are obtained as the limiting cases of our solutions. Moreover, in the special cases when $\alpha = \beta = 1$. While the MATHEMATICA package is used to draw the figures velocity components in the plane.

Keywords: Fractional derivative, Laplace transform, Fourier transform.

تأثيـــر المجال المغناطيسي الهيدروديناميكـــي على التدفقات المتسارعة للموائع اللزجة المرنة مـــع أنمـــوذج بيركـــر

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الخلاصة

في هذا البحث, درسنا تأثير المجال المغناطيسي الهيدروديناميكي على التدفقات المتسارعة للموائع اللزجة مع أنموذج "بيركر". والذي يصف حقل سرعة التدفق بواسطة معادلة تفاضليه جزئية كسرية. استخدمنا تحويلات كل من فورير و لابلاس, للحصول على الحلول الدقيقة لتوزيع السرعة للمسألتين الا تيتين : التدفق الناجم عن لوحة التسريع الثابتة, و التدفق الناجم عن لوحة التسريع المتغيرة. هذه الحلول, كتبت بصيغة التكامل والمتسلسلات بدلالة الدالة ميتاج لفلر,كما انها تظهر بصيغة جمع للجزئين. الجزء الاول يمثل حقل السرعة للموائع النيوتونية لأداء الحركة نفسها, والجزء الثاني يمثل الاضافة الى حقل سرعة المائع النيوتوني

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وهذا لسبب كونه المائع الذي درسناه هو مائع لانيوتوني. تم الحصول على حلول مماثلة لموائع من الرتبة الثانية مثل ماكسويل, و اولدرويد من النمط بي.ذات مشتقات كسرية, بالأضافة الى ذلك, وكحالات خاصة, تم تغطيتها, هي عندما $\alpha=\beta=1$
كما كان متوقعا, حلولنا تميل الى حلول مماثلة لموائع بيركر الأولية. تم استخدام الماثيماتيكا لرسم أشكال مكونات السرعة في المستوي.

Introduction

As to the history of fractional calculus, already in 1965 L'Hospital [1] raised the question as to meaning of $d^n y/dx^n = 1/2$, that is "what if n is fractional?". "This is an apparent paradox from which, one day, useful consequences will be drawn", Leibniz [2] replied, together with "$d^{1/2}x$ will be equal to $x\sqrt{dx:x}$". Lacroix [3] was the first to mention in some two pages a derivative of arbitrary order in a 700 page text book of 1819. Thus for $y=x^a, a\in\Re_+$, he showed that $\frac{d^{1/2}y}{dx^{1/2}}=\frac{\Gamma(a+1)}{\Gamma(a+1/2)}x^{a-1/2}$. In particular he had $(d/dx)^{1/2}x=2\sqrt{x/\pi}$ (the same result as by the present day Riemann-Liouville definition below) Fourier, who in 1822 [4] derived an integral representation for $f(x)$,

$$f(x)=\frac{1}{2\pi}\int_{\Re} f(\alpha)\,d\alpha \int_{\Re}\cos p(x-\alpha)\,dp$$

obtained (formally) the derivative version

$$\frac{d^v}{dx^v}f(x)=\frac{1}{2\pi}\int_{\Re} f(\alpha)d\alpha\int_{\Re} p^v\cos\{p(x-\alpha)+\frac{v\pi}{2}\}dp,$$

where "the number v will be regarded as any quantity whatever, positive or negative". It is usually claimed that

Abel resolved in 1823 [5] the integral equation arising from the brachistochrone problem, namely

$$\frac{1}{\Gamma(\alpha)}\int_0^x \frac{g(u)}{(x-u)^{1-\alpha}}du = f(x), \quad 0<\alpha<1$$

with the solution

$$g(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x\frac{f(u)}{(x-u)^{\alpha}}du$$

As Lutzen [6] first showed, Abel never solved the problem by fractional calculus but merely showed how the solution, found by other means, could be written as a fractional derivative. Lutzen also briefly summarized what Abel actually did. Liouville [7], however, did solve the integral equation in 1832. Fractional calculus has developed especially intensively since 1974 when the first international conference in the field took place. It was organized by Betram Ross and took place at the university of New Haven, Connecticut in 1974. It had an exceptional turnout of 94 mathematicians; the proceedings contain 26 papers by the experts of the time. It was followed by the conferences conducted by Adam Mc Bride and Garry Roach (University of Strathclyde, Glasgow, Scotland) of 1989, by Katsuyuki Nishimoto (Nihon University, Tokyo, Japan) of 1989, and by Peter Rusev, Ivan Dimovski and Virginia Kiryakova (Varna, Bulgaria) of 1996. In the period 1975 to the present, about 600 papers have been published relating to fractional calculus [1].

Understanding non- Newtonian fluid flows behavior becomes increasingly important as the application of non-Newtonian fluids perpetuates through various industries, Including polymer processing and electronic packaging , paints , oils liquid polymers,glycerin ,chemical,geophysics,biorheology. However, there is no model which can alone predict the behaviors of all non-Newtonian fluids. Amongst the existing model, rate type models have special importance and many researchers are using equations of motion of Maxwell and Oldroyd-fluid flows. Khan, Hyder Ali, Haitao Qi. (2007) [2] construct the exact solutions for the accelerated flows of a generalized Oldroyd-B fluid. The fractional calculus approach is used in the constitutive relationship of a viscoelastic fluid. The velocity field and the adequate tangential stress that is induced by the flow due to constantly accelerating plate and flow due to variable accelerating plate are determined by means of discrete Laplace transform. Khan, Huder Ali, Haitao Qi. (2009) [3] Studied the accelerated flows for a viscoelastic fluid governed by the fractional Burgers' model. The velocity field of the flow is described by a fractional partial differential equation.et.al.(2011) [4] research for the magnetohydrodynamic (MHD) flow of an incompressible generalized Oldroyd-B fluid due to an infinite accelerating plate. The motion of the fluid is produced by the infinite plate, which at time $t = 0^{+}$ begins to slide in its plane with a velocity At. The solutions are established by means of Fourier sine and Laplace transforms.

Problem statement

Consideration is given to a conducting fluid permeated by an imposed magnetic field Bo which acts in the positive y- direction. In the low-magnetic Reynolds number approximation, the magnetic body force is represented by $\sigma B_{\circ}^{2}u$. Consider an incompressible fractional Burgers' fluid lying over an infinitely extended plate which is situated in the (x,z) plane. Initially, the fluid is at rest and at time $t = 0^{+}$, the infinite plate to slide in its own plane with a motion of the constant acceleration A. Owing to the shear, the fluid above the plate is gradually moved. Under these considerations, the governing equation, in the absence of pressure gradient in the flow direction, is given by

$$(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})\frac{\partial u}{\partial t}=\upsilon(1+\lambda_3^{\beta}D_t^{\beta})\frac{\partial^2 u}{\partial y^2}$$
$$-M(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})u$$

Where $\upsilon = \dfrac{\mu}{\rho}$ is the kinematics' viscosity of the fluid and $M = \dfrac{\sigma B_{\circ}^{2}u}{\rho}$.

The associated initial and boundary condition are followed

Initial condition:

$$u(y,0)=\frac{\partial u(y,0)}{\partial t}=0\ , \quad y>0$$

Boundary conditions:

$$u(0,t)=At \quad , \quad t>0$$

Moreover, the following natural conditions

$$u(y,t)\ ,\frac{\partial u(y,t)}{\partial y}\to 0 \ \ as\ y\to\infty\ and \ \ t>0$$

Have to be also satisfied. In order to solve this problem, we shall use the Fourier sine and Laplace transforms.

Solution of the problem

The constitutive equations for an incompressible fractional Burger's fluid is given by

$$T=-PI+S,(1+\lambda_1^{\alpha}\widetilde{D}_t^{\alpha}+\lambda_2^{2\alpha}\widetilde{D}_t^{2\alpha})S$$
$$=\mu(1+\lambda_3^{\beta}\widetilde{D}_t^{\beta})A \qquad (1)$$

Where T is the Cauchy stress tensor,-PI denotes the indeterminate spherical stress, S the extra stress tensor, and also S the first Rivlin-Ericksen tensor, where L is the velocity gradient,μ the dynamic viscosity of the fluid, λ_1 and λ_3 ($<\lambda_1$) the relaxation and retardation times, respectively, λ_2 is the new material parameter of the Burgers' fluid, α and β the fractional calculus parameters such that $0\le\alpha\le\beta\le 1$ and $\widetilde{D}_t^{p}$ the upper connected fractional derivative defined by

$$\widetilde{D}_t^{p}S=D_t^{p}S+v\cdot\nabla S-LS-SL^{T},$$
$$\widetilde{D}_t^{p}A=\widetilde{D}_t^{p}A+v\cdot\nabla A-LA-AL^{T} \qquad (2)$$

In which $D_t^{p}(=\partial_t^{p})$ is the fractional differentiation operator of order p with respect to t and may be defined as [5]

$$D_t^{p}[f(t)]=\frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t\frac{f(\tau)}{(1-\tau)}d\tau\ ,0\le p\le 1 \quad (3)$$

Here $\Gamma(\cdot)$ denotes the Gamma function and

$$\widetilde{D}_t^{2p}S=\widetilde{D}_T^{P}(\widetilde{D}_T^{P}S)\ , \qquad (4)$$

The equations of motion in absence of body force can be described as

$$\rho\frac{d\vec{v}}{dt}=\nabla\cdot\vec{T}, \qquad (5)$$

Where ρ is the density of the fluid and d/dt represents the material time derivative. Since the fluid is incompressible, it can undergo only is isochoric motion and hence

$$\nabla\cdot\vec{v}=0, \qquad (6)$$

For the following problems of unidirectional flow the intrinsic velocity field takes the form

$$\vec{v}=[u(y,t),0,0] \qquad (7)$$

Where $u(y,t)$ is the velocity in the x-coordinatesdirection. For this velocity field, the constraint of incompressibility (6) is automatically satisfied, we also assume that the extra stress S depends on y and t only. Substituting equation. (7) into (1), (5) and taking account of the initial conditions $S(y,0)=\partial_t S(y,0)=0, y>0$. i.e. the fluid being at rest up to time t = 0.

For the components of the stress field S, we have $S_{yy} = S_{zz} = S_{xz} = S_{yz} = 0$ and $S_{xy} = S_{yx}$, this yields

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial S_{xy}}{\partial y}, \tag{8}$$

$$(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) S_{xy} = \mu (1+\lambda_3^{\beta} D_t^{\beta}) \frac{\partial u}{\partial y} \tag{9}$$

Consider that the conducting fluid is permeated by an imposed magnetic field B_0 which acts in the positive y- direction. In the low-magnetic Reynolds number approximation, the magnetic body force is represented by $\sigma B_0^2 u$ .Then, in the absence of a pressure gradient in the x-direction, the equation of motion yields the following scalar equations:

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{xy}}{\partial y} - \sigma B_0^2 u \tag{10}$$

Where ρ is the constant density of the fluid. Eliminating S_{xy} between equations.(9) and (10), we arrive at the following fractional differential equation

$$\rho (1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\varepsilon}) \frac{\partial u}{\partial t} = -(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\varepsilon}) \frac{\partial p}{\partial x}$$

$$+ \mu (1+\lambda_3^{\beta} D_t^{\beta}) \frac{\partial^2 u}{\partial y^2} - (1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) \sigma B_0^2 u \tag{11}$$

Consider an incompressible fractional Burgers' fluid lying over an infinitely extended plate which is situated in the (x,z) plane. Initially, the fluid is at rest and at time $t = 0$ the infinite plate to slide in its own plane with a motion of the constant acceleration A. Owing to the shear, the fluid above the plate is gradually moved. Under these considerations, the governing equation, in the absence of pressure gradient in the flow direction, is given by

$$(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) \frac{\partial u}{\partial t} = \upsilon (1+\lambda_3^{\beta} D_t^{\beta}) \frac{\partial^2 u}{\partial y^2}$$

$$- M (1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) u \tag{12}$$

Where $\upsilon = \frac{\mu}{\rho}$ is the kinematics' viscosity of the fluid and $M = \frac{\sigma B_0^2 u}{\rho}$.

The associated initial and boundary condition are as follows:

Initial condition:

$$u(y,0) = \frac{\partial u(y,0)}{\partial t} = 0\ , \quad y > 0 \tag{13}$$

Boundary conditions:

$$u(0,t) = At \quad , \quad t > 0 \tag{14}$$

Moreover, the natural conditions are

$$u(y,t)\ , \frac{\partial u(y,t)}{\partial y} \to 0 \ \ as \ \ y \to \infty \ \ and \ \ t > 0 \tag{15}$$

Have to be also satisfied. In order to solve this problem, we shall use the Fourier sine and Laplace transforms.

Employing the non-dimensional quantities

$$U = \frac{u}{(\upsilon A)^{1/3}}\ , \ \eta = y(\frac{A}{\upsilon^2})^{1/3}\ , \ \tau = t(\frac{A^2}{\upsilon})^{1/3}\ ,$$

$$\hat{\lambda}_1 = \lambda_1 (\frac{A^2}{\upsilon})^{1/3}\ , \ \hat{\lambda}_2 = \lambda_2 (\frac{A^4}{\upsilon^2})^{1/3}$$

$$and \ \hat{\lambda}_3 = \lambda_3 (\frac{A^2}{\upsilon})^{1/3} \tag{16}$$

Eqs. (12) - (15) in dimensionless form are

$$(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) \frac{\partial U}{\partial \tau} = (1+\lambda_3^{\beta} D_t^{\beta}) \frac{\partial^2 U}{\partial \eta^2}$$

$$- M (1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) U \tag{17}$$

$$U(\eta,0) = \frac{\partial U(\eta,0)}{\partial \tau} = \frac{\partial^2 U(\eta,0)}{\partial \tau^2} = 0\ , \ \eta > 0 \tag{18}$$

$$U(0,\tau) = \tau \ \ , \tau > 0 \tag{19}$$

$$U(\eta,\tau)\ , \ \frac{\partial U(\eta,\tau)}{\partial \eta} \to 0\ , \ \ as \ \eta \to \infty$$

$$and \ \tau \succ 0 \tag{20}$$

Where the dimensionless mark that has been omitted for simplicity.

Now, applying Fourier sine transform [6] to equations.(17) and taking into account the boundary conditions (19) and (20), we find that

$$(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) \frac{\partial U_S(\xi,\tau)}{\partial \tau}$$

$$= (1+\lambda_3^{\beta} D_t^{\beta}) \ (\sqrt{\frac{2}{\pi}}\ \zeta \tau - \xi^2 U_s(\xi,\tau)\ )$$

$$- M (1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha})\ U_s(\xi,\tau) \tag{21}$$

Where the Fourier sine transform $U_s(\xi,\tau)\ of\ U(\eta,t)$ has to satisfy the conditions

$$U_s(\xi,0) = \frac{\partial U_s(\xi,0)}{\partial \tau} = \frac{\partial^2 U_s(\xi,0)}{\partial \tau^2} = 0\ ; \quad \xi > 0. \tag{22}$$

Let $\overline{U}_s(\xi,s)$ be the Laplace transform of $U_s(\xi,\tau)$ defined by

$$\overline{U}_s(\xi,s)=\int_0^\infty U_s(\xi,\tau)\exp(-st)\,d\tau,\quad s>0. \qquad (23)$$

Taking the Laplace transform of equation.(21), having in mind the initial conditions (22), we get

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}\frac{\xi(1+\lambda_3^\beta s^\beta)}{s^2}\times \frac{1}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_2^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^{2\alpha}s^{2\alpha})} \qquad (24)$$

In order to obtain $U_s(\xi,\tau)=L^{-1}\{\overline{U}_s(\xi,s)\}$ with L^{-1} as the inverse Laplace transform operator and to avoid the lengthy procedure of residues and contour integral, we apply the discrete Laplace transform method. However, for a more suitable presentation of the final results, we rewrite equation. (24) in the equivalent form

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}\frac{\xi(1+\lambda_3^\beta s^\beta)(s+\xi^2)}{s^2}\times\left(\frac{1}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_2^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^{2\alpha}s^{2\alpha})}\right)$$

$$=\sqrt{\frac{2}{\pi}}\frac{\xi(s+\xi^2)+\xi\lambda_3^\beta s^\beta(s+\xi^2)}{s^2(s+\xi^2)}\times\frac{1}{(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_2^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^{2\alpha}s^{2\alpha})}$$

By adding and subtracting the quantities $\xi\lambda_1^\alpha s^{\alpha+1},\xi\lambda_2^\alpha s^{2\alpha+1},\xi M,\xi M\lambda_1^\alpha s^\alpha$ *and* $\xi M\lambda_2^\alpha s^{2\alpha}$.

We get:

$$=\sqrt{\frac{2}{\pi}}\frac{1}{s^2(s+\xi^2)}\times\frac{\xi s+\xi^3+\xi\lambda_3^\beta s^{\beta+1}+\xi^3\lambda_3^\beta s^\beta\pm\xi\lambda_1^\alpha s^{\alpha+1}\pm\xi\lambda_2^\alpha s^{2\alpha+1}\pm\xi M\pm\xi M\lambda_1^\alpha s^\alpha\pm\xi M\lambda_2^\alpha s^{2\alpha}}{(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_2^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^{2\alpha}s^{2\alpha})}$$

$$=\sqrt{\frac{2}{\pi}}(\xi(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M$$
$$+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})-s^2\xi(\lambda_1^\alpha s^{\alpha-1}+\lambda_2^\alpha s^{2\alpha-1}-\lambda_3^\beta s^{\beta-1}$$
$$+Ms^{-2}+M\lambda_1^\alpha s^{\alpha-2}+M\lambda_2^\alpha s^{2\alpha-2})$$
$$\times\frac{1}{s^2(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}\frac{\xi}{s^2(s+\xi^2)}-\frac{\xi(\lambda_1^\alpha s^{\alpha-1}+\lambda_2^\alpha s^{2\alpha-1}-\lambda_3^\beta s^{\beta-1}+Ms^{-2}+M\lambda_1^\alpha s^{\alpha-2}+M\lambda_2^\alpha s^{2\alpha-2})}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}\frac{s+\xi^2-s}{\xi s^2(s+\xi^2)}-\frac{\xi(\lambda_1^\alpha s^{\alpha-1}+\lambda_2^\alpha s^{2\alpha-1}-\lambda_3^\beta s^{\beta-1}+Ms^{-2}+M\lambda_1^\alpha s^{\alpha-2}+M\lambda_2^\alpha s^{2\alpha-2})}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^2}-\frac{1}{\xi s(s+\xi^2)}-\frac{\xi(\lambda_1^\alpha s^{\alpha-1}+\lambda_2^\alpha s^{2\alpha-1}-\lambda_3^\beta s^{\beta-1}+Ms^{-2}+M\lambda_1^\alpha s^{\alpha-2}+M\lambda_2^\alpha s^{2\alpha-2})}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^2}-\frac{\xi^2}{\xi^3 s(s+\xi^2)}-\frac{\xi(\lambda_1^\alpha s^{\alpha-1}+\lambda_2^\alpha s^{2\alpha-1}-\lambda_3^\beta s^{\beta-1}+Ms^{-2}+M\lambda_1^\alpha s^{\alpha-2}+M\lambda_2^\alpha s^{2\alpha-2})}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^2}-(\frac{s+\xi^2-s}{s(s+\xi^2)})\frac{1}{\xi^3}-\frac{\xi(\lambda_1^\alpha s^{\alpha-1}+\lambda_2^\alpha s^{2\alpha-1}-\lambda_3^\beta s^{\beta-1}+Ms^{-2}+M\lambda_1^\alpha s^{\alpha-2}+M\lambda_2^\alpha s^{2\alpha-2})}{(s+\xi^2)(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})})$$

$$=\sqrt{\frac{2}{\pi}}(\frac{1}{\xi s^2}-(\frac{1}{s}-\frac{1}{s+\xi^2})\frac{1}{\xi^3}-\frac{\xi(\lambda_1^\alpha s^{\alpha-1}+\lambda_2^\alpha s^{2\alpha-1}-\lambda_3^\beta s^{\beta-1}+Ms^{-2}+M\lambda_1^\alpha s^{\alpha-2}+M\lambda_2^\alpha s^{2\alpha-2})}{(s+\xi^2)}) \qquad (25)$$
$$\times\frac{1}{(s+\lambda_1^\alpha s^{\alpha+1}+\lambda_2^\alpha s^{2\alpha+1}+\xi^2+\xi^2\lambda_3^\beta s^\beta+M+M\lambda_1^\alpha s^\alpha+M\lambda_2^\alpha s^{2\alpha})}$$

Hence, the Eq.(25) can be written under the form of a series as

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}\{[\frac{1}{\xi s^2}-(\frac{1}{s}-\frac{1}{s+\xi^2})\frac{1}{\xi^3}]$$
$$-\xi(\lambda_1^\alpha s^{\alpha-1}+\lambda_2^\alpha s^{2\alpha-1}-\lambda_3^\beta s^{\beta-1}+Ms^{-2}$$
$$+M\lambda_1^\alpha s^{\alpha-2}+M\lambda_2^\alpha s^{2\alpha-2})\sum_{m=0}^{\infty}(-1)^m$$
$$\frac{\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}}{(s+\xi^2)(s^{\alpha+1}+\frac{1}{\lambda_1^\alpha}(\xi^2+M))^{m+1}}\}$$
$$\lambda_1^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}\ M^{j-d}\ \xi^{2d}\ m!\,s^\delta \qquad (26)$$

In which $\delta=m+2\alpha l-j-\alpha i+\beta d-\alpha d$.

Now, applying the inversion formula term by term for the Laplace transform, equation.(26) yields

$$U_s(\xi,\tau)=\sqrt{\frac{2}{\pi}}\,[\frac{\tau}{\xi}-\frac{1}{\xi^3}\,(1-\exp(-\xi^2\tau)]$$
$$-\int_0^\tau \xi\sqrt{\frac{2}{\pi}}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}$$
$$\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}$$
$$\lambda_1^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}M^{\,j-d}\ \xi^{2d}$$
$$\times[\lambda_1^{\alpha}\sigma^{(\alpha+1)m+(2-\delta)-1}\ E^{(m)}_{(\alpha+1),(2-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})$$
$$+\lambda_2^{\alpha}\ \sigma^{(\alpha+1)m+(2-\alpha-\delta)-1}\ E^{(m)}_{(\alpha+1),(2-\alpha-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\ \sigma^{\alpha+1})$$
$$-\lambda_3^{\beta}\ \sigma^{(\alpha+1)m+(2+\alpha-\beta-\delta)-1}E^{(m)}_{(\alpha+1),(2+\alpha-\beta-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})$$
$$+M\ \sigma^{(\alpha+1)m+(3-\delta+\alpha)-1}\ E^{(m)}_{(\alpha+1),(3-\delta+\alpha)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})$$
$$+M\lambda_1^{\alpha}\sigma^{(\alpha+1)m+(3-\delta)-1}\ E^{(m)}_{(\alpha+1),(3-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})$$
$$+M\lambda_2^{\alpha}\sigma^{(\alpha+1)m+(3-\alpha-\delta)-1}E^{(m)}_{(\alpha+1),(3-\alpha-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})]$$
$$*\exp(-\xi^2(\tau-\sigma)]\,d\sigma \qquad (27)$$

Where " $*$ " represents the convolution of two functions and

$$E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\lambda n+\mu)}\,,\quad \lambda,\mu>0, \qquad (28)$$

Denotes the generalized Mittag-Leffler function with

$$E^{(k)}_{\lambda,\mu}(z)=\frac{d^k}{dz^k}\ E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{(n+k)!z^n}{n!\Gamma(\lambda n+\lambda k+\mu)} \qquad (29)$$

Here, we used the following property of the generalized Mittad-Leffler function

$$L^{-1}\{\frac{k!s^{\lambda-\mu}}{(s^{\lambda}\mp c)^{k+1}}\}=t^{\lambda k+\mu-1}E^{(k)}_{\lambda,\mu}\ (\mp ct^{\lambda}),$$
$$(\mathrm{Re}(s)\succ |c|^{1/\lambda}). \qquad (30)$$

Finally, inverting (27) by the Fourier transform we find for the velocity $U(\xi,\tau)$ the expression

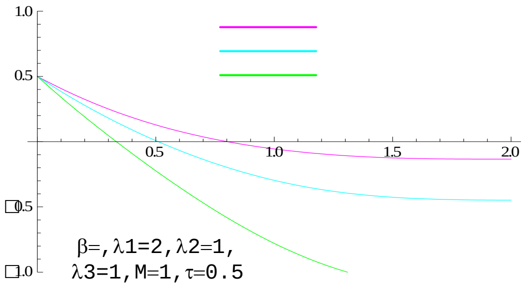
$$U(\eta,\tau)=U_N(\eta,\tau)-\frac{2}{\pi}\int_0^{\infty}\int_0^{\tau}\frac{\sin(\xi\eta)}{\xi}\sum_{m=0}^{\infty}(-1)^m$$
$$\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}$$
$$\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\lambda^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}$$
$$M^{\,j-d}\ \xi^{2d}\ \times[\lambda_1^{\alpha}\,\sigma^{(\alpha+1)m+(2-\delta)-1}$$
$$\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(2-\delta))}$$
$$+\lambda_2^{\alpha}\,\sigma^{(\alpha+1)m+(2-\alpha-\delta)-1}$$
$$\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(2-\alpha-\delta)}$$
$$+\lambda_3^{\beta}\,\sigma^{(\alpha+1)m+(2+\alpha-\beta-\delta)-1}$$
$$\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(2+\alpha-\beta-\delta))}$$
$$+M\,\sigma^{(\alpha+1)m+(3-\delta+\alpha)-1}$$
$$\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(3-\delta+\alpha))}$$
$$+M\,\lambda_1^{\alpha}\,\sigma^{(\alpha+1)m+(3-\delta)-1}$$
$$\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(3-\delta))}$$
$$+M\,\lambda_2^{\alpha}\,\sigma^{(\alpha+1)m+(3-\alpha-\delta)-1}$$
$$\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(3-\alpha-\delta))}]$$
$$*\exp(-\xi^2(\tau-\sigma)\,d\sigma d\xi \qquad (31)$$

Whence,

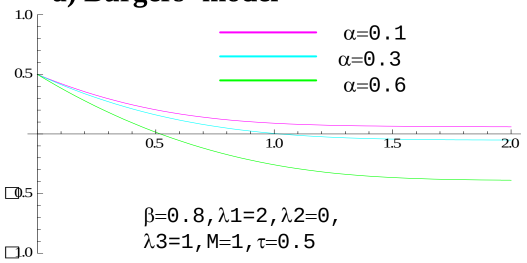
$$U_N(\eta,\tau)=\tau-\frac{2}{\pi}\int_0^{\infty}(1-\exp(-\xi^2\tau)\frac{\sin(\xi\eta)}{\xi^3}d\xi$$
$$=4\tau i^2\,Erfc(\frac{\eta}{2\sqrt{\tau}}), \qquad (32)$$

Represents the velocity field corresponding to a Newtonian fluid performing the same motion.

In the above relation $i^n\,Erfc(.)$ are the integrals of the complementary error function of Gauss.

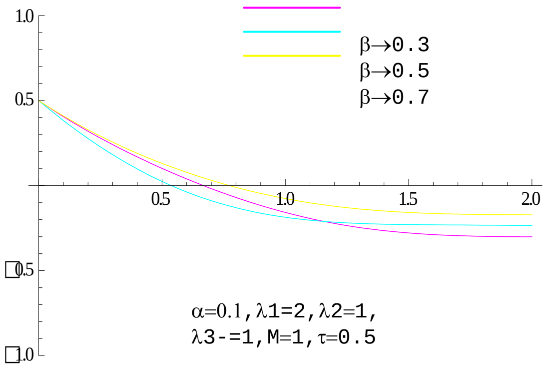


a) Burgers' model

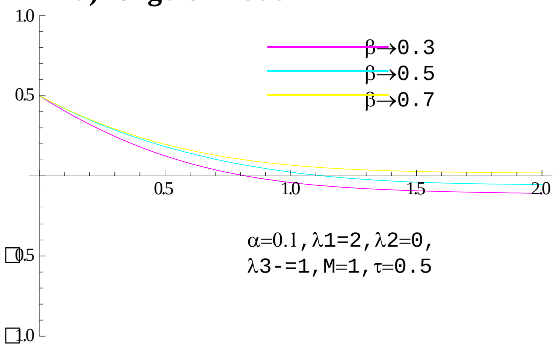


b) Oldroyd-B fluid

Figure 1.- Velocity $U(\eta,\tau)$ versus η for different values of α when other parameters are fixed.



a)Burgers' model



b)Oldroyd-B fluid

Figure 2- Velocity $U(\eta,\tau)$ versus η for different values of β when other parameters are fixed.

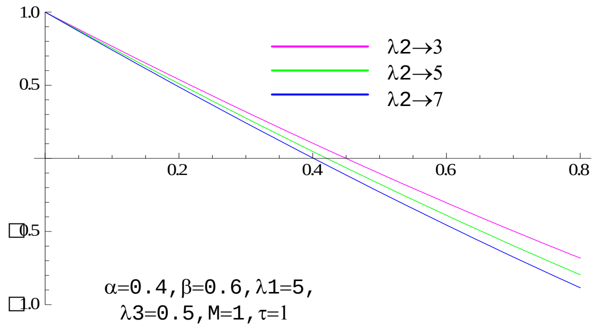


Figure 3- Velocity $U(\eta,\tau)$ versus η for different values of λ_2 when other parameters are fixed.

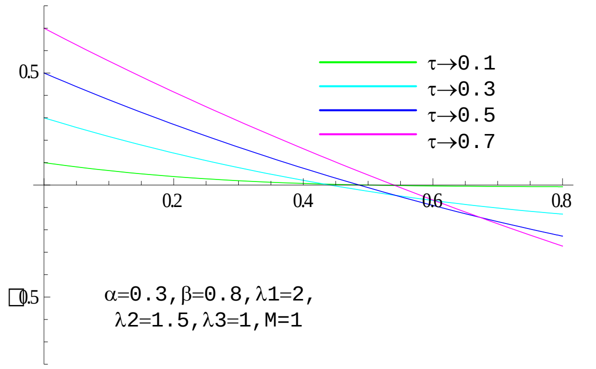


Figure 4- Velocity $U(\eta,\tau)$ versus η for different values of τ when other parameters are fixed.

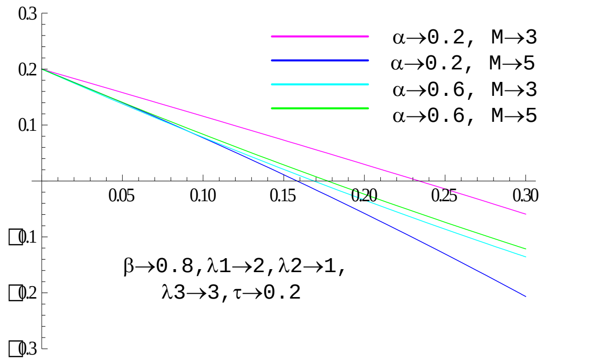


Figure 5- Velocity $U(\eta,\tau)$ versus η for different values of α, M when other parameters are fixed.

Results and discussion

We interpret these results with respect to the variation of emerging parameters of interest. The exact analytical solutions for accelerated flows have been obtained for a Burgers' fluid and a comparison is made with the results for those of the fractional Oldroyd-B fluid.

Figure 1 is prepared to show the effects of non-integer fractional parameters α on the velocity field, as well as a comparison between the fractional Oldroyd-B fluid and fractional Burgers' fluid for fixed values of other parameters. As seen from this figures that for time τ =0.5 the smaller the α, the more slowly the velocity decays for both the fluids. Moreover, for time τ =0.5 the velocity profiles for an Oldroyd-B fluid are greater than those for a Burgers' fluid. Its also observed that for time τ =0.5 the velocity profiles for Burgers' fluids approach the velocity profile of the fractional Oldroyd-B fluid and after some time it will become the same. Thus, it's obvious that the relaxation and retardation times and the orders of the fractional parameters have strong effects on the velocity field.

Figure 2 is prepared to show the effects of non-integer fractional parameters β on the velocity field, as well as a comparison between the fractional Oldroyd-B fluid and fractional Burgers' fluid for fixed values of other parameters. It is observed that for time τ =0.5 the velocity will increase by the increase in the parameter β. It is also observed that for time τ =0.5 the velocity profiles for Burgers' fluids approach the velocity profile of the fractional Oldroyd-B fluid and after some time it will become the same.

Figure 3 shows the effects of new material parameter on the velocity field for fixed values of other parameters. It is observed that for time τ =1 the velocity will decrease by the increase in new material parameter λ_2.

Figure 4 shows the variation of time on the velocity field for fixed values of other parameters. It's observed that the velocity will increase by the increase in time and after some time it will become the same.

Figure 5 shows the velocity changes with the fractional parameters and the magnetic field parameter. It is observed that for $\alpha \le 0.2$ the velocity will decrease by the increase in the magnetic field M. However, one can see that an increase in the magnetic field M for $\alpha \le 0.6$ has quite the opposite effect to that of $\alpha \le 0.2$.

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