Stability Analysis with Bifurcation of an *SVIR* Epidemic Model Involving Immigrants

Raid .K. Naji and Ahmed.A. Muhseen*
Department of mathematics, College of science, University of Baghdad. Baghdad,Iraq.

Abstract
There are many factors effect on the spread of infectious disease or control it, some of these factors are (*immigration and vaccination).* The main objective of this paper is to study the effect of those factors on the dynamical behavior of an *SVIR* model. It is assumed that the disease is spread by contact between members of populations individuals. While the recovered individuals gain permanent immunity against the disease. The existence, uniqueness and boundedness of the solution of this model are investigated. The local and global dynamical behaviors of the model are studied. The local bifurcations and Hopf bifurcation of the model are investigated. Finally, in order to confirm our obtained results and specify the effects of model's parameters on the dynamical behavior, numerical simulation of the *SVIR* model is performed.

Keywords: Epidemic models, Stability, Vaccinated, Immigrants, Local and Hopf bifurcation.

تحليل الاستقرارية مع التفرع للنظام الوبائي SVIR مع المهاجرين

رائد كامل ناجي و احمد علي محسن*
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق.

الخلاصة
هناك العديد من العوامل التي تؤثر على انتشار الامراض المعدية والتحكم بها, وبعض هذه العوامل هي المهاجرين والتطعيم. الهدف الرئيسي من البحث هو دراسة تأثير هذه العوامل على السلوك الديناميكي للنموذج *SVIR*. افترضنا انتشار المرض عن طريق الاتصال المباشر بين افراد المجتمع مع اكتساب الأفراد المتعافين مناعة دائمية ضد المرض. تمت مناقشة وجود و وحدانية وقيود الحل للأنموذج المقترح. قمنا بدراسة السلوك المحلي والشامل له. كذلك بحثنا التفرعات المحلية وتفرع هوبف. واخيرا من اجل تأكيد نتائجنا وتحديد تأثير معلمات الأنوذج الوبائي *SVIR* على السلوك الديناميكي له اجرينا محاكاة عددية له.

*Email:aamuhseen@gmail.com

1. Introduction

It is well known that infectious diseases have tremendous influence on human life and hence controlling these diseases is very important issue. Consequently, many epidemic models, which used mathematics for describing the evolution of infectious diseases within the populations, are constructed and investigated in literature, see for example ref. [1] and the references their in. These models are different from each other in their formulation depending on the type of transmission of disease, latent period, resistance, immigrants, vaccination and many other factors. The existence of infectious disease divided the population into many compartments, depending on the type of disease, such as susceptible (*S*), infected (*I*), removal (*R*), vaccinated (*V*) and others. Since the well known epidemic model *SIR*, which proposed originally by Kermark and Mckendrick in 1927 [2], many mathematical models are formulated to describe the spread of infectious disease using the frame work of Kermark-Mckendrick model. Kribs-Zaleta and Velasco-Hernandez in 2000 [3] proposed and studied the *SIS* epidemic model with vaccine for the diseases such as pertussis and tuberculosis, later on Arino et al. [4], generalized this model by allowing individuals recovering from the diseases to go into a temporarily immune class rather than directly back in to the susceptible class. Kribs-Zaleta and Martcheva [5] investigated the effects of a vaccination campaign upon the spread of a non-fatal diseases such as Hepatitis A, B. Alexander et al. [6] and Shim [7] are discussing the transmission dynamics of influenza with vaccination through using SVIR models. d' Onofrio et al. [8] gave a family of models for information related vaccinating behavior.

On the other hand, it is well known that immigrants play a critical role in disease dynamics see for example ref.[9-11] and the references their in. Keeping the above in view, *Shim* [10], proposed and studied an *SVIR* epidemic model with the existence of constant flow of incoming immigrant, in his model it is assumed that the disease transmitted between the compartments depending on simple mass action incident rate. He discussed the local stability analysis through eliminating the susceptible compartment from the model. In this paper however the *Shim* model is modified so that the disease transmitted between the compartments depending on the standard mass action incident rate which is more biologically realistic, instead of simple mass action incident rate. The local as well as global stability analysis of the modified model is investigated. Also, the local bifurcations, as well as, Hopf bifurcation are discussed.

2. The mathematical model

Consider a simple *SIR* disease transmission model involving a constant birth rate $\Lambda > 0$ in the susceptible class with a proportional natural death rate $\mu > 0$ in each class while there is no death caused by disease. This model can be written as follows:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \beta SI - \mu S \\ \frac{dI}{dt} &= \beta SI - (\mu + \alpha) I \\ \frac{dR}{dt} &= \alpha I - \mu R \end{aligned} \qquad (1)$$

Here *S(t)* , *I(t)* and *R(t)* represent the number of susceptible individuals, infected individuals and removal individuals at time t respectively and hence the total number of population at time t is $N = S(t) + I(t) + R(t)$. Further, in model (1) it is assumed that the disease transmitted from class S to class I by contact according to simple mass action interaction between them with infection rate constant $\beta > 0$, Finally, $\alpha > 0$ represents the recovery rate constant.

Now, by assuming, there is a constant flow, say $A > 0$, of a new members arriving into the population in unit time with the fraction p of A arriving infected ($0 \le p \le 1$). Also, since the number of contacts between the susceptible and infected depends on the total number of population *N*, hence we will use standard mass action interaction for describing the direct contact between *S* and *I* instead of simple mass action interaction. Therefore the above system (1) can be rewritten as follows:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda + (1-p)A - \frac{\beta SI}{N} - \mu S \\ \frac{dI}{dt} &= pA + \frac{\beta SI}{N} - (\mu + \alpha) I \\ \frac{dR}{dt} &= \alpha I - \mu R \end{aligned} \qquad (2)$$

Keeping the above in view, in order to study the effect of vaccination on the system (2) the following assumptions are made:

❖ The susceptible class is vaccinated at per capita rate $\psi > 0$ and then the number of vaccinated individuals at time t can be represented by *V(t).*

❖ The infection can invade the susceptible class or vaccinated class depending on vaccine efficiency.

❖ The vaccine reduces the possibility of infection by a factor of σ, where $0 \le \sigma \le 1$.

❖ Varying the losing vaccine immunity rate (failure in vaccine), that is wears off at the per capita rate $0 \le \theta \le 1$.

Accordingly, the flow of disease in system (2) along with the above assumptions can be representing in the following block diagram:

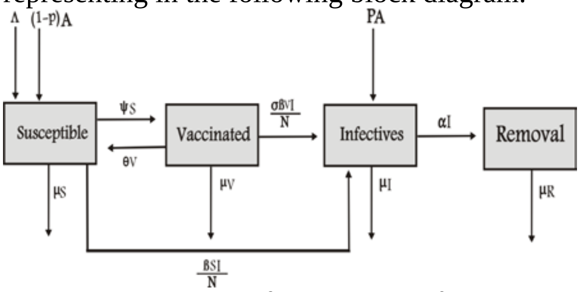


Figure 1- *Block diagram of system (3).*Therefore system (2) can be modified to:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda + (1-p)A - \frac{\beta SI}{N} - (\mu+\psi)S + \theta V \\
\frac{dV}{dt} &= \psi S - \frac{\sigma\beta VI}{N} - (\mu+\theta)V \\
\frac{dI}{dt} &= pA + \frac{\beta SI}{N} + \frac{\sigma\beta VI}{N} - (\mu+\alpha)I \\
\frac{dR}{dt} &= \alpha I - \mu R
\end{aligned} \qquad (3)$$

Clearly for $\sigma = 0$ the vaccine is completely affective. While, $\sigma = 1$ stand for the situation where the vaccine is totally ineffective. On the other hand, $\theta = 0$ denotes to the case when immunity is life-long while $\theta = 1$ corresponds to the case where there is absolutely no vaccine induced immunity. Therefore the total number population becomes $N = S(t) + V(t) + I(t) + R(t)$.

Obviously, due to the biological meaning of the variables *S(t), V(t), I(t), and R(t),* system (3) has the domain

$$\Re_+^4 = \left\{(S,V,I,R) \in \Re^4, S \ge 0, V \ge 0, I \ge 0, R \ge 0\right\}$$

which is positively invariant for system (3).

Further, all the solutions of system (3) with non-negative initial conditions are uniformly bounded as it is proved in the following theorem.

Theorem 1: All the solutions of system (3), which are initiate in $\Re_+^4$, are uniformly bounded.

Proof:

Let (*S(t), V(t), I(t), R(t)*) be any solution of the system (3) with non-negative initial condition (*S(0), V(0), I(0), R(0)*), since *N(t) = S(t) + V(t) + I(t) + R(t),* then :

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dV}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$$

which gives

$$\frac{dN}{dt} + \mu N = \Lambda + A$$

Now, by solving the above linear differential equation, we get that the total population is asymptotically constant by:

$$N(t) = \frac{\Lambda + A}{\mu}$$

Hence all the solution of system (3) that initiate in $\Re_+^4$, are confined in the region:

$$\Omega = \left\{(S,V,I,R) \in \Re_+^4 : N \le \frac{\Lambda + A}{\mu} + \varepsilon;\ \varepsilon > 0\right\} \qquad ■$$

3. Existence of equilibrium points of system (3)

In this section, we discuss the existence of all possible equilibrium points of the system (3). Now, since the removal class R is related with infected class only, hence knowing the value of I leads directly to determine the value of R from solving the fourth equation in system (3). In fact, if $I = 0$ then R approaches to zero asymptotically. However, if $I = I_c$, where I_c is a positive constant, then R approaches to:

$$R = \frac{\alpha I_c}{\mu} \qquad (4)$$

Consequently, the first three equations of system (3) those given below, will be analyzed and then equation 4 can be used to give the value of R.

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda + (1-p)A - \frac{\beta SI}{N} - (\mu+\psi)S + \theta V = f_1(S,V,I) \\
\frac{dV}{dt} &= \psi S - \frac{\sigma\beta VI}{N} - (\mu+\theta)V = f_2(S,V,I) \\
\frac{dI}{dt} &= pA + \frac{\beta SI}{N} + \frac{\sigma\beta VI}{N} - (\mu+\alpha)I = f_3(S,V,I)
\end{aligned} \qquad (5)$$

Clearly if $I = 0$, (in this case $p = 0$) then the system (5) has an equilibrium point called a disease free equilibrium point and denoted by $E^{\circ}=(S^{\circ},V^{\circ},0)$ where:

$$\left.\begin{aligned} S^{\circ} &= \frac{(\Lambda+A)(\mu+\theta)}{\mu(\mu+\theta+\psi)} \\ V^{\circ} &= \frac{\psi(\Lambda+A)}{\mu(\mu+\theta+\psi)} \end{aligned}\right\} \quad (6)$$

However, if $I \neq 0$ then system (5) has an endemic equilibrium point denoted by $E^{*}=(S^{*},V^{*},I^{*})$ where S^{*},V^{*} and I^{*} represent the positive solution for the following equations:

$$\begin{aligned} &\Lambda+(1-p)A-\frac{\beta SI}{N}-(\mu+\psi)S+\theta V=0 \\ &\psi S-\frac{\sigma\beta VI}{N}-(\mu+\theta)V=0 \\ &pA+\frac{\beta SI}{N}+\frac{\sigma\beta VI}{N}-(\mu+\alpha)I=0 \end{aligned} \quad (7)$$

Now, from the second equation of (7) we get:

$$V=\frac{N\psi S}{\sigma\beta I+N(\mu+\theta)}$$

Substituting in the first equation of system (7) and then doing some computations give that:

$$\left.\begin{aligned} S^{*} &= \frac{N[\Lambda+(1-p)A]\left[\sigma\beta I^{*}+N(\mu+\theta)\right]}{\sigma\beta^{2}I^{*2}+\beta N[(\mu+\theta)+\sigma(\mu+\psi)]I^{*}+\mu N^{2}(\mu+\psi+\theta)} \\ V^{*} &= \frac{N^{2}\psi[\Lambda+(1-p)A]}{\sigma\beta^{2}I^{*2}+\beta N[(\mu+\theta)+\sigma(\mu+\psi)]I^{*}+\mu N^{2}(\mu+\psi+\theta)} \end{aligned}\right\} \quad (8)$$

While I^{*} is a positive root for the following equation

$$D_1 I^3+D_2 I^2+D_3 I+D_4=0 \quad (9)$$

here:

$$\begin{aligned} D_1 &= -\sigma\beta^{2}N(\mu+\alpha)<0 \\ D_2 &= N\beta\big(\sigma\beta(\Lambda+A)-N(\mu+\alpha)[(\mu+\theta+\sigma(\mu+\psi)]\big) \\ D_3 &= N^{2}\big(\beta[pA\sigma\mu+(\Lambda+A)(\mu+\theta+\sigma\psi] \\ &\qquad -N\mu(\mu+\alpha)(\mu+\psi+\theta)\big) \\ D_4 &= N^{3}pA\mu(\mu+\psi+\theta)>0 \end{aligned}$$

Clearly equation 9 has a unique positive root given by I^{*} if and only if one of the following conditions hold.

$$\sigma\beta(\Lambda+A)<N(\mu+\alpha)[(\mu+\theta)+\sigma(\mu+\psi] \quad (10a)$$

or

$$\beta[pA\sigma\mu+(\Lambda+A)(\mu+\theta+\sigma\psi)]>N\mu \times(\mu+\alpha)(\mu+\psi+\theta) \quad (10b)$$

4. Local Stability analysis

In this section, the local stability analysis of the equilibrium points E° and E^{*} of system (5) is studied as shown in the following theorems.

Theorem 2: The disease free equilibrium point $E^{\circ}=(S^{\circ},V^{\circ},0)$ of system (5) is locally asymptotically stable provided that:

$$\frac{\beta(S^{\circ}+\sigma V^{\circ})}{N(\mu+\alpha)}<1 \quad (11a)$$

While it is a saddle point provided that:

$$\frac{\beta(S^{\circ}+\sigma V^{\circ})}{N(\mu+\alpha)}>1 \quad (11b)$$

Proof: The Jacobian matrix of system (5) at (E°) can be written as:

$$J(E^{\circ})=\begin{bmatrix} -(\mu+\psi) & \theta & \frac{-\beta S^{\circ}}{N} \\ \psi & -(\mu+\theta) & \frac{-\sigma\beta V^{\circ}}{N} \\ 0 & 0 & \frac{\beta S^{\circ}+\sigma\beta V^{\circ}}{N}-(\mu+\alpha) \end{bmatrix}$$

Then the characteristic equation of the Jacobian matrix *J(*E°*)* is given by :

$$\left(\frac{\beta(S^{\circ}+\sigma V^{\circ})}{N}-(\mu+\alpha)-\lambda\right)\left[\lambda^{2}+H\lambda+B\right]=0 \quad (12)$$

Where:

$$\left.\begin{aligned} H &= [(\mu+\psi)+(\mu+\theta)]>0 \\ B &= \mu(\mu+\theta+\psi)>0 \end{aligned}\right\} \quad (13)$$

Consequently equation 12 have the following roots (eigenvalues) of $J(E^{\circ})$:

$$\lambda_{S,V}=\frac{-H}{2}\pm\frac{1}{2}(\psi+\theta)<0 \quad (14)$$

$$\lambda_{I}=\frac{\beta(S^{\circ}+\sigma V^{\circ})}{N}-(\mu+\alpha) \quad (15)$$

Where λ_S,λ_V and λ_I describe the dynamics in the S , V and I direction respectively. Clearly λ_S and λ_V are negative. However the third eignvalue in the *I*-direction λ_I is negative or positive depending on conditions (11a) and (11b) respectively.

Therefore, E° is asymptotically stable equilibrium point provided that condition (11a) holds, while it is saddle point provided that condition (11b) holds and hence the proof is complete. ■

Theorem 3

Assume that, The endemic equilibrium point $E^{*}=(S^{*},V^{*},I^{*})$ of system (5) exists then it is locally asymptotically stable provided the:

$$\frac{\beta(S^{*}+\sigma V^{*})}{N}<(\mu+\alpha) \quad (16a)$$

$$\frac{\beta \mathrm{M}}{N}<2(\theta+\sigma\psi)(\mu+\alpha) \quad (16b)$$

Where $\mathrm{M} = \left[S^*(2\theta + 3\sigma\psi) + \sigma V^*(3\theta + 2\sigma\beta)\right]$.

Proof :

The Jacobian matrix of system (5) at the endemic equilibrium point E^* can be written :

$$J(E^*) = \begin{bmatrix} -\frac{\beta I^*}{N} - (\mu+\psi) & \theta & \frac{-\beta S^*}{N} \\ \psi & -\frac{\sigma\beta I^*}{N} - (\mu+\theta) & \frac{-\sigma\beta V^*}{N} \\ \frac{\beta I^*}{N} & \frac{\sigma\beta I^*}{N} & \frac{\beta S^* + \sigma\beta V^*}{N} - (\mu+\alpha) \end{bmatrix}$$

$$= \left[b_{ij}\right]_{3\times3}$$

Then the characteristic equation of Jacobian matrix is given by:

$$\lambda^3 + \Omega_1\lambda^2 + \Omega_2\lambda + \Omega_3 = 0 \qquad (17)$$

here:

$$\Omega_1 = -\left[b_{11} + b_{22} + b_{33}\right]$$

$$= \left(\frac{\beta I^*}{N} + (\mu+\psi)\right) + \left(\frac{\sigma\beta I^*}{N} + (\mu+\theta)\right) - \left(\frac{\beta(S^* + \sigma V^*)}{N} - (\mu+\alpha)\right)$$

$$\Omega_2 = b_{11}b_{22} - b_{12}b_{21} + b_{11}b_{33} - b_{13}b_{31} + b_{22}b_{33} - b_{23}b_{32}$$

$$\Omega_3 = -b_{33}\left(b_{11}b_{22} - b_{12}b_{21}\right) - b_{12}b_{23}b_{31} - b_{13}b_{21}b_{32} + b_{13}b_{22}b_{31} + b_{11}b_{23}b_{32}$$

$$= \left(\frac{\beta(S^* + \sigma V^*)}{N} - (\mu+\alpha)\right)\left(\frac{\sigma\beta^2 I^{*^2}}{N^2} + \frac{\beta I^*}{N}(\mu+\theta) + \frac{\sigma\beta I^*}{N}(\mu+\psi) + \mu^2 + \mu\theta + \mu\psi\right)$$
$$+ \left(\frac{\sigma\beta^2\theta V^* I^*}{N^2}\right) + \left(\frac{\sigma\beta^2\psi S^* I^*}{N^2}\right) + \left(\frac{\beta^2 S^* I^*}{N^2}\right)\left(\frac{\sigma\beta I^*}{N} + (\mu+\theta)\right) + \left(\frac{\sigma^2\beta^2 V^* I^*}{N^2}\right)\left(\frac{\beta I^*}{N} + (\mu+\psi)\right)$$

Further:

$$\Delta = \Omega_1\Omega_2 - \Omega_3$$

$$= -(b_{11} + b_{22})(b_{11}b_{22} - b_{12}b_{21}) - (b_{11} + b_{33})(b_{11}b_{33} - b_{13}b_{31}) - (b_{22} + b_{33})(b_{22}b_{33} - b_{23}b_{32}) - 2b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32}$$

$$= \left[\left(\frac{\beta I^*}{N} + (\mu+\psi)\right) + \left(\frac{\sigma\beta I^*}{N} + (\mu+\theta)\right)\right]\left[\frac{\sigma\beta^2 I^*}{N} + \frac{\beta I^*}{N}(\mu+\theta) + \frac{\sigma\beta I^*}{N}(\mu+\psi) + \mu^2 + \mu\theta + \mu\psi\right]$$
$$+ \left[\left(\frac{\beta I^*}{N} + (\mu+\psi)\right) + \left((\mu+\alpha) - \frac{\beta(S^* + \sigma V^*)}{N}\right)\right]\left[\left(\frac{-\beta I^*}{N} - (\mu+\psi)\right)\left(\frac{\beta(S^* + \sigma V^*)}{N} - (\mu+\alpha)\right) + \left(\frac{\beta^2 S^* I^*}{N^2}\right)\right]$$
$$+ \left[\left(\frac{\sigma\beta I^*}{N} + (\mu+\theta)\right) + \left((\mu+\alpha) - \frac{\beta(S^* + \sigma V^*)}{N}\right)\right]\left[\left(\frac{-\sigma\beta I^*}{N} - (\mu+\theta)\right)\left(\frac{\beta(S^* + \sigma V^*)}{N} - (\mu+\alpha)\right) + \left(\frac{\sigma^2\beta^2 V^* I^*}{N^2}\right)\right]$$
$$- 2\left[\left(\frac{-\beta I^*}{N} - (\mu+\psi)\right)\left(\frac{-\sigma\beta I^*}{N} - (\mu+\theta)\right)\left(\frac{\beta(S^* + \sigma V^*)}{N} - (\mu+\alpha)\right)\right]$$
$$+ \left(\frac{-\sigma\beta^2\theta V^* I^*}{N^2}\right) + \left(\frac{-\sigma\beta^2\psi S^* I^*}{N^2}\right)$$

Now according to Routh-Hurwitz criterion E^* will be locally asymptotically stable provided that $\Omega_1 > 0$; $\Omega_3 > 0$ and $\Delta = \Omega_1\Omega_2 - \Omega_3 > 0$. Clearly: $\Omega_1 > 0$ and $\Omega_3 > 0$ provided that condition 16a holds. While $\Delta = \Omega_1\Omega_2 - \Omega_3 > 0$, provided that conditions 16 (a-b) hold. Hence the proof is completed. ■

5. Global stability analysis of system (5)

In this section, the global dynamics of system (5) is studied with the help of Lyapunov function as shown in the following theorems.

Theorem 4:

Assume that, the disease free equilibrium point E° of system (5) is locally asymptotically stable. Then the basin of attraction of E°, say $\mathrm{B}(E^\circ) \subset \Re_+^3$, satisfy the following conditions

$$\left(\frac{\theta}{S} + \frac{\psi}{V}\right)^2 \le 4\left(\frac{\mu+\psi}{S}\right)\left(\frac{\theta+\mu}{V}\right) \qquad (18a)$$

$$\frac{pAS^\circ}{S(\alpha+\mu)I} + \frac{\beta(S^\circ + \sigma V^\circ)}{N(\alpha+\mu)} < 1 \qquad (18b)$$

Proof: Consider the following positive definite function:

$$W_1 = (S - S^\circ - S^\circ Ln\frac{S}{S^\circ}) + (V - V^\circ - V^\circ Ln\frac{V}{V^\circ}) + I$$

Clearly, $W_1 : R_+^3 \to R$ is a continuously differentiable function such that $W_1(S^\circ, V^\circ, 0) = 0$,

and $W_1(S,V,I) > 0$, $\forall (S,V,I) \neq (S^\circ, V^\circ, 0)$. Further, we have:

$$\frac{dW_1}{dt} = \left(\frac{S - S^\circ}{S}\right)\frac{dS}{dt} + \left(\frac{V - V^\circ}{V}\right)\frac{dV}{dt} + \frac{dI}{dt}$$

By simplifying this equation we get:

$$\begin{aligned}\frac{dW_1}{dt} =& -\frac{(\mu+\psi)}{S}(S - S^\circ)^2 + \left(\frac{\theta}{S} + \frac{\psi}{V}\right)(S - S^\circ)(V - V^\circ)\\ & -\frac{(\theta+\mu)}{V}(V - V^\circ)^2 + \frac{pAS^\circ}{S}\\ & + \frac{\beta S^\circ I}{N} + \frac{\sigma\beta V^\circ I}{N} - (\alpha+\mu)I\end{aligned}$$

Therefore, according to condition (18a) it is obtained that:

$$\begin{aligned}\frac{dW_1}{dt} \le & -\left[\sqrt{\frac{\mu+\psi}{S}}(S - S^\circ) - \sqrt{\frac{\mu+\theta}{V}}(V - V^\circ)\right]^2\\ & + \frac{pAS^\circ}{S} + \frac{\beta I(S^\circ + \sigma V^\circ)}{N} - (\alpha+\mu)I\end{aligned}$$

Obviously $\frac{dW_1}{dt} < 0$ for every initial points satisfying condition (18b) and then W_1 is a Lyapunov function provided that conditions (18a-18b) hold. Thus E° is globally asymptotically stable in the interior of $\mathrm{B}(E^\circ)$, which means that $\mathrm{B}(E^\circ)$ is the basin of attraction and that complete the proof. ■

Theorem 5: Let the endemic equilibrium point (E^*) of system (5) is locally asymptotically stable. Then the basin of attraction of E^*, say $\mathrm{B}(E^*) \subset \Re_+^3$, satisfy the following conditions

$$\beta(S^* + \sigma V^*) < N(\mu+\alpha) \tag{19a}$$

$$(\theta+\psi)^2 < \left[\frac{\beta I + (\mu+\psi)N}{N}\right]\left[\frac{\sigma\beta I + (\mu+\theta)N}{N}\right] \tag{19b}$$

$$\left(\beta I - \beta S^*\right)^2 < (\beta I + (\mu+\psi)N)(-\beta S^* - \sigma\beta V^* + (\mu+\alpha)N) \tag{19c}$$

$$\left(\sigma\beta I - \sigma\beta V^*\right)^2 < (\sigma\beta I + (\mu+\theta)N)(-\beta S^* - \sigma\beta V^* + (\mu+\alpha)N) \tag{19d}$$

Proof: Consider the following positive definite function:

$$W_2 = \frac{(S - S^*)^2}{2} + \frac{(V - V^*)^2}{2} + \frac{(I - I^*)^2}{2}$$

Clearly, $W_2 : R_+^3 \to R$ is a continuously differenttiable function such that $W_2(S^*, V^*, I^*) = 0$ and $W_2(S,V,I) > 0$, $\forall (S,V,I) \neq (S^*, V^*, I^*)$. Further, we have:

$$\frac{dW_2}{dt} = (S - S^*)\frac{dS}{dt} + (V - V^*)\frac{dV}{dt} + (I - I^*)\frac{dI}{dt}$$

By simplifying this equation we get:

$$\begin{aligned}\frac{dW_2}{dt} =& -\frac{1}{2}q_{11}(S - S^*)^2 - \frac{1}{2}q_{22}(V - V^*)^2\\ & + q_{12}(S - S^*)(V - V^*)\\ & -\frac{1}{2}q_{11}(S - S^*)^2 - \frac{1}{2}q_{33}(I - I^*)^2\\ & + q_{13}(S - S^*)(I - I^*)\\ & -\frac{1}{2}q_{22}(V - V^*)^2 - \frac{1}{2}q_{33}(I - I^*)^2\\ & + q_{23}(V - V^*)(I - I^*)\end{aligned}$$

With

$$q_{11} = \left[\frac{\beta I}{N} + (\mu+\psi)\right], q_{12} = (\theta+\psi), q_{22} = \left[\frac{\sigma\beta I}{N} + (\mu+\theta)\right]$$

$$q_{13} = \left[\frac{\beta I}{N} - \frac{\beta S^*}{N}\right], q_{33} = \left[\frac{-\beta S^*}{N} - \frac{\sigma\beta V^*}{N} + (\mu+\alpha)\right]$$

$$q_{23} = \left[\frac{\sigma\beta I}{N} - \frac{\sigma\beta V^*}{N}\right]$$

Therefore, according to the conditions 19a-19d we obtain that:

$$\begin{aligned}\frac{dW_2}{dt} \le & -\left[\sqrt{\frac{q_{11}}{2}}(S - S^*) - \sqrt{\frac{q_{22}}{2}}(V - V^*)\right]^2\\ & -\left[\sqrt{\frac{q_{11}}{2}}(S - S^*) - \sqrt{\frac{q_{33}}{2}}(I - I^*)\right]^2\\ & -\left[\sqrt{\frac{q_{22}}{2}}(V - V^*) - \sqrt{\frac{q_{33}}{2}}(I - I^*)\right]^2\end{aligned}$$

Clearly, $\frac{dW_2}{dt} < 0$, and then W_2 is a Lyapunov function provided that the given conditions hold. Therefore, E^* is globally asymptotically stable in the interior of $\mathrm{B}(E^*)$, which means that $\mathrm{B}(E^*)$ is a basin of attraction of E^* and the proof is complete . ■

6. The local bifurcation analysis of system (5)

In this section, the occurrence of local bifurcations (such as saddle-node, transcritical and pitchfork) near the equilibrium point of system (5) is studied in the following theorem.

Theorem 6: System (5) has a transcritical bifurcation near the disease free equilibrium point E°, but neither saddle-node bifurcation, nor pitchfork bifurcation can accrue at the parameter

$$\alpha_{\circ}=\frac{\beta(S^{\circ}+\sigma V^{\circ})}{N}-\mu \,. \qquad (20)$$

Proof: It is easy to verify that the Jacobian matrix of system (5) at $(E^{\circ},\alpha_{\circ})$ can be written as:

$$J=Df(E^{\circ},\alpha_{\circ})$$
$$=\begin{bmatrix}-(\mu+\psi) & \theta & \frac{-\beta S^{\circ}}{N}\\ \psi & -(\mu+\theta) & \frac{-\sigma\beta V^{\circ}}{N}\\ 0 & 0 & 0\end{bmatrix}$$

Clearly, the third eigenvalue λ_I in the I – direction is zero ($\lambda_I=0$), while λ_S and λ_V those are given in equation 14 are negative. Further, the eigenvector (say $K=(k_1,k_2,k_3)^T$) corresponding to λ_I satisfy the following:

$$JK=\lambda K \quad then \quad JK=0$$

Thus

$$\begin{bmatrix}-(\mu+\psi) & \theta & \frac{-\beta S^{\circ}}{N}\\ \psi & -(\mu+\theta) & \frac{-\sigma\beta V^{\circ}}{N}\\ 0 & 0 & 0\end{bmatrix}\cdot\begin{bmatrix}k_1\\ k_2\\ k_3\end{bmatrix}=0$$

From which we get that:

$$-(\mu+\psi)k_1+\theta k_2-\frac{\beta S^{\circ}}{N}k_3=0 \qquad (21a)$$

$$\psi k_1-(\mu+\theta)k_2-\frac{\sigma\beta V^{\circ}}{N}k_3=0 \qquad (21b)$$

So by solving the above system of equations we get:

$$k_1=zk_3;\ k_2=qk_3$$

Where :

$$z=\frac{-\beta}{\mu(\mu+\psi+\theta)N}\left[\sigma\theta(\mu+\psi)^2V^{\circ}+S^{\circ}\{\mu^3+\mu^2(2\psi+\theta)+\theta\psi(2\mu+\psi)+\mu\psi^2\}\right]$$

$$q=\frac{-\beta[S^{\circ}\psi+\sigma(\mu+\psi)V^{\circ}]}{\mu[\mu+\psi+\theta]N}$$

Here k_3 be any non zero real number. Thus

$$K=\begin{bmatrix}zk_3\\ qk_3\\ k_3\end{bmatrix}$$

Similarly the eigenvector $W=(w_1,w_2,w_3)^T$ corresponding to λ_I of J^T can be written:

$$\begin{bmatrix}-(\mu+\psi) & \psi & 0\\ \theta & -(\mu+\theta) & 0\\ \frac{-\beta S^{\circ}}{N} & \frac{-\sigma\beta V^{\circ}}{N} & 0\end{bmatrix}\cdot\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix}=0$$

This gives:

$$W=\begin{bmatrix}0\\ 0\\ w_3\end{bmatrix}$$

Here w_3 is any non-zero real number. Now rewrite system (5) in a vector form as:

$$\frac{dX}{dt}=f(X)$$

Where $X=(S,V,I)^T$ and $f=(f_1,f_2,f_3)^T$ with f_i ,$i=1,2,3$ given in system (5), and then determine $\frac{df}{d\alpha}=f_{\alpha}$ we get that:

$$f_{\alpha}=\begin{bmatrix}0\\ 0\\ -I\end{bmatrix} \quad then \quad f_{\alpha}(E^{\circ},\alpha_{\circ})=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Therefore:

$$W^T\cdot f_{\alpha}(E^{\circ},\alpha_{\circ})=0$$

Consequently, according to Sotomayor theorem [12] the system has no saddle-node bifurcation near E° at $\alpha_{\circ}$.

Now in order to investigate the accruing of other types of bifurcation, the derivative of f_{α} with respect to vector X, say $Df_{\alpha}(E^{\circ},\alpha_{\circ})$, is computed

$$Df_{\alpha}(E^{\circ},\alpha_{\circ})=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}$$

So

$$W^T\cdot\left[Df_{\alpha}(E^{\circ},\alpha_{\circ})\cdot K\right]=-k_3w_3\neq 0$$

Again, according to Sotomayor theorem, if in addition to the above the following holds

$$W^T\cdot\left[D^2f(E^{\circ},\alpha_{\circ})\cdot(K,K)\right]\neq 0$$

Here $Df(E^{\circ},\alpha_{\circ})$ is the Jacobian matrix at E° and $\alpha_{\circ}$, then the system (5) possesses a transcritical bifurcation but no pitch-fork bifurcation can occur. Now since we have that:

$$\left[D^2 f(E^\circ,\alpha_\circ)\cdot(K,K)\right]=\begin{bmatrix}\dfrac{2z\beta k_3^2}{N}\\ \dfrac{2q\sigma\beta k_3^2}{N}\\ \dfrac{-2\beta(z+q\sigma)k_3^2}{N}\end{bmatrix}$$

Therefore:

$$W^T\cdot[D^2 f(E^\circ,\alpha_\circ)\cdot(K,K)]=\frac{-2\beta(q+z\sigma)k_3^2 w_3}{N}\neq 0$$

Then the system (5) has a transcritical bifurcation at E° when the parameter α passes through the bifurcation value $\alpha_\circ$.■

7. The Hopf-bifurcation analysis of system (5)

In this section, the occurrence of Hopf-bifurcation near the endemic equilibrium point is studied below.

According to the local stability analysis of system (5) at E^*, we have that the coefficients of the characteristic equation $\Omega_i, i=1,2,3$ are positive provided that condition 16a holds. However, $\Delta=\Omega_1\Omega_2-\Omega_3$ is positive provided that condition 16b holds and hence there is no Hopf-bifurcation in this case.

Now, suppose that $\Delta=\Omega_1\Omega_2-\Omega_3=0$ then there is possibility to occurrence of Hopf-bifurcation if and only if the Jacobian matrix of system (5) near E^* has two complex conjugate eigenvalues, say $\lambda_i=\rho_1\pm i\rho_2$ with the third eigenvalue is real and negative. In addition the following two conditions are hold at the specific parameter say $l=l^*$:

$$\rho_1(l^*)=0 \tag{22a}$$

$$\left.\frac{d\rho_1}{dl}\right|_{l=l^*}\neq 0 \tag{22b}$$

Now, from $\Delta=\Omega_1\Omega_2-\Omega_3=0$, We obtain that

$$Mb_{11}^2+Bb_{11}+C=0 \tag{23a}$$

Where :

$$M=-(b_{22}+b_{33})>0$$
$$B=b_{12}b_{21}+b_{13}b_{31}-(b_{22}+b_{33})^2$$
$$C=(b_{22}+b_{33})\left[b_{23}b_{32}-b_{22}b_{33}\right]+b_{12}\left[b_{21}b_{22}+b_{23}b_{31}\right]+b_{13}\left[b_{31}b_{33}+b_{21}b_{32}\right]$$

Clearly for $C<0$ we have two real roots of equation 23:

$$b_{11}=\frac{-B}{2M}\pm\frac{1}{2M}\sqrt{B^2-4MC}$$

Since $b_{11}=-\frac{\beta I^*}{N}-(\mu+\psi)<0$, then we get $b_{11}=-\frac{B}{2M}-\frac{1}{2M}\sqrt{B^2-4MC}$ and hence

$$\frac{\beta I^*}{N}+(\mu+\psi)-\left(\frac{B}{2M}+\frac{1}{2M}\sqrt{B^2-4MC}\right)=0 \tag{23b}$$

Which gives $f(\psi^*)=0$, and hence $\psi=\psi^*$ represents root of equation 23b. Consequently for $\psi=\psi^*$ we have $\Omega_1\Omega_2=\Omega_3$ from which the characteristic equation can be written as:

$$p_3(\lambda)=(\lambda+\Omega_1)(\lambda^2+\Omega_2)=0 \tag{24}$$

Hence in such case (i.e $\psi=\psi^*$) the eigenvaluesare $\lambda_1=-\Omega_1<0$ *and* $\lambda_{2,3}=\pm i\sqrt{\Omega_2}$. So the first condition 22a for the Hopf-bifurcation is satisfied at $\psi=\psi^*$, that is $\rho_1(\psi^*)=0$, while $\rho_2=\sqrt{\Omega_2}$. Let as now check the second condition 22b Since, in general, the complex eigenvalues for any value of ψ can be written as:

$$\lambda_{2,3}=\rho_1(\psi)\pm i\rho_2(\psi)$$

Then by substituting $\lambda_2=\rho_1(\psi)+i\rho_2(\psi)$ into the equation 24, and calculating the derivative with respect to the bifurcation parameter ψ, that is $\frac{d}{d\psi}p_3(\lambda)=p_3'(\lambda)=0$ and then comparing the two sides of this equation with equating their real and imaginary parts, it is obtain that :

$$\left.\begin{aligned}\eta(\psi)\rho_1'(\psi)-\Phi(\psi)\rho_2'(\psi)&=-\theta(\psi)\\ \Phi(\psi)\rho_1'(\psi)+\eta(\psi)\rho_2'(\psi)&=-\Gamma(\psi)\end{aligned}\right\} \tag{25}$$

where :

$$\eta(\psi)=3(\rho_1(\psi))^2+2\Omega_1(\psi)\rho_1(\psi)+\Omega_2(\psi)-3(\rho_2(\psi))^2,$$
$$\Phi(\psi)=6\rho_1(\psi)\rho_2(\psi)+2\Omega_1(\psi)\rho_2(\psi),$$
$$\theta(\psi)=(\rho_1(\psi))^2\Omega_1'(\psi)+\Omega_2'(\psi)\rho_1(\psi)+\Omega_3'(\psi)-\Omega_1'(\psi)(\rho_2(\psi))^2,$$
$$\Gamma(\psi)=2\rho_1(\psi)\rho_2(\psi)\Omega_1'(\psi)+\Omega_2'(\psi)\rho_2(\psi).$$

Solving the linear system (25) for the unknown $\rho_1'(\psi)$ and $\rho_2'(\psi)$, it is obtain that :

$$\rho_1'(\psi)=\frac{d}{d\psi}\rho_1(\psi)=-\frac{\eta\theta+\Gamma\Phi}{\eta^2+\Phi^2} \tag{26}$$

Hence, the second condition 22b of Hopf-bifurcation will be reduces to verifying that:

$$\eta(\psi^*)\theta(\psi^*)+\Gamma(\psi^*)\Phi(\psi^*)\neq 0 \tag{27}$$

Straight forward computation shows that:

$\Omega_1'=-1$; $\Omega_2'=-(b_{22}+b_{12}+b_{33})$

$\Omega_3'=-\Omega_2-\Omega_1(b_{22}+b_{12}+b_{33})$

Thus for $\psi = \psi^*$ we have:

$$\eta = -2\Omega_2 \; ; \quad \Phi = 2\Omega_1\sqrt{\Omega_2} \; ;$$
$$\theta = -\Omega_1\left(b_{22} + b_{12} + b_{33}\right) \;\; and$$
$$\Gamma = -\left(b_{22} + b_{12} + b_{33}\right)\sqrt{\Omega_2}$$

Therefore, substituting in equation 27, we get that $\eta\theta + \Gamma\Phi = 0$. Hence the system (5) dose not undergoes a Hopf-bifurcation around E^*.

8. Numerical analysis of system (3)

In this section the global dynamics of system (3) is studied. The objectives of this study are confirming our analytical results and understand the effects of immigration and the existence of vaccine on the dynamics of *SVIR* epidemic system. Consequently, system (3) is solved numerically for different sets of initial conditions and different sets of parameters. It is observed that, for the following set of hypothetical parameters, system (3) is solved numerically for different sets of initial values and then the trajectories of system (3) as a function of time are drawn in figure 2

$$E = 400,\ A = 100,\ p = 0,\ \beta = 0.4,\ \psi = 0.5, \quad \theta = 0.05,\ \mu = 0.1,\ \sigma = 0.1,\ \alpha = 0.2 \tag{28}$$

<u>Note that :</u> In the following figures, we will used the following representations: Solid line for **S**; dashed line for **V**; dash dot line for I; dotted line for R.

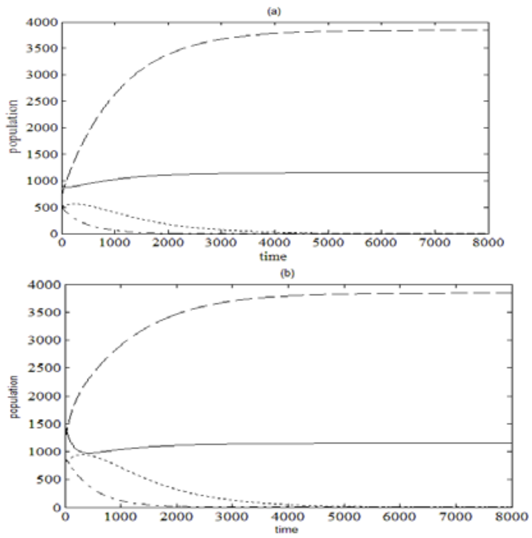


Figure 2- Time series of trajectories of system (3) for data given in equation 28. (a) trajectories starting at (900,700,500,500) and (b) trajectories starting at (1500,1200,900,800).

Obviously, figure 2 shows clearly the converg-ence of system (3) to the disease free equilibrium point $E^{\circ} = (1153,3846,0,0)$ from two different initial data.

However, for the data given equation 28 with *p=0.1*. The trajectories of system (3) starting from different sets of initial data are drawn in figure.. 3

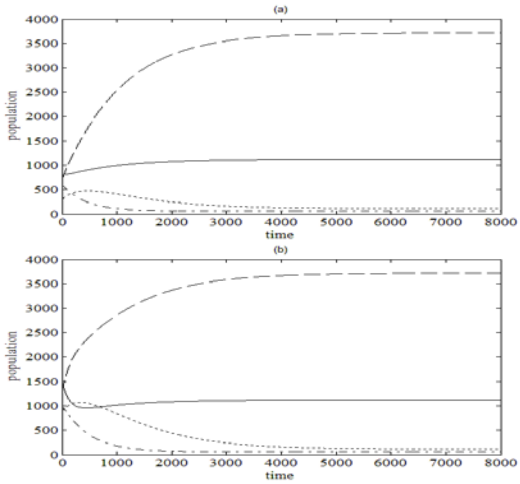
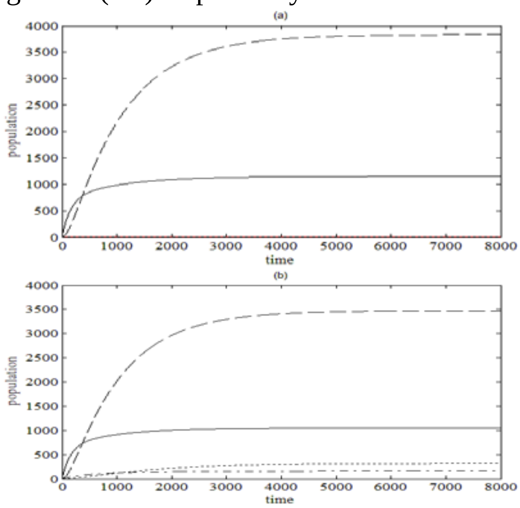


Figure 3-Time series of the solution of system (3). (a) trajectories starting at (800,700,600,300) and (b) trajectories starting (1500,1300,1000,900).

Similarly, figure 3 shows the approaching of system (3) to the endemic equilibrium point $E^* = (1118, 3716, 55, 110)$ from two different initial data.

Now the effect of varying the fraction of immigrant individuals, which arrive infected, on the dynamics of system (3), is studied. So, system (3) is solved for the parameters values $p = 0.01, 0.3, 0.75$ respectively, keeping other parameters fixed as given in equation 28, and then the trajectories of system (3) are drawn in figures 4 (a-c) respectively.



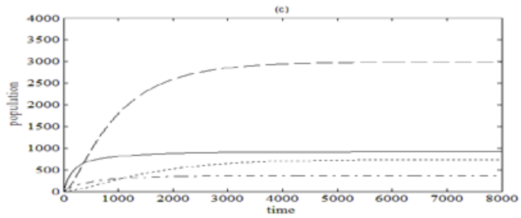
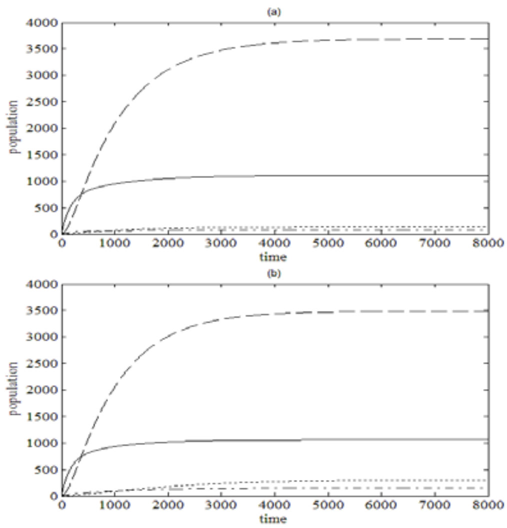


Figure 4- Time series of the solution of system (3). (a) For p=0.01, (b) For p=0.3, (c) For p=0.75

According to these figures, as the fraction of infected immigrant individuals increases *(through increasing p)*, the disease free equilibrium point of system (3) becomes unstable point and the trajectory of system (3) approaches asymptotically to the endemic equilibrium point. In fact as p increases it is observed that the number of susceptible and vaccinated individuals decrease and the number of removal individuals increases whereas the number of infected individuals increases slightly.

Now, in order to discuss the effect of varying the infection rate on the dynamical behavior of system (3), the system is solved for different values of infection rate $\beta = 0.01$, 0.6, 0.9 respectively, keeping other parameters fixed as given in equation 28 with $p = 0.1$, and then the solution of system (3) is drawn in figures 5 (a-c) respectively.



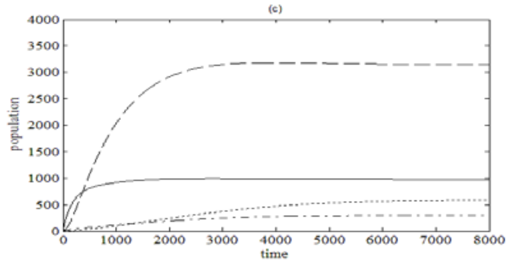


Figure 5-Time series of the solution of system (3). (a) For $\beta = 0.01$, (b) For $\beta = 0.6$ (c)For $\beta = 0.9$

Obviously from these figures, as the infection rate increases the endemic equilibrium point of system (3), still coexists and stable but the number of susceptible and vaccinated individuals decrease while the number of the infected and removal individuals increases.

The effect of varying the vaccination coverage rate on the dynamical behavior of system (3) is studied too. The system is solved for different values of $\psi = 0.02$, 0.6, 0.9 keeping other parameters fixed as given in equation 28 with *p=0.1* , and then the solution of system (3) are drawn in figures 6 (a-c) respectively.

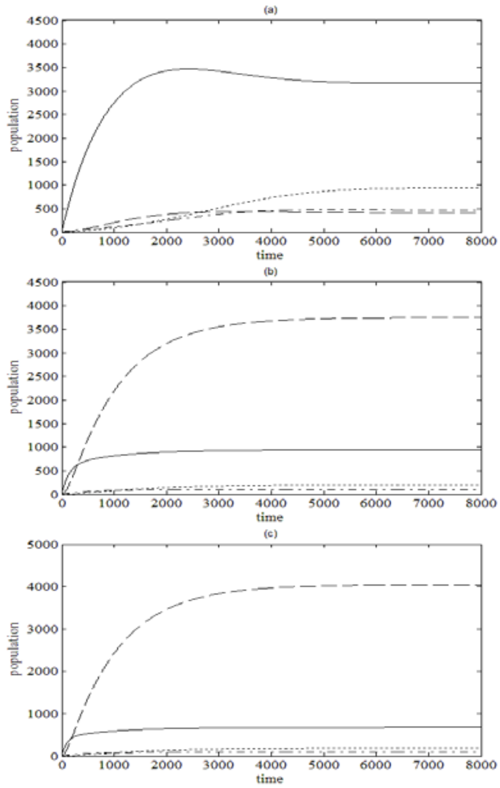


Figure 6-Time series of the solution of system (3). *(a) For* $\psi = 0.02$ *, (b) For* $\psi = 0.6$ *, (c)For* $\psi = 0.9$.

From these figures, as the rate of vaccination coverage increases the endemic equilibrium point of system (3), still coexists and stable but the number of susceptible, infected and removal individuals decrease whereas the number of vaccinated individuals increases.

Similarly the effect of varying the number of individuals who lose vaccine immunity and return to susceptible (failure in vaccine), on the dynamical behavior of system (3) is investigated. The system is solved for the value $\theta = 0.01$, 0.2 and 0.5 keeping the rest of parameters fixed as given in equation 28 with *p= 0.1* and then the trajectories are drawn in figures 7 (a-c). In this case, it is observed that increasing θ causes increasing in the susceptible, infected and removal while the number of vaccinated decreases but the system (3) in this case still approaches to endemic equilibrium point.

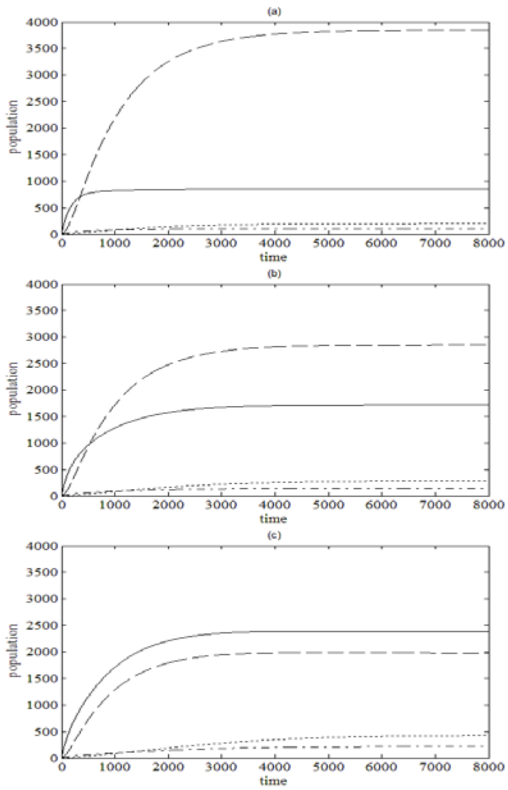


Figure 7-Time series of the solution of system (3). (a) For $\theta = 0.01$,(b) For $\theta = 0.2$,(c) For $\theta = 0.5$.

Finally the effect of vaccine efficiency against the disease on the dynamical behavior of system (3), is studied too then the system is solved for different values of $\sigma = 0.02$, 0.5, 0.8, keeping other parameters as given in equation 28 with *p= 0.1*, and then the solutions of system (3) are drawn in figures 8 (a-c) respectively.

From these figures, as the vaccine efficiency decreases, the endemic equilibrium point of system (5), still coexists and stable, but the number of susceptible and vaccinated individuals decrease and the number of infected and removal individuals increase.

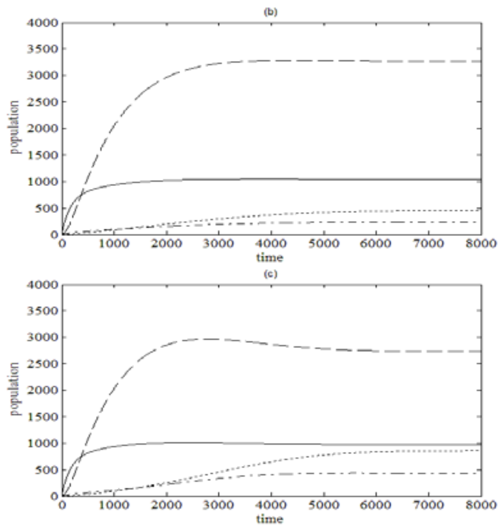


Figure 8- *Time series of the solutions of system (3). (a) For* $\sigma = 0.02$ *, (b) For* $\sigma = 0.5$ *(c) For* $\sigma = 0.8$.

9. Discussion and conclusions

In this paper, a mathematical model has been studied and analyzed to study the effect of vaccine and immigrants on the dynamical behavior of *SIR* epidemic model. The existence and the stability analysis of all possible equilibrium point are studied analytically as well as numerically. It is observed that the system (3) has a transcritical bifurcation near the disease free equilibrium point, but neither saddle node nor pitchfork bifurcation can accrue. Further the system dose not has a Hopf bifurcation near the endemic equilibrium point. Finally according to the numerically simulation the following results are obtained:

The *SVIR* system (3) dose not have periodic dynamic, instead it is approaches either to the disease free equilibrium point or else to endemic equilibrium point.

As the fraction of the infected immigrant individuals increases, the asymptotic behavior of the system transfers from approaching to disease free equilibrium point to the endemic equilibrium point.

As the losing vaccine immunity rate (θ) increases, then *SVIR* system still coexist at the endemic equilibrium point with increasing in the *S, I and R* while the number of vaccinated individual decreases.

As the vaccine efficiency against the disease decreases then the *SVIR* system still approaches to the endemic equilibrium point with increasing in the *I* and *R*, while the number of *S* and *V* decreases.

Reference

1. Hethcote, H. W.,**2000**. The mathematics of infectious disease, Siam review. 42 (4),pp: 599-653.

2. Kermark, M. and Mckendrick, A.,**1927**. Contributions to the mathematical theory of epidemics Part I. Proc. Roy. Soc. A 115,pp:700-721.

3. Kribs-Zaleta, C.M. and Velasco-Hern´andez, J.X., **2000**. A simple vaccination model with multiple endemic states. *Math. Biosci*. **164**: ,pp183-201.

4. Arino, J., Mccluskey, C.C.and van den Driessche, P., **2003**. Global results for an epidemic model with vaccination that exhibits backward bifurcation. *SIAM J. APPL*. MATH. **64**,pp: 260-276.

5. Kribs-Zaleta, C.M. and Martcheva, M., **2002**. Vaccination strategies and backward bifurcation in an age-since-infection structured model. *Math. Biosci*.177-178: ,pp:317-332.

6. Alexander, M.E., Bowman, C., Moghadas, S.M., Summers, R., Gumel, A.B. and Sahai, B.M., **2004**. A vaccination model for transmission dynamics of influenza. *SIAM J. Appl. Dyn. Syst*. **3**,pp:503-524.

7. Shim, E.,**2004**. An epidemic model with immigration of infectives and vaccination. M.sc. thesis. in. mathematics, The University of Britsh Columbia.

8. d'Onofrio, A., Manfredi, P. and Salinelli, E., **2007**. Vaccinating behaviour, information, and the dynamics of SIR vaccine preventable diseases. Theor. Popul. *Biol.* **71**,pp:301-317.

9. Brauer, **2011,** Backward bifurcation in simple vaccination / treatment models. *Bio. Dyn*. Vol.**5**, no.**5**,pp:410-418.

10. Shurowq k. Shafeeq**, 2011,**. The effect of treatment, immigrants and vaccina,ted on the dynamic of SIS epidemic model. M.sc. thesis. Department of Mathematics, College of Science, University of Baghdad. Baghdad, Iraq.

11. Sotomayor, J.,**1973**. Generic bifurcations of dynamical systems, in dynamical systems, M. M. Peixoto, New York, academic press.

12. Mainul Haque and Ezio Venturino, **2004**. Increase of the prey may decrease the healthy predator population in presence of a disease in the predator.