Modeling and Stability Analysis of an Eco-epidemiological Model

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Abstract

In this paper,a prey-predator model with infectious disease in predator population is proposed and studied. Nonlinear incidence rate is used to describe the transition of disease. The existence, uniqueness and boundedness of the solution are discussed. The existences and the stability analysis of all possible equilibrium points are studied. Numerical simulation is carried out to investigate the global dynamical behavior of the system.

Keywords: eco-epidemiological model, *SIS* epidemics disease, prey-predator model, stability analysis, nonlinear incidence rate.

نمذجة و تحليل الاستقرارية لنظام بيئي وبائي

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الخلاصة

في هذا البحث أقترحنا و درسنا نظام الفريسة والمفترس عند وجود مرض معدي في مجتمع المفترس. تم استخدام معدل اصابة غير خطي لوصف انتقال المرض. ناقشنا وجود، وحدانية وقيد الحل. قمنا بدراسة وجود وتحليل الاستقرارية لجميع نقاط التوازن الممكنة. كما تم استخدام المحاكاة العددية لبحث السلوك الديناميكي الشامل للنظام.

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1. Introduction

Mathematical modeling becomes important tools to analyze the spread and control of infectious diseases. These models are known as epidemiological models which are used to study of the spread and control of diseases in human or animal populations. One of the major mathematical model in the field of epidemiology which describe the spread of disease from susceptible to infected and then to removal individuals has been formulated by Kermack and McKendric in 1927[1]. On the other hand, its well known that in nature no species can survive alone; and the species not only spreads the disease but also competes with other species for space or food or is predated by other species.

The mathematical models which describe the dynamical behavior of an interacting species in ecology are known as ecological models. The first mathematical model in the field of ecology that describes the interactions between biological species was formulated, independently, by Lotka (American physical chemist) in 1925 [2] and Volterra (Italian mathematician) in 1926 [3]. The researchers studied the dynamics of the mathematical models of these two fields epidemiology and ecology independently along the years until now [1-5], however during the last four decades the ideas oriented to study the dynamical behavior of eco-epidemiological models, which represented by mathematical models merging of the two phenomena, that is means the demographics of interaction of interacting species and an epidemic evolution in different environment. In 1978 Anderson and May were the first who merged the above two fields, ecological system and epidemiology system, they formulated a prey-predator model with infectious disease spread among prey by contact between them. In the subsequent time many researchers proposed and studied different prey-predator models with disease spread in prey population [6-9]. In addition to the above there are many investigations about prey-predator model with disease in the predator population. Haque [10] proposed a prey-predator model includes an Susceptible-Infected-Susceptible (SIS) parasitic infection in the predator population with linear functional response and nonlinear disease incidence rate. Haque and Venturino [11] considered a prey-predator model with SI epidemic disease spread in predators involving linear functional response. Das [12] studied a prey-predator model with SI epidemic disease in predators included Holling type-II as a functional response. Venturino [13] proposed and analyzed prey-predator model with SIS disease in predators included linear functional response and linear disease incidence. Haque and Venturino [14] considered a prey-predator model with SI epidemic disease spread in predators included ratio-dependent functional response and linear disease's incidence rate.

Keeping the above in view, there are many diseases such as influenza, typhoid fever, bird flu and strep throat are the most diseases spread in the human population and they classified to be SIS epidemic diseases, which transmitted by contact between susceptible and infected individuals (i.e. contact nasal secretions or inhalation of aerosols…etc).

So in this paper we proposed and analyzed a mathematical model describing prey-predator model having SIS epidemic disease in the predator population with nonlinear functional response, represented by Holling type-II and ratio-dependent incidence rate.

2. Mathematical Model

In this section an eco-epidemiological model consisting of a prey-predator model with infectious disease in the predator is proposed for study. In order to construct our model the following assumptions have been assumed.

1. Let $X(T)$ and $N(T)$ be the population densities of the prey species and predator species at time T respectively.
2. The prey grows logistically with intrinsic growth rate $a > 0$ and carrying capacity $\frac{a}{b} > 0$.
3. The predator preys upon the prey according to Holling type-II functional response with maximum attack rate $c > 0$ and half saturation constant $d > 0$. While in the absence of the prey the predator decay exponentially with natural death rate $\theta > 0$.
4. In addition to the above it is assumed that the predator has other food sources represented by the constant $\beta > 0$.

   Accordingly the following prey-predator model is obtained:

$$\frac{dX}{dT} = X\left(a - bX - \frac{c\,N}{d+X}\right)$$
$$\frac{dN}{dT} = N\left(\frac{e\,c\,X}{d+X} - \theta + \beta\right) \qquad (1)$$

here $e > 0$ represent the conversion rate constant.

In addition to the above assumptions, let us consider the following:

5. There is an SIS epidemic disease spreads among the predator population and it transmitted between the predator individuals (but not the prey) by contact, according to ratio-dependent incidence rate with infection rate constant $\alpha > 0$. Therefore the total predator population is divided into two classes: susceptible that denoted by $Y(T)$ and infected that denoted by $Z(T)$. Hence at any time T the total predator population is $N(T) = Y(T) + Z(T)$. Furthermore it is assumed that the infected predator depends in its feeding on the prey species only with attack rate $p > 0$ and the disease induced mortality rate represented by $\delta > 0$.
6. Finally the infected predator can be recovered and becomes susceptible again with recovery rate constant $\omega > 0$.

Consequently, the prey-predator model 1 with the above two assumptions can be rewritten in the following form:

$$\frac{dX}{dT} = X\left(a - bX - \frac{cY}{d+X} - \frac{pc\,Z}{d+X}\right)$$
$$\frac{dY}{dT} = Y\left(\frac{ec\,X}{d+X} - \frac{\alpha Z}{Y+Z} - \theta + \beta\right) + \omega Z \qquad (2)$$
$$\frac{dZ}{dT} = Z\left(\frac{\alpha Y}{Y+Z} + \frac{epc\,X}{d+X} - \theta - \delta\right) - \omega Z$$

Clearly system (2) can be represented by the following block diagram.

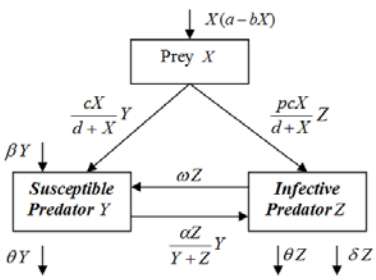

Figure 1- Block diagram for system (2).

Obviously, system (2) contains 12 parameters in all, which makes the analysis difficult, so to reduce the number of parameters and determine which combinations of parameters control the behavior of the system, it is assumed that $\psi = \theta + \delta > 0$ and $\mu = \beta - \theta \in \Re$. Then, the following dimensionless variables are used in the system (2)

$$x = \frac{X}{d},\ y = \frac{Y}{ed},\ z = \frac{Z}{ed},\ t = ecT$$

Now straight forward computation on system (2) gives the following dimensionless system

$$\frac{dx}{dt} = x\left(r - Hx - \frac{y}{1+x} - \frac{pz}{1+x}\right)$$
$$\frac{dy}{dt} = y\left(\frac{x}{1+x} - \frac{mz}{y+z} + n\right) + hz \qquad (3)$$
$$\frac{dz}{dt} = z\left(\frac{my}{y+z} + \frac{px}{1+x} - d\right) - hz$$

where

$$r = \frac{a}{ec} > 0\ ,\ H = \frac{db}{ec} > 0\ ,\ m = \frac{\alpha}{ec} > 0\ ,$$
$$h = \frac{\omega}{ec} > 0\ ,\ n = \frac{\mu}{ec} \in \Re\ ,\ d = \frac{\psi}{ec} > 0$$

represent the dimensionless parameters. Clearly the dimensionless system (3) has seven parameters. Further, the interaction functions $F_i(x, y, z),\ i = 1,2,3$ are continuously differentiable on the $Int.\Re^3_+ = \{(x, y, z) \in \Re^3, x > 0, y > 0$ $z > 0\}$. In addition to that:

$$\lim_{(x,y,z)\to(0,0,0)} F_i(x, y, z) = 0 \quad \forall i = 1, 2, 3$$

and

$$\lim_{(x,y,z)\to(x,0,0)} F_i(x, y, z) = 0 \quad \forall i = 1, 2, 3,\ x \in \Re_+$$

So, if we define that

$$F_i(0, 0, 0) = F_i(x, 0, 0) = 0 \quad \forall i = 1,2,3$$

Then with this assumption the interaction functions of system (3), $F_i\ ;\ i = 1,2,3$ are continuously differentiable on the extended domain

$$\Re^3_+ = \left\{(x, y, z) \in \Re^3, x \ge 0, y \ge 0, z \ge 0\right\}.$$

In fact, they are Lipschizian on $\Re^3_+$. Accordingly, the solution of the system (3) with non negative initial condition exists and is unique. Therefore, $\Re^3_+$ is invariant for the system (3). In the following theorem the sufficient condition for uniformly bounded of the system (3) is established.

Theorem (1): All the trajectories of the system (3) are uniformly bounded provided that $(n<0)$.

Proof: From the first equation of system (3) we get:

$$\frac{dx}{dt} \le x(r-Hx)$$

Now, by solving the above differential inequality we get that:

$$\limsup_{t\to\infty} x(t) \le \frac{r}{H}$$

Define the function $\mathrm{M}(t)=x(t)+y(t)+z(t)$ and take its time derivative along the solution of the system (3).

$$\frac{d\mathrm{M}}{dt}=\frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}$$
$$=rx-Hx^2+ny-dz$$
$$\le rx-\phi y-\phi z \quad where \quad \phi=\min\{-n,d\}$$
$$\le (r+\phi)x-\phi \mathrm{M}$$
$$\le \pi-\phi\mathrm{M} \quad where \quad \pi=(r+\phi)\frac{r}{H}$$

Now, by using Gronwall lemma [15] it is obtained that:

$$0<\mathrm{M}(t)\le \mathrm{M}(0)e^{-\phi t}+\frac{\pi}{\phi}\left(1-e^{-\phi t}\right)$$

Hence, $\limsup_{t\to\infty} \mathrm{M}(t)\le \frac{\pi}{\phi}$ that is independent of the initial conditions. Thus the proof is complete. ■

3. Existence of equilibrium points

The system (3) has at most six non negative equilibrium points, namely $E_i=(x_i,y_i,z_i)$, $i=0,1,...,5$ the existence conditions for each of these equilibrium points are established in the following:

1. The ***vanishing equilibrium point*** $E_0=(0,0,0)$ always exists.
2. The ***predator free equilibrium point*** $E_1=\left(\frac{r}{H},0,0\right)$ always exists.
3. The ***axial equilibrium point*** $E_2=(0,y_2,0)$ where y_2 is any positive number, exists if and only if $n=0$.
4. The ***disease free equilibrium point*** $E_3=(x_3,y_3,0)$ where:
$$x_3=\frac{-n}{n+1} \text{ and } y_3=(r-Hx_3)(1+x_3) \qquad (4)$$
exists uniquely in interior of first quadrant of $xy-$plane under the following necessary and sufficient conditions:
$$n\in(-1,0) \text{ and } x_3<\frac{r}{H} \qquad (5)$$
5. The ***prey free equilibrium point*** $E_4=(0,y_4,z_4)$ where:
$$z_4=By_4 \qquad (6)$$
here y_4 is any positive number and $B=\frac{m-(d+h)}{(d+h)}$, exists uniquely in the interior of first quadrant of $yz-$plane under the following necessary and sufficient conditions:
$$m>(d+h) \text{ and } n+hB=\frac{mB}{1+B} \qquad (7)$$
6. The ***coexistence equilibrium point*** $E_5=(x_5,y_5,z_5)$ where:
$$\left.\begin{aligned} y_5&=\frac{(r-Hx_5)(1+x_5)}{(1+pQ)}\\ z_5&=Qy_5=Q\frac{(r-Hx_5)(1+x_5)}{(1+pQ)}\end{aligned}\right\} \qquad (8)$$

with $Q=\dfrac{m-\left(d+h-\frac{px_5}{1+x_5}\right)}{\left(d+h-\frac{px_5}{1+x_5}\right)}$

However x_5 represents the positive root of the following quadratic equation:

$$\begin{aligned}&[(A_1+A_2)(A_3-p)+h(A_4+p)]x^2\\ &+[(A_1+A_2)A_3+A_2(A_3-p)+h(2A_4+p)]x \qquad (9)\\ &+[A_2A_3+hA_4]=0\end{aligned}$$

where:

$$A_1=1-p\,;\ A_2=n-A_4\,;\ A_3=d+h\,;$$
$$A_4=m-(d+h)$$

Obviously, E_5 exists uniquely in the int. $\Re_+^3$ if and only if the following conditions are hold.

$$\left.\begin{aligned}&(A_1+A_2)(A_3-p)+h(A_4+p)>0\\ &A_2A_3+hA_4<0\end{aligned}\right\} \qquad (10a)$$

Or

$$\left.\begin{aligned}&(A_1+A_2)(A_3-p)+h(A_4+p)<0\\ &A_2A_3+hA_4>0\end{aligned}\right\} \qquad (10b)$$

with

$$x_5<\frac{r}{H} \qquad (11a)$$

$$m>\left(d+h-\frac{px_5}{1+x_5}\right)>0 \qquad (11b)$$

4. The stability analysis

In this section, the local dynamical behavior of the system (3) around each of these equilibrium points is studied, first the Jacobian matrix of the system (3) at each of the equilibrium points E_i ; $\forall i = 2,...,5$ is computed and then their eigenvalues are determined.

It is easy to verify that the Jacobian matrices at the equilibrium points E_0 and E_1 are not define, while that at E_2 has zero eigenvalue and hence E_2 is nonhyperbolic point. Therefore, the stability analysis at these points will be study by using other methods as shown below.

4.1 The stability analysis at $E_0 = (0,0,0)$:

Since, the system (3) cannot be linearized at E_0, so in order to study the dynamical behavior of system (3) near E_0, the technique of Arino et .al. [7] is used. Now, rewrite system (3) in form:

$$\frac{dB}{dt} = \Lambda(\mathrm{B}(t)) + \Phi(\mathrm{B}(t)) \tag{12}$$

In which Λ is C^1 outside the origin and homogeneous of degree 1.

$$\Lambda(s\mathrm{B}) = s\Lambda(\mathrm{B})$$

for all $s \geq 0, \mathrm{B} \in \Re^3$ and Φ is a C^1 function such that in the vicinity of the origin we have:

$$\Phi(\mathrm{B}) = o(\mathrm{B}).$$

To study the behavior of the system (12) at the origin point, we use $\|\cdot\|$ that denotes the Euclidian norm on $\Re^3$ and $\langle \cdot,\cdot \rangle$ denotes the associated inner product.

Let

$$\mathrm{B} = (b_1, b_2, b_3) = (x, y, z)$$
$$\Lambda(\mathrm{B}) = (\Lambda_1(\mathrm{B}), \Lambda_2(\mathrm{B}), \Lambda_3(\mathrm{B}))$$
$$\Phi(\mathrm{B}) = (\Phi_1(\mathrm{B}), \Phi_2(\mathrm{B}), \Phi_3(\mathrm{B})).$$

Therefore, the functions Λ_i and Φ_i $(i = 1,2,3)$ are given by:

$$\Lambda_1(\mathrm{B}) = rb_1,\ \Lambda_2(\mathrm{B}) = \frac{-mb_2b_3}{b_2+b_3} + nb_2 + hb_3,$$

$$\Lambda_3(\mathrm{B}) = \frac{mb_2b_3}{b_2+b_3} - db_3 - hb_3;$$

$$\Phi_1(\mathrm{B}) = -Hb_1^2 - \frac{b_1b_2}{1+b_1} - \frac{pb_1b_3}{1+b_1},$$

$$\Phi_2(\mathrm{B}) = \frac{b_1b_2}{1+b_1},\ \Phi_3(\mathrm{B}) = \frac{pb_1b_3}{1+b_1}$$

Let $\mathrm{B}(t)$ be a solution of system (12). Assume that:

$\liminf_{t\to\infty} \|\mathrm{B}(t)\| = 0$ and B is bounded. One can extract from the family $(\mathrm{B}(t+.))_{t\geq 0}$ sequences $\mathrm{B}(t_n +.)$, $t_n \to \infty$, such that $\mathrm{B}(t_n +.) \to 0$ locally uniformly on $s \in \Re$. Define

$$q_n(s) = \frac{\mathrm{B}(t_n+s)}{\|\mathrm{B}(t_n+s)\|} \tag{13}$$

Recall that: $\Phi(\mathrm{B}) = o(\mathrm{B})$ in the vicinity of the origin. Then we can write Φ as:

$$\Phi(\mathrm{B}) = \|\mathrm{B}\|^2 \mathrm{O}(1) \tag{14}$$

We have:

$$\frac{d\mathrm{B}(t_n+s)}{ds} = \Lambda(\mathrm{B}(t_n+s)) + \Phi(\mathrm{B}(t_n+s)). \tag{15}$$

From (13) we have:

$$\mathrm{B}(t_n+s) = q_n(s)\|\mathrm{B}(t_n+s)\| = q_n(s) \times \langle \mathrm{B}(t_n+s), \mathrm{B}(t_n+s)\rangle^{1/2} \tag{16}$$

Now, using the derivative of $\langle \mathrm{B}(t_n+s), \mathrm{B}(t_n+s)\rangle$ with respect to s.

$$\frac{d}{ds}(\langle \mathrm{B}(t_n+s), \mathrm{B}(t_n+s)\rangle) = 2\left\langle \mathrm{B}(t_n+s), \frac{d\mathrm{B}(t_n+s)}{ds}\right\rangle$$

Put it in (16) and take derivative of $\mathrm{B}(t_n+s)$ with respect to s, we obtain:

$$\frac{d\mathrm{B}(t_n+s)}{ds} = \frac{dq_n(s)}{ds}\|\mathrm{B}(t_n+s)\| + \frac{q_n(s)}{\|\mathrm{B}(t_n+s)\|}\left\langle \mathrm{B}(t_n+s), \frac{d\mathrm{B}(t_n+s)}{ds}\right\rangle.$$

Therefore, we have:

$$\Lambda(\mathrm{B}(t_n+s)) + \Phi(\mathrm{B}(t_n+s)) = \frac{dq_n(s)}{ds}\|\mathrm{B}(t_n+s)\| + \frac{q_n(s)}{\|\mathrm{B}(t_n+s)\|}\langle \mathrm{B}(t_n+s), \Lambda(\mathrm{B}(t_n+s)) + \Phi(\mathrm{B}(t_n+s))\rangle$$

Now dividing by $\|\mathrm{B}(t_n+s)\|$ and replacing $\mathrm{B}(t_n+s)/\|\mathrm{B}(t_n+s)\|$ by $q_n(s)$, we obtain:

$$\frac{\Lambda(\mathrm{B}(t_n+s))}{\|\mathrm{B}(t_n+s)\|} + \frac{\Phi(\mathrm{B}(t_n+s))}{\|\mathrm{B}(t_n+s)\|} = \frac{dq_n(s)}{ds} + q_n(s)\left\langle \frac{\mathrm{B}(t_n+s)}{\|\mathrm{B}(t_n+s)\|}, \frac{\Lambda(\mathrm{B}(t_n+s)) + \Phi(\mathrm{B}(t_n+s))}{\|\mathrm{B}(t_n+s)\|}\right\rangle$$

Hence:

$$\frac{dq_n(s)}{ds} = \Lambda(q_n(s)) + \frac{\Phi(\mathrm{B}(t_n+s))}{\|\mathrm{B}(t_n+s)\|}$$
$$- q_n(s)\left\langle q_n(s), \Lambda(q_n(s)) + \frac{\Phi(\mathrm{B}(t_n+s))}{\|\mathrm{B}(t_n+s)\|}\right\rangle$$
$$= \Lambda(q_n(s)) - q_n(s)\langle q_n(s), \Lambda(q_n(s))\rangle$$
$$+ \|\mathrm{B}(t_n+s)\|\left[\frac{\Phi(\mathrm{B}(t_n+s))}{\|\mathrm{B}(t_n+s)\|^2} - q_n(s)\right.$$
$$\left.\times\left\langle q_n(s), \frac{\Phi(\mathrm{B}(t_n+s))}{\|\mathrm{B}(t_n+s)\|^2}\right\rangle\right]$$

Which is equivalent to:

$$\frac{dq_n(s)}{ds} = [\Lambda(q_n(s)) - q_n(s)\langle q_n(s), \Lambda(q_n(s))\rangle]$$
$$+ \|\mathrm{B}(t_n+s)\|[\Phi(q_n(s)), q_n(s)$$
$$\times\langle q_n(s), \Phi(q_n(s))\rangle$$

Clearly, $q_n(s)$ is bounded, $\|q_n(s)\| = 1, \forall s$, and $\frac{dq_n(s)}{ds}$ is bounded too.

So, applying the Ascoli–Arzela [16] theorem, one can extract from $q(s)$ a subsequence – also denoted by $q(s)$ – which converges locally uniformly on $\Re$ towards some function q, such that:

$$\|\mathrm{B}(t_n+s)\|[\Phi(q_n(s)) - q_n(s)$$
$$\langle q_n(s), \Phi(q_n(s))\rangle] \xrightarrow{t_n\to\infty} 0$$

And q satisfy the following system:

$$\frac{dq}{dt} = \Lambda(q(t)) - q(t)\langle q(t), \Lambda(q(t))\rangle \qquad (17)$$

here $\|q(t)\| = 1\ \forall t$. Equation (17) is defined for all $t \in \Re$.

So by study of equation (17) we get that, the steady state of Λ are vectors K satisfying:

$$\Lambda(\mathrm{K}) = \mathrm{K}\langle \mathrm{K}, \Lambda(\mathrm{K})\rangle.$$

This is a so–called nonlinear eigenvalue. Note that the equation can be alternatively written as:

$$\Lambda(\mathrm{K}) = \eta\mathrm{K}. \qquad (18)$$

with $\|\mathrm{K}\| = 1$; it then holds that $\eta = \langle \mathrm{K}, \Lambda(\mathrm{K})\rangle$. These stationary solutions correspond to fixed directions that the trajectories of equation (17) may reach asymptotically. Now, equation (18) can be written as:

$$(\eta - r)k_1 = 0; \qquad (19a)$$
$$[(\eta + m - n - h)k_3 + (\eta - n)k_2]k_2 - hk_3^2 = 0 \qquad (19b)$$
$$[(\eta - m + d + h)k_2 + (\eta + d + h)k_3]k_3 = 0. \qquad (19c)$$

Now, we are in a position to discuss in detail the possibility of reaching the origin following fixed direction.

Case 1*:* when $k_1 = 0$

(a) $k_2 = 0$ **and** $k_3 \neq 0$.

In this case, there is a possibility to reach the origin following the $z-$axis, with $h = 0$ and $\eta = -d$.

(b) $k_2 \neq 0$ **and** $k_3 = 0$.

In this case, there is a possibility to reach the origin following the $y-$axis, with $\eta = n$ when $n < 0$;

OR

cannot reach the origin when $n > 0$.

(c) $k_2 \neq 0$ **and** $k_3 \neq 0$.

In this case, we obtain different results depending on the parameters:

Sub case 1*:* If $(-m + d + n) < 0$ then:

(i) We reach the origin if $(-m + d + n)d + (d + n)h < 0$.

(ii) We cannot reach the origin if $(-m + d + n)d + (d + n)h > 0$.

Sub case 2*:* If $(-m + d + n) > 0$ then:

(i) We cannot reach the origin if $(-m + d + n)d + (d + n)h < 0$.

(ii) We reach the origin if $(-m + d + n)d + (d + n)h > 0$.

Case 2*:* when $k_1 \neq 0$

(a) $k_2 = 0$ and $k_3 = 0$.

In this case, we cannot reach the origin following the $x-$axis, with $\eta = r$.

(b) $k_2 = 0$ and $k_3 \neq 0$.

In this case, there is a possibility to reach the origin following the $z-$axis, with $\eta = -d$ when $h = 0$. while it cannot reach the origin following the $x-$axis with $\eta = r$.

(c) $k_2 \neq 0$ **and** $k_3 = 0$.

In this case, there is a possibility to reach the origin following the $y-$axis, with $\eta = n$ when $n < 0$; and cannot reach the origin following $x-$axis, with $\eta = r$.

OR

it cannot be reaching the origin following both axes ($y-$axis and $x-$axis) with $\eta = n$ when $n > 0$ and $\eta = r$.

(d) $k_2 \neq 0$ **and** $k_3 \neq 0$.

In this case, we obtain different results depending on the parameters as these in sub case 1 and sub case 2 above:

So, under the above conditions, the trajectories may follow a fixed direction, that is contained in the positive octant to reach the fixed point E_0.

4.2 The stability analysis at $E_1 = \left(\frac{r}{H}, 0, 0\right)$:

In this subsection the stability of the system (3) near E_1 is studied using the method of Lyapunov function as shown in the following theorem.

Theorem 2: The predator free equilibrium point E_1 is globally asymptotically stable in $\Re_+^3$ if and only if:

$$x_1 < \min\left\{-n, \frac{d}{p}\right\} \tag{20}$$

Proof: Consider the function

$$V^{[1]} = \left(x - x_1 - x_1 \ln\frac{x}{x_1}\right) + y + z \tag{21}$$

Clearly, $V^{[1]} : \Re_+^3 \to \Re$ and $V^{[1]}(E_1) = 0$ with $V^{[1]}(E) \neq 0 \quad \forall\, E \neq E_1$ and $E \in \Re_+^3$. Hence it is positive definite function in the $\Re_+^3$.

Now, the derivative of $V^{[1]}$ with respect to the time t is given as follows.

$$\begin{aligned}
\frac{dV^{[1]}}{dt} &= \left[\frac{x - x_1}{x}\right]\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} \\
&= x(r - Hx) - x_1\left(r - Hx - \frac{y}{1+x} - \frac{pz}{1+x}\right) + ny - dz \\
&= -H(x - x_1)^2 + y\left(\frac{x_1}{1+x} + n\right) - z\left(d - \frac{px_1}{1+x}\right) \\
&\leq -H(x - x_1)^2 + y(x_1 + n) - z(d - px_1)
\end{aligned}$$

Hence, $\frac{dV^{[1]}}{dt} < 0$ under the sufficient condition (20), then $V^{[1]}$ is a Lyapunov function. Therefore by using the Lyapunov Theorem for stability E_1 is locally asymptotically stable in $\Re_+^3$. Further more, since $\frac{dV^{[1]}}{dt} < 0$ on the domain $\Re_+^3$ then E_1 is a globally asymptotically stable too. ■

4.3 The stability analysis at E_2:

The Jacobian matrix of the system (3) at E_2 is given by:

$$J_2 \equiv J(E_2) = \begin{pmatrix} r - y_2 & 0 & 0 \\ y_2 & 0 & -m + h \\ 0 & 0 & m - d - h \end{pmatrix} \tag{22}$$

Clearly J_2 has zero eigenvalue and hence E_2 is nonhyperbolic point. Therefore in the following theorem we will study the stability of E_2 using Lyapunov function.

Theorem 3: The axial equilibrium point E_2 is locally asymptotically stable in the $\Re_+^3$ if the following conditions hold:

$$y_2 > (r - Hx)(1 + x); \tag{23a}$$

$$d > \frac{my_2}{y+z} - \frac{hy_2}{y}; \tag{23b}$$

Proof: Consider the following function:

$$V^{[2]} = x + \left(y - y_2 - y_2 \ln\frac{y}{y_2}\right) + z \tag{24}$$

Clearly, $V^{[2]} : \Re_+^3 \to \Re$ and $V^{[2]}(E_2) = 0$ with $V^{[2]}(E) \neq 0 \quad \forall\, E \neq E_2$ and $E \in \Re_+^3$. Hence it is positive definite function in the $\Re_+^3$.

Now, the derivative of $V^{[2]}$ with respect to the time t can be written as

$$\begin{aligned}
\frac{dV^{[2]}}{dt} &= \frac{dx}{dt} + \left(\frac{y - y_2}{y}\right)\frac{dy}{dt} + \frac{dz}{dt} \\
&= x\left(r - Hx - \frac{y_2}{1+x}\right) + \frac{n(y - y_2)^2}{y} \\
&\quad - z\left(\frac{hy_2}{y} - \frac{my_2}{y+z} + d\right)
\end{aligned}$$

Since E_2 exists if and only if $n = 0$, further conditions $(23a)$ and $(23b)$ guarantees that $\frac{dV^{[2]}}{dt} < 0$ then $V^{[2]}$ is a Lyapunov function. Therefore by using the Lyapunov Theorem for stability E_2 is locally asymptotically stable in the $\Re_+^3$. ■

Note that, according to condition (23) it is easy to verify that E_2 is locally asymptotically stable but not globally.

4.4 The stability analysis at E_3:

The Jacobian matrix of the system (3) at E_3 is given by:

$$J_3 \equiv J(E_3) = \begin{pmatrix} r-2Hx_3-\frac{y_3}{(1+x_3)^2} & n & np \\ \frac{y_3}{(1+x_3)^2} & 0 & -m+h \\ 0 & 0 & m-np-d-h \end{pmatrix} \quad (25)$$

So, the characteristic equation of J_3 can be written by

$$\left((m-np-d-h)-\mu_z\right) \times \left(\mu^2-\left(r-2Hx_3-\frac{y_3}{(1+x_3)^2}\right)\mu + \left(-n\frac{y_3}{(1+x_3)^2}\right)\right)=0$$

from which, we obtain that:

$$\mu_z=(m-np-d-h) \quad (26a)$$

and

$$\mu_x+\mu_y=\left(r-2Hx_3-\frac{y_3}{(1+x_3)^2}\right)$$
$$\mu_x\cdot\mu_y=\left(\frac{-ny_3}{(1+x_3)^2}\right)>0 \quad (26b)$$

Here μ_x, μ_y and μ_z denote to the eigenvalues in the $x-$direction, $y-$direction and $z-$direction, respectively.

So, it is easy to verify that, all the eigenvalues have negative real parts if and only if

$$m<np+d+h \quad (27a)$$

$$r<2Hx_3+\frac{y_3}{(1+x_3)^2} \quad (27b)$$

Therefore, the equilibrium point E_3 is locally asymptotically stable in the $\Re^3_+$ if and only if the conditions 27a and 27b hold.

4.5 The stability analysis at E_4 :

The Jacobian matrix of the system (3) at E_4 is given by:

$$J_4 \equiv J(E_4)$$
$$\begin{pmatrix} r-y_4(1+pB) & 0 & 0 \\ y_4 & \frac{-(m-(d+h))^2}{m}+n & \frac{-(d+h)^2}{m}+h \\ pBy_4 & \frac{(m-(d+h))^2}{m} & \frac{(d+h)^2}{m}-d-h \end{pmatrix} \quad (28)$$

Similarly, the characteristic equation of J_4 is given by:

$$\left(r-y_4(1+pB)-\delta_x\right) \times \left(\delta^2-\left(n-\left(m-(d+h)\right)\right)\delta + \left(\frac{(m-(d+h))(d+h)}{m}(dB-n)\right)\right)=0$$

Therefore, the eigenvalues of J_4 satisfy the following:

$$\delta_x=r-y_4(1+pB)$$

and

$$\delta_y+\delta_z=\left(n-\left(m-(d+h)\right)\right)$$
$$\delta_y\cdot\delta_z=\left(\frac{(m-(d+h))(d+h)}{m}(dB-n)\right)$$

Here δ_x, δ_y and δ_z denote to the eigenvalues in the $x-$direction, $y-$direction and $z-$direction, respectively.

So, all the eigenvalues have negative real parts if and only if

$$r<y_4(1+pB) \quad (29a)$$

$$n<\frac{d}{d+h}\left(m-(d+h)\right) \quad (29b)$$

Thus, the equilibrium point E_4 is locally asymptotically stable in $\Re^3_+$, if and only if the conditions 29a and 29b hold.

4.6 The stability analysis at E_5 :

The Jacobian matrix of the system (3) at E_5 is given by:

$$J_5 \equiv J(E_5)=\left(\beta_{ij}^{[5]}\right)_{3\times3} \quad (30)$$

where:

$$\beta_{11}^{[5]}=r-2Hx_5-\frac{(r-Hx_5)}{(1+x_5)}; \quad \beta_{12}^{[5]}=\frac{-x_5}{(1+x_5)};$$
$$\beta_{13}^{[5]}=\frac{-px_5}{(1+x_5)}; \quad \beta_{21}^{[5]}=\frac{y_5}{(1+x_5)^2};$$
$$\beta_{22}^{[5]}=\frac{x_5}{(1+x_5)}-\frac{1}{m}\left(m-\left(d+h-\frac{px_5}{(1+x_5)}\right)\right)^2+n;$$
$$\beta_{23}^{[5]}=\frac{-1}{m}\left(d+h-\frac{px_5}{(1+x_5)}\right)^2+h;$$
$$\beta_{32}^{[5]}=\frac{1}{m}\left(m-\left(d+h-\frac{px_5}{(1+x_5)}\right)\right)^2; \beta_{31}^{[5]}=\frac{pz_5}{(1+x_5)^2};$$
$$\beta_{33}^{[5]}=\frac{-1}{m}\left(m-\left(d+h-\frac{px_5}{(1+x_5)}\right)\right)\left(d+h-\frac{px_5}{(1+x_5)}\right)$$

Theorem 4: The coexistence equilibrium point E_5 of the system (3) is locally asymptotically stable in the $Int.\Re^3_+$ if and only if the following conditions are satisfied:

$$r-2Hx_5<\min\Big\{\frac{(r-Hx_5)}{(1+x_5)}\ ,$$

$$\frac{(r-Hx_5)}{(1+x_5)}\Big[1-mpx_5/\Big[(1+x_5)(1+pQ) \qquad (31a)$$

$$\times\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big]\Big]$$

$$n+\frac{x_5}{(1+x_5)}<\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)^2 \qquad (31b)$$

$$h<\frac{1}{m}\Big(d+h-\frac{px_5}{(1+x_5)}\Big)^2-\frac{Q}{m}\Big(d+h-\frac{px_5}{(1+x_5)}\Big)^2 \qquad (31c)$$

Proof: According to the Jacobian matrix J_5 at the equilibrium point E_5, the characteristic equation of J_5 can be written by:

$$\gamma^3+D_1\gamma^2+D_2\gamma+D_3=0 \qquad (32)$$

where the coefficients:

$$\left.\begin{aligned}
D_1&=-\left(\beta_{11}^{[5]}+\beta_{22}^{[5]}+\beta_{33}^{[5]}\right)\\
D_2&=\left(\beta_{11}^{[5]}\beta_{22}^{[5]}-\beta_{12}^{[5]}\beta_{21}^{[5]}\right)+\left(\beta_{11}^{[5]}\beta_{33}^{[5]}\right.\\
&\quad\left.-\beta_{13}^{[5]}\beta_{31}^{[5]}\right)+\left(\beta_{22}^{[5]}\beta_{33}^{[5]}-\beta_{23}^{[5]}\beta_{32}^{[5]}\right)\\
D_3&=\beta_{21}^{[5]}\left(\beta_{12}^{[5]}\beta_{33}^{[5]}-\beta_{13}^{[5]}\beta_{32}^{[5]}\right)-\beta_{22}^{[5]}\left(\beta_{11}^{[5]}\beta_{33}^{[5]}\right.\\
&\quad\left.-\beta_{13}^{[5]}\beta_{31}^{[5]}\right)+\beta_{23}^{[5]}\left(\beta_{11}^{[5]}\beta_{32}^{[5]}-\beta_{12}^{[5]}\beta_{31}^{[5]}\right)\\
\Delta&=D_1D_2-D_3=-\beta_{11}^{[5]}\left(\beta_{22}^{[5]}+\beta_{33}^{[5]}\right)^2-\beta_{22}^{[5]}\\
&\times\beta_{33}^{[5]}\left(\beta_{22}^{[5]}+\beta_{33}^{[5]}\right)+\beta_{11}^{[5]}\left(\beta_{12}^{[5]}\beta_{21}^{[5]}+\beta_{13}^{[5]}\beta_{31}^{[5]}\right)\\
&-\left(\beta_{11}^{[5]}\right)^2\left(\beta_{22}^{[5]}+\beta_{33}^{[5]}\right)+\beta_{22}^{[5]}\left(\beta_{12}^{[5]}\beta_{21}^{[5]}\right.\\
&\left.+\beta_{23}^{[5]}\beta_{32}^{[5]}\right)+\beta_{33}^{[5]}\left(\beta_{13}^{[5]}\beta_{31}^{[5]}+\beta_{23}^{[5]}\beta_{32}^{[5]}\right)\\
&+\beta_{21}^{[5]}\beta_{13}^{[5]}\beta_{32}^{[5]}+\beta_{12}^{[5]}\beta_{31}^{[5]}\beta_{23}^{[5]}
\end{aligned}\right\}\ (33)$$

Now, by substituting the elements of J_5 in the equation.33 and then simplifying the resulting terms we obtain that:

$$D_1=-\Big(r-2Hx_5-\frac{(r-Hx_5)}{(1+x_5)}\Big)$$

$$-\Big(\frac{x_5}{(1+x_5)}-\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)^2+n\Big)$$

$$+\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)\Big(d+h-\frac{px_5}{(1+x_5)}\Big)$$

$$D_2=\Big[\Big(\frac{x_5}{(1+x_5)}-\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)^2+n\Big)$$

$$\times\Big(r-2Hx_5-\frac{(r-Hx_5)}{(1+x_5)}\Big)+\frac{x_5(r-Hx_5)}{(1+pQ)(1+x_5)^2}\Big]$$

$$+\Big[\frac{-1}{m}\Big(r-2Hx_5-\frac{(r-Hx_5)}{(1+x_5)}\Big)\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)$$

$$\times\Big(d+h-\frac{px_5}{(1+x_5)}\Big)+\frac{p^2Qx_5(r-Hx_5)}{(1+pQ)(1+x_5)^2}\Big]$$

$$-\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)\Big(d+h-\frac{px_5}{(1+x_5)}\Big)$$

$$\times\Big[\Big(\frac{x_5}{(1+x_5)}-\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)^2+n\Big)$$

$$+Q\Big(h-\frac{1}{m}\Big(d+h-\frac{px_5}{(1+x_5)}\Big)^2\Big)\Big]$$

$$D_3=\frac{x_5(r-Hx_5)}{m(1+x_5)^2}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)\Big(d+h-\frac{px_5}{(1+x_5)}\Big)$$

$$+\Big(\frac{x_5}{(1+x_5)}-\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)^2+n\Big)$$

$$\Big[\frac{1}{m}\Big(r-2Hx_5-\frac{(r-Hx_5)}{(1+x_5)}\Big)\Big(d+h-\frac{px_5}{(1+x_5)}\Big)$$

$$\times\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)-\frac{p^2Qx_5(r-Hx_5)}{(1+x_5)^2(1+pQ)}\Big]$$

$$+\Big(h-\frac{1}{m}\Big(d+h-\frac{px_5}{(1+x_5)}\Big)^2\Big)\Big[\frac{1}{m}\Big(r-2Hx_5-\frac{(r-Hx_5)}{(1+x_5)}\Big)$$

$$\times\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)^2+\frac{pQx_5(r-Hx_5)}{(1+x_5)^2(1+pQ)}\Big]$$

$$\Delta=-\Big[\Big(\frac{x_5}{(1+x_5)}-\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)^2+n\Big)$$

$$-\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big]^2$$

$$\times\Big(r-2Hx_5-\frac{(r-Hx_5)}{(1+x_5)}\Big)+\frac{1}{m}\Big(\frac{x_5}{(1+x_5)}+n$$

$$-\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)^2\Big)\Big(d+h-\frac{px_5}{(1+x_5)}\Big)$$

$$\times\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)\Big[\Big(\frac{x_5}{(1+x_5)}+n$$

$$-\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)^2\Big)$$

$$-\frac{1}{m}\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)\Big]\Big]$$

$$\times-\frac{x_5(r-Hx_5)}{(1+pQ)(1+x_5)^2}\Big(1+p^2Q\Big)\Big(r-2Hx_5-\frac{(r-Hx_5)}{(1+x_5)}\Big)$$

$$-\Big(r-2Hx_5-\frac{(r-Hx_5)}{(1+x_5)}\Big)^2$$

$$\times\Big[\Big(\frac{x_5}{(1+x_5)}-\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)^2+n\Big)$$

$$-\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big]$$

$$-\Big(\frac{x_5}{(1+x_5)}-\frac{1}{m}\Big(m-\Big(d+h-\frac{px_5}{(1+x_5)}\Big)\Big)^2+n\Big)$$

$$\times\left[\frac{x_5(r-Hx_5)}{(1+pQ)(1+x_5)^2}-\frac{1}{m}\left(m-\left(d+h-\frac{px_5}{(1+x_5)}\right)\right)^2\right.$$
$$\left.\times\left(h-\frac{1}{m}\left(d+h-\frac{px_5}{(1+x_5)}\right)^2\right)\right]+\frac{1}{m}\left(d+h-\frac{px_5}{(1+x_5)}\right)$$
$$\times\left(m-\left(d+h-\frac{px_5}{(1+x_5)}\right)\right)\left[\frac{p^2Q\,x_5(r-Hx_5)}{(1+pQ)(1+x_5)^2}\right.$$
$$\left.-\frac{1}{m}\left(m-\left(d+h-\frac{px_5}{(1+x_5)}\right)\right)^2\left(h-\frac{1}{m}\left(d+h-\frac{px_5}{(1+x_5)}\right)^2\right)\right]$$
$$-\frac{p\,Q\,x_5(r-Hx_5)}{m(1+pQ)(1+x_5)^2}\left[\left(d+h-\frac{px_5}{(1+x_5)}\right)\right.$$
$$\left.\times\left(m-\left(d+h-\frac{px_5}{(1+x_5)}\right)\right)+m\left(h-\frac{1}{m}\left(d+h-\frac{px_5}{(1+x_5)}\right)^2\right)\right]$$

Therefore, it is easy to verify that $D_i > 0$ for $i = 1, 3$ and $\Delta > 0$ provided that the conditions 31a , 31b and 31c hold.
So according to Routh-Hurwitz criterion the equilibrium point E_5 is locally asymptotically stable. ■

5. Numerical Simulation

In this section the dynamical behavior of system(3) is studied numerically. The system (3) is solved numerically for different sets of parameters values and different sets of initial points. The objectives are confirming our analytical results and investigate the effect of varying the infection rate parameter on the dynamical behavior of system (3). Now for the following set of hypothetical parameters values:

$$r = 1 \quad , H = 0.1 \quad , p = 0.7 \quad , m = 0.35$$
$$n = -0.8 \quad , h = 0.1 \quad , d = 0.6 \qquad (34)$$

The trajectories of the system (3) are drawn in the figure 2.

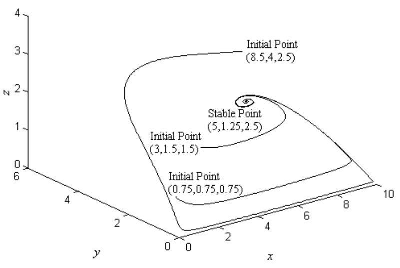

Figure 2- Phase plot of system (3) starting from different initial points.

According to the above figure, the system (3) approaches asymptotically to the stable coexistence equilibrium point $E_5 = (5,1.25,2.5)$ starting at different initial points.

Now observe the dynamical behavior of the system (3) for the set of parameter values given by Equation. (34) while the infection rate m varying at the values $m = 0.1, 0.2, 0.8$ respectively and then the trajectories of system (3) are drawn in figure 3a-3c and I figure 4a-4c respectively.

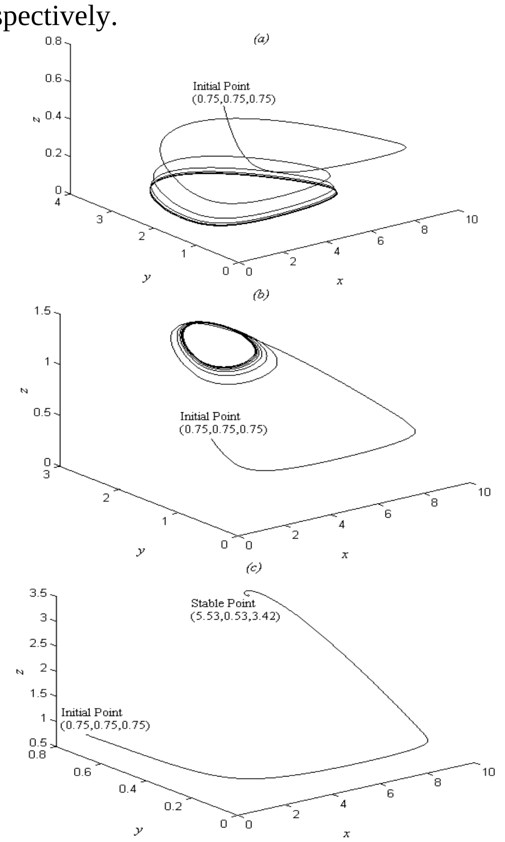

Figure 3-Phase plots of system (3). (a) periodic attractor of system(3) for $m = 0.1$, (b) periodic attractor of system(3) for $m = 0.2$, (c) asymptotically stable of system (3) for $m = 0.8$.

Note that, we will use the following lines types (—); (–•–) and (▬) in the figure 4a-4c to describe the prey species, susceptible predator and infected predator species respectively.

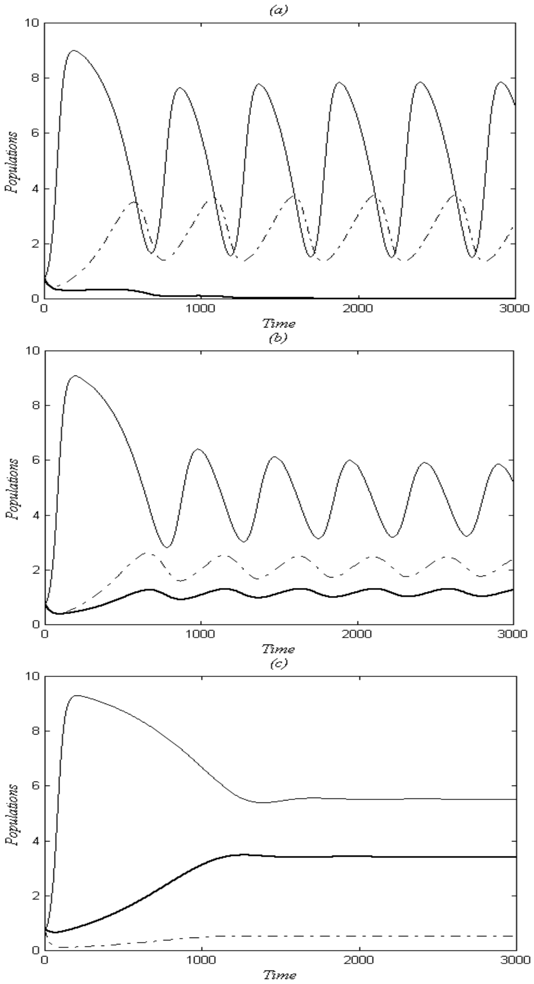

Figure 4- Time series of the solution of system(3);
(a) time series for the attractor in Figure $(3a)$,
(b) time series for the attractor in Figure $(3b)$,
(c) time series for the attractor in Figure $(3c)$.

Observe that, as the infection rate increases, say $m > 0.217$, the dynamical behavior of the system transfers from periodic dynamics to asymptotically stable at the coexistence equilibrium point. Thus the infection rate constant works as a stabilizing parameter in the system (3).

Now to understand of dynamical behavior of the system (3) at the vanishing equilibrium point E_0 the following set of hypothetical parameter values is chosen:

$$\begin{array}{l} r = 0.01 \quad , H = 0.0001 \quad , p = 0.7 \quad , m = 0.3 \\ n = -0.01 \quad , h = 0.1 \quad , d = 0.1 \end{array} \qquad (35)$$

and then the trajectories of the system (3) are drawn in the figure 5 and figure 6a-6c starting from different initial values. Again, in figure 6 we will use (—) to describe the trajectory starting at (0.75,0.75,0.75); (▬) to describe the trajectory starting at (0.65,0.65,0.65) and (–•–) to describe the trajectory starting at (0.45,0.45,0.45).

According to these figures it is clear that, the system (3) approaches asymptotically to the vanishing equilibrium point $E_0 = (0\,,0\,,0)$, which insure our analytical result.

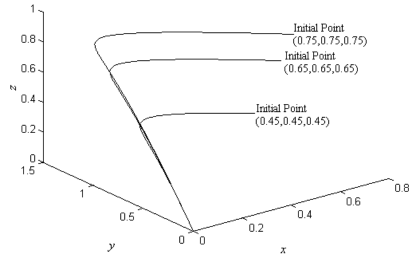

Figure 5- Phase plot of the system (3) starting from different initial points.

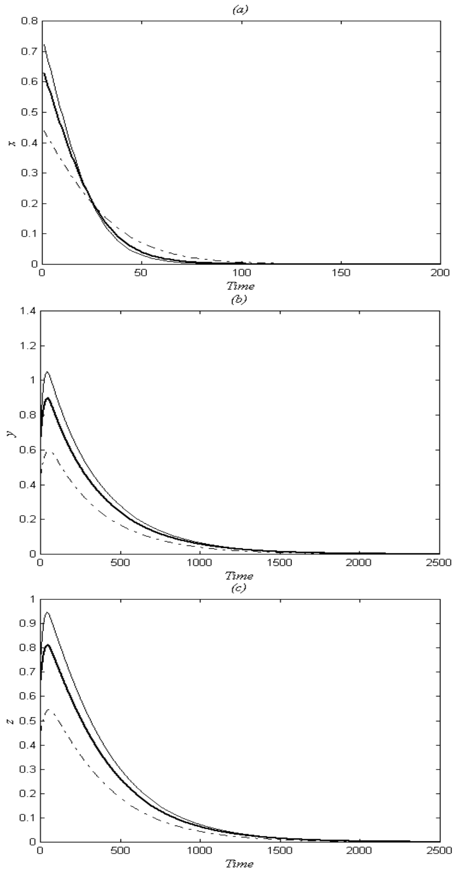

Figure 5-Time series of figure 5; (a) trajectory of x as a function of time, (b) trajectory of y as a function of time, (c) trajectory of z as a function of time.

6. Conclusions

In this paper, an eco-epidemiological model has been proposed and analyzed to study the dynamical behavior of a Holling-type II prey–predator model with ratio-dependent incidence rate for the disease in predator species. The model consists of three non-linear autonomous differential equations that describe the dynamics of three different populations namely prey (x), susceptible predator (y), infected predator (z). It is observed that, the system (3) is bounded if the net growth rate of the susceptible predators n is negative (i.e. the naturally mortality rate larger than the alternative source of food). The conditions for existence and stability for each equilibrium points are obtained.

In order to confirm our analytical results and understand the effect of varying the infection rate m parameter on the dynamical behavior of the system (3), system (3) has been solved numerically for different sets of initial points and different sets of parameters and the following observations are made:

1. For the set of hypothetical parameters values given by Equation.34, the system (3) approaches asymptotically to globally stable point $E_5 = (5,1.25,2.5)$.
2. It is observed that, for the values of infection rate m in the range $0.21 < m < 1$, system (3) has asymptotically stable point in the $Int.\Re^3_+$. While decreasing the value of m further leads to periodic dynamics in the $Int.\Re^3_+$. Therefore, the infection rate parameter play vital role in controlling the stability of system (3).
3. For the set of parameters values given by Equation 35 the system approaches asymptotically to vanishing equilibrium point E_0 which conform our analytic.

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