Proving The Existence and the Uniqueness Solutions of fractional Integro-Differential Equations

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Abstract

In this paper, we will study and prove the existence and the uniqueness theorems of solutions of the generalized linear integro-differential equations with unequal fractional order of differentiation and integration by using Schauder fixed point theorem. This type of fractional integro-differential equation may be considered as a generalization to the other types of fractional integro-differential equations Considered by other researchers, as well as, to the usual integro-differential equations.

Keyword: fractional integro-differential equation, Schauder fixed point.

مبرهنة وجود ووحدانية الحلول للمعادلات التفاضلية التكامليةالكسرية

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الخلاصة:

في هذا البحث, سنقوم بدراسة وبرهان مبرهنة وجود ووحدانية حلول الصيغة العامة للمعادلات التفاضلية التكاملية ولرتب اشتقاق وتكامل كسري غير متساوية وذلك باستخدام مبرهنة شاودر للنقطة الصامدة .
هذا النوع من معادلات تفاضلية تكاملية ذات رتب كسرية يمكن اعتباره أعمام إلى الأنواع الأخرى من المعادلات التفاضلية التكاملية ذات الرتب الكسرية والتي درست من قبل بقية الباحثين , بالإضافة إلى المعادلات التفاضلية التكاملية الاعتيادية.

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1- Introduction

Consider the linear integro-differential equation of fractional order:

$D^q u(t) = f(t) + J^p u(t),\ 0 < q, p \leq 1$ (1.)

$u(0) = u_0$ (1.)

where D^q is the Caputo fractional derivative operator of order q, J^p denotes the Riemann-Liouville fractional integral operator of order p, $f \in C\,[0, T]$.

in recent years, the study of fractional integro-fractional differential equations as a basic theoretical part of some applications are investigated by many authors and therefore there have been interest in the study of fractional integro-differential equation of the type

$$D^q u(t) = f(t,u(t)) + \int_0^t k(t,s,u(s))ds\ ,0<q\leq 1$$

With initial condition u (0) = u.0

Where f is a continuous function on (t, u) for $u\in R$,a > 0 and 0 < t <a , k is a continuous function on (t, s, u) for $u\in R$ and 0< t ,s < a , u0 is a real positive constant and D^q denotes the Caputo fractional derivative , (see[1,2,3,4 and 5]). In ref. [6] the author justify the existence and the uniqueness of equations 1 and 2 of the same order p and q, while in this paper we concern with the existence and the uniqueness of the solutions of equations 1 and 2 with different fractional orders p and q, and we shall use Schauder fixed point theorem to prove the existence of solution, while the Grounwall's inequality have been used to obtain the uniqueness of solutions of the fractional integro-differential equation.

Moreover , the operator of the fractional integro-differential equations 1 and 2 , becomes:

$Au(t) = f(t)$ (3)

where:

$Au(t) = D^q u(t) - J^p u(t)$ (4)

2- Preliminaries

Before proving the existence and the uniqueness theorems of fractional integro-fractional differential equations, some basic and fundamental concepts which are necessary for this work must be given first.

Definition 2.1: [7,8]

The Riemann-Liouville fractional integral operator of order $p \geq 0$, of a function f(x), $x \in R$ is defined by:

$$J^p f(x) = \frac{1}{\Gamma(p)}\int_0^x (x-t)^{p-1} f(t)\,dt\ ,\ p > 0 \quad (5)$$

and $J^0 f(x) = f(x)$.

properties of the operator J^p , for $f \in C[0,T]$, q , p≥ 0 and ∂ >-1 , we have:

1. $J^p J^q f(x) = J^{p+q} f(x)$
2. $J^p J^q f(x) = J^q J^p f(x)$
3. $\Gamma(p+1) = p\,\Gamma(p)$
4. $J^p x^{\partial} = \dfrac{\Gamma(\partial+1)}{\Gamma(p+\partial+1)} x^{p+\partial}$

Definition 2.2: [9 , 10]

The Caputo fractional derivative of f(x) of order q can be written as:

$$D^q f(x) = \frac{1}{\Gamma(m-q)}\int_a^x (x-t)^{m-q-1} f^{(m)}(t)\,dt \quad (6)$$

For $m - 1 < q \leq m$, $m \in R$, $x > 0$, $f \in C[0, T]$.

Definition 2.3: [11]:

A subset S of C[0, T] is said to be equicontinuous, if for each $\varepsilon > 0$, there is a **δ**> 0, such that $|t - t_1| < \delta$ and $u\in M$ imply $\|u(t) - u(t_1)\|_{C[0,T]} < \varepsilon$.

Next, equations. 1 and 2 may be written into an equivalent form, as in the following lemma:

Lemma 2.1: [5]

The solution of the initial-value problem given by equations.1 and 2 has the form:

$$u(t) = u_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)\,ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\{\frac{1}{\Gamma(p)}\int_0^t (s-v)^{p-1} u(v)\,dv\right\}ds \quad (7)$$

Also, the following theorems are used later on.

Theorem 2.1(Schauder Fixed Point) [12]:

Let X be a nonempty, closed, bounded and convex subset of Banach space B and $T: X \longrightarrow X$ is a compact operator. Then T has at least one fixed point in X.

Theorem 2.2 (Arzela-Ascoli Theorem)[13]**:**

Suppose F is a Banach space and E is a compact metric space. In order that a subset H of the Banach space ƮF(E) be relatively compact, if and only if H be equicontinuous and that , for each $x \in E$, the set $H(x) = \{f(x): f \in H\}$ be relatively compact in F.

Theorem 2.3 (Grounwall's Inequality)[5]**:**

Let u(t) and b(t) be a nonnegative continuous functions for $t \geq \alpha$ and let:

$$u(t) \leq a + \int_{\alpha}^{t} b(s)u(s)\,ds, \quad t \geq \alpha$$

where a is a nonnegative constant, then:

$$u(t) \leq a\, e^{\int_{\alpha}^{t} b(s)\,ds}, \quad t \geq \alpha$$

3- The Main Results

This section concerned with the proof of the main theorems of the existence and the uniqueness of solutions of equations.1 and 2

Theorem 3.1 (The Existence Theorem):

Let u and $u^{(m)}$ be a real nonnegative function in C[0, T], and that $t \in [0, T]$, $0 < q , p \leq 1$. Then eqs. (2.1)-(2.2) has a solution u.

Proof:

First, let us define B = C[0, T] to be the Banach space with the supremum norm. In order to discuss the condition for the existence for the solution of eqs.(1.1) and (1.2), so let:

$U = \{u \in C[0, T] : \|u\| \leq c_1, \|u^{(m)}\| \leq c_2, c_1, c_2 > 0, m \geq 0\}$

and suppose that $f \in C[0, T]$ is bounded function at t_0, there exist $M \in R^+$, such that:

$\|f(t)\| \leq M, \forall t \in [0, T]$.

Now, in order to use Schauder fixed point theorem, then it sufficient to prove that U is a nonempty closed, bounded and convex subset of the Banach space B and then the operator $A : U \longrightarrow U$ is compact operator, where the operator A was defined in equation.4.

It can be seen that the set U is nonempty since from the properties of the norm we have $0 \in U$; on the other hand it is closed and bounded subset of Banach space (from the definition of U).

To prove that U is convex set, let $u_1, u_2 \in U$, such that:

$\|u_1\| \leq c_1$, $\|u_1^{(m)}\| \leq c_2$

$\|u_2\| \leq c_1$, $\|u_2^{(m)}\| \leq c_2$

i.e., to prove that u(t), $u^{(m)}(t) \in C[0, T]$ and Moreover , to prove that $\|u(t)\| \in U$ and $\|u^{(m)}(t)\| \in U$,where

$u(t) = \lambda u_1(t) + (1 - \lambda)u_2(t)$

$u^m(t) = \lambda u_1^{(m)}(t) + (1 - \lambda) u_2^{(m)}(t)$, $\lambda \in [0,1]$

since we can prove easily:

$\|u(t)\| \leq c_1$ and $\|u^{(m)}(t)\| \leq c_2$

as follows:

$\|u(t)\| = \|\lambda u_1(t) + (1 - \lambda)u_2(t)\|$

$\leq \|\lambda u_1(t)\| + \|(1 - \lambda)u_2(t)\|$

$= |\lambda|\|u_1(t)\| + |1 - \lambda|\,\|u_2(t)\|$

$\leq \lambda c_1 + (1 - \lambda) c_1$

$= c_1$

and

$\|u^m(t)\| = \|\lambda u_1^{(m)}(t) + (1 - \lambda) u_2^{(m)}(t)\|$

$\leq \|\lambda u_1^{(m)}(t)\| + \|(1 - \lambda) u_2^{(m)}(t)\|$

$= |\lambda|\,\|u_1^{(m)}(t)\| + |1 - \lambda|\,\|u_2^{(m)}(t)\|$

$\leq \lambda c_2 + (1-\lambda) c_2$

$= c_2$

Therefore, u satisfies the conditions of U, so:

$u(t) = \lambda u_1(t) + (1 - \lambda)u_2(t) \in U$

Hence, U is a convex set.

Now, we have to show that the operator A in equation. 4 is completely continuous, in order to see that equations.1 and 2 has a solution first, one can prove that A is relatively compact.

Let v(t)=Au(t),to prove $v(t) \in U$

$$|v(t)| = \left| \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} u^{(m)}(s)\,ds - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} u(s)ds \right|$$

$$\leq \frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1} \| u^{(m)}(s) \| ds + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} \| u(s) \| ds$$

$$= \frac{\sup_{t\in[0,T]} | u^{(m)}(t) |}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1} ds + \frac{\sup_{t\in[0,T]} | u(t) |}{\Gamma(p+1)}\int_0^t (t-s)^{p-1} ds$$

$$= \frac{\sup_{t\in[0,T]} | u^{(m)}(t) |}{\Gamma(m-q+1)} t^{m-q} + \frac{\sup_{t\in[0,T]} | u(t) |}{\Gamma(p+1)} t^{p}$$

$$\leq \frac{\sup_{t\in[0,T]} | u^{(m)}(t) |}{\Gamma(m-q+1)} T^{m-q} + \frac{\sup_{t\in[0,T]} | u(t) |}{\Gamma(p+1)} T^{p}$$

$$\leq c_2 \frac{T^{m-q}}{\Gamma(m-q+1)} + c_1 \frac{T^{p}}{\Gamma(p+1)}$$

$$= c$$

That is v(t) is bounded.

$$| v^{(k)}(t) | = \left| \frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1} u^{(km)}(s)\, ds - \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} u^{(k)} ds \right| \leq$$

$$\frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1} \| u^{(km)}(s) \| ds + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} \| u^{(k)}(s) \| ds$$

$$= \frac{\sup_{t\in[0,T]} | u^{(km)}(t) |}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1} ds + \frac{\sup_{t\in[0,T]} | u^{(k)}(t) |}{\Gamma(p)}\int_0^t (t-s)^{p-1} ds$$

$$= \frac{\sup_{t\in[0,T]} | u^{(km)}(t) |}{\Gamma(m-q+1)} t^{m-q} + \frac{\sup_{t\in[0,T]} | u^{(k)}(t) |}{\Gamma(p+1)} t^{p}$$

$$\leq c_2 \frac{T^{m-q}}{\Gamma(m-q+1)} + c_2 \frac{T^{p}}{\Gamma(p+1)}$$

$$\leq c^* .$$

Proving that A maps U into itself. Moreover, A is bounded operator.

To prove that A is continuous, let u, v $\in$ U, then:

$|Au(t) - Av(t)| = |(D^q u(t) - J^p u(t)) - (D^q v(t) - J^p v(t))|$

$$= \left| \frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1} u^{(m)}(s)\, ds - \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} u(s) ds \right| - \left| \frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1} v^{(m)}(s)\, ds - \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} v(s) ds \right|$$

$$\leq \left| \frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1} (u^{(m)}(s) - v^{(m)}(s))\, ds + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} (u(s) - v(s)) ds \right|$$

$$\leq \frac{\sup_{t\in[0,T]} | u^{(m)}(t) - v^{(m)}(t) |}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} ds + \frac{\sup_{t\in[0,T]} | u(t) - v(t) |}{\Gamma(p)} \int_0^t (t-s)^{p-1} ds$$

$$\leq \frac{\sup_{t\in[0,T]} | u^{(m)}(t) - v^{(m)}(t) |}{\Gamma(m-q+1)} T^{m-q} + \frac{\sup_{t\in[0,T]} | u(t) - v(t) |}{\Gamma(p+1)} T^{p}$$

Let w=u-v

$$\leq \frac{\sup_{t\in[0,T]} |w^{(m)}(t)|}{\Gamma(m-q+1)} T^{m-q} + \frac{\sup_{t\in[0,T]} |w(t)|}{\Gamma(p+1)} T^{p} \leq c.$$

Which means that Au is bounded operator led to Au is continuous.

Now, to prove that A is equicontinuous,

Let $u \in U$ and $t_1, t_2 \in [0, T]$, then:

$|Au(t_1)-Au(t_2)| = |[D^q u(t_1)-J^q u(t_1)] - [D^q u(t_2)-J^q u(t_2)]|$

$$= \left\| \left[\frac{1}{\Gamma(m-q)} \int_0^{t_1} (t_1-s)^{m-q-1} u^{(m)}(s)\,ds - \frac{1}{\Gamma(m-q)} \int_0^{t_2} (t_2-s)^{m-q-1} u^{(m)}(s)\,ds \right] + \frac{1}{\Gamma(p)} \int_0^{t_1} (t_1-s)^{p-1} u(s)\,ds - \frac{1}{\Gamma(p)} \int_0^{t_2} (t_2-s)^{p-1} u(s)\,ds \right]\right\|$$

$$\leq \frac{\sup_{t\in[0,T]} |u^{(m)}(s)|}{\Gamma(m-q)} \left| \int_0^{t_1} (t_1-s)^{m-q-1}\,ds - \int_0^{t_2} (t_2-s)^{m-q-1}\,ds \right| + \frac{\sup_{t\in[0,T]} |u(s)|}{\Gamma(p)} \left| \int_0^{t_1} (t_1-s)^{p-1}\,ds - \int_0^{t_2} (t_2-s)^{p-1}\,ds \right|$$

$$\leq \frac{\sup_{t\in[0,T]} |u^{(m)}(s)|}{\Gamma(m-q+1)} \| t_1^{m-q} - t_2^{m-q} \| + \frac{\sup_{t\in[0,T]} |u(s)|}{\Gamma(p+1)} \| t_1^{p} - t_2^{p} \|$$

$$\leq \frac{2c_2}{\Gamma(m-q+1)} T^{m-q} + \frac{2c_1}{\Gamma(p+1)} T^{P}$$

$\leq c.$

Au is equicontinuous operator, is relatively compact and this implies that A is completely continuous by Arzela-Ascoli theorem.

Then by Schauder fixed point theorem, A has a fixed point, which corresponds the solution of equation. 3. ■

Now, to study the uniqueness of the solution of equations.1 and 2, we shall prove the following theorem:

Theorem 3.2 (The Uniqueness Theorem):

The initial value problem given by equations.1 and 2 has a unique solution on the interval [0, T] if u and $u^{(m)}$ are continuous functions in the region:

$D = \{(t, u) \mid 0 < t < T, |t - t_0| \leq b\}$ and $u(0)= u_0$ and satisfy the conditions:

$$\int_0^t \left| \frac{1}{\Gamma(q)} (s-\sigma)^{p-1} u(\sigma) - \frac{1}{\Gamma(q)} (s-\sigma)^{p-1} y(\sigma) \right| d\sigma \leq L\,|u-y|$$

for some positive constant L.

Proof:

Let u and y be two solutions to equations 1 and 2, then:

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\,ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left\{ \frac{1}{\Gamma(p)} \int_0^s (s-\sigma)^{p-1} u(\sigma)\,d\sigma \right\} ds$$

$$y(t) = y_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\,ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left\{ \frac{1}{\Gamma(p)} \int_0^s (s-\sigma)^{p-1} y(\sigma)\,d\sigma \right\} ds$$

this implies to:

$$|u(t)-y(t)| \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left\{ \frac{1}{\Gamma(p)} \int_0^t (s-\sigma)^{p-1} |u(\sigma)-y(\sigma)|\,d\sigma \right\} ds$$

for any $\varepsilon > 0$ and $0 < t < T$. Hence:

$$|u(t)-y(t)| \le \varepsilon + \frac{L}{\Gamma(p)} \int_0^t (t-s)^{p-1} |u(s)-y(s)| ds$$

Where $(t-s)^{p-1}$ is non negative since the limit of the integration $\in$ [0, T], and it is differentiation for all t not equal to s ,hence it is continuous function with respect to t.
Then, by using theorem 2.3, we get:

$$|u(t) - y(t)| \le \varepsilon\, e^{\frac{L}{\Gamma(p)} \int_0^t (t-s)^{p-1} ds}$$

Since ε is arbitrary, then as $\varepsilon \longrightarrow 0$, which implies to u(t) = y(t), for all t $\in$ [0, T]. ■

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