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Existence and Uniqueness of the Solution to the Reaction-Diffusion System of FHN type

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Abstract
In this study, we delve into the intricacies of the reaction-diffusion system associated with neuronal activities, focusing on open bounded three dimensions convex domain. Employing the renowned Faedo-Galerkin method, alongside compactness techniques, we establish the uniqueness, existence, and initial data sensitivity of both weak and strong solutions within this framework. Furthermore, a comprehensive case analysis is presented, demonstrating the practical application of this methodology to the reaction-diffusion system under consideration.

Keywords: Existence, Faedo-Galerkin, Neumann boundary conditions, uniqueness, weak solution, strong solutions, reaction-diffusion, Picard's Theorem.

وجود ووحدانية الحل لنظام الانتشار – التفاعل من نوع *FHN*

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الخلاصة
أجري التحليل لنظام التفاعل–الانتشار لنوع المرتبط بالأنشطة العصبية في المجالات المحدبة ذات الحدود المفتوحة ثلاثية الابعاد مع شروط نيومان الحدودية. تم استخدام اسلوب Faedo–Galerkin ومفاهيم التراص لأثبات الوجود ووحدانية الحل والاعتماد المستمر على البيانات الأولية للحلول الضعيفة, تم تقديم تحليل شامل للنظام.

1. Introduction

Mathematics is a fundamental tool across scientific disciplines, aiding analysis and application [1-4]. Reaction-diffusion systems, comprising nonlinear parabolic partial differential equations, have garnered significant attention over the years. These systems find diverse applications in fields such as chemistry, ecology, physics, and biology. For an in-depth exploration, references such as [5-7] provide extensive insights. Our study is mostly about a certain reaction-diffusion system that mimics how action potentials spread in heart muscle cells, similar to the Hodgkin-Huxley model and studying Ca+2 waves in Xenopus oocytes [8]. The Hodgkin-Huxley framework, originally comprising four ordinary differential equations to describe potential changes across a nerve cell's membrane, particularly in the squid's giant axon, has been simplified by Fitzhugh, Nagumo et al. [9-14] into a two-equation

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system, known as the FHN equations and Medaka eggs [11]. These equations have seen widespread application, such as in modelling CO oxidation on Pt(110) [12] , and analysing cardiac tissue re-entry [15] . For further application examples, refer to [16-18].

In our study, we examine a reaction-diffusion system represented by four coupled reaction-diffusion equations, subject to Neumann boundary conditions, formulated as follows:

$(\mathcal{M})$ Find $\{u_1, v_1, u_2, v_2\}$ such that

$$\epsilon \frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + af(u_1) - v_1 + \alpha(u_2 - u_1), \qquad in\ Q_T, \qquad (1.1)$$

$$\frac{\partial v_1}{\partial t} = d_2 \Delta v_1 + u_1 - \delta v_1 + \beta(v_2 - v_1), \qquad in\ Q_T, \qquad (1.2)$$

$$\epsilon \frac{\partial u_2}{\partial t} = d_1 \Delta u_2 + af(u_2) - v_2 - \alpha(u_2 - u_1), \qquad in\ Q_T, \qquad (1.3)$$

$$\frac{\partial v_2}{\partial t} = d_2 \Delta v_2 + u_2 - \delta v_2 - \beta(v_2 - v_1), \qquad in\ Q_T, \qquad (1.4)$$

$$\frac{\partial u_1}{\partial v} = 0, \frac{\partial v_1}{\partial v} = 0, \frac{\partial u_2}{\partial v} = 0, \frac{\partial v_2}{\partial v} = 0, \qquad on\ S_T, \qquad (1.5)$$

$$u_1(\cdot\,,0) = u_{1,0}, v_1(\cdot\,,0) = v_{1,0}, u_2(\cdot\,,0) = u_{2,0}, v_2(\cdot\,,0) = v_{2,0}, \qquad in\ \Omega. \qquad (1.6)$$

Where $Q_T = \Omega \times (0, T)$, Ω is an open bounded convex domain in $\mathbb{R}^\eta (\eta = 1,2,3)$, with smooth boundary $\partial\Omega, S_T = \partial\Omega \times (0, T)$, v denotes the exterior unit normal to $\partial\Omega$, d_1 and d_2 are known as the diffusion coefficients for u_1, v_1, u_2 and v_2, respectively, Δ is the Laplace operator, $f(u) = 3u - u^3$. The ϵ and δ are small parameters; and α, β measure the coupling strength, which represents the interactions between neurons. All parameters are considered to be real and finite, as outlined in [19]. The concept of elliptic problems with Neumann boundary conditions, initially introduced in [20], was not widely recognized as a variational problem within a Hilbert space context until later developments. Showalter's work in [21] demonstrated that these elliptic boundary value problems could indeed be interpreted as weak forms in Hilbert spaces.

The study of reaction-diffusion systems with Neumann boundary conditions is important because there aren't many studies that only look at Neumann conditions, even though there is a lot of research on systems with Dirichlet and mixed conditions. Sherratt's work [22] delves into "oscillatory" reaction-diffusion equations, particularly relevant in ecological contexts where Dirichlet conditions often approximate more practical Neumann conditions. Al-Ofl's study [23] provides a mathematical analysis of reaction-diffusion equations under Neumann conditions, establishing the existence, uniqueness, and initial data dependency of both weak and strong solutions. This importance is further emphasized by recent studies incorporating Neumann boundary conditions in their analysis (see [24-32]).

The primary objective of this investigation is to establish the existence and uniqueness of a weak solution for the system described by equations (1.1) -(1.6). This goal is achieved through the application of the Alaoglu Compactness Theorem [33] and the Faedo-Galerkin method [13]. The approach involves approximating the infinite dimensional dynamical system with a finite-dimensional model through a truncated Eigen function expansion. Utilizing Picard's existence theorem, we establish the local existence of a finite weak form of the reaction-diffusion equation. Furthermore, for weak solutions in $L^2(\Omega)$, global existence, uniqueness, and dependence on initial data are demonstrated using the Alaoglu Compactness Theorem and various bound estimates. The methods outlined in [34] are employed to obtain refined results for Neumann boundary value problems. Additionally, the existence, uniqueness, and initial data dependency of strong solutions in $H^1(\Omega)$ are proven, leveraging the low regularity of the initial data [14].

The structure of this study is as follows: Section 2 outlines the essential notation, while Section 3 delves into the existence and uniqueness of weak solutions, Section 4 delves into the existence and continuous dependence.

2. Notation and auxiliary results

In this research, the symbol Ω represents a bounded domain within $\mathbb{R}^{\eta}$, where $\eta = 1,2,3$, characterized by a Lipschitz continuous boundary, denoted as $\partial\Omega$. We employ standard Sobolev spaces, expressed as $W^{h,\beta}(\Omega)$, where h belongs to the set of natural numbers N and β lies within the interval $[1,\infty]$. The norms and semi-norms associated with these spaces are represented by $\|\cdot\|_{h,\beta}$ and $|\cdot|_{h,\beta}$, respectively. Specifically, when $\beta = 2$, the space $W^{h,2}(\Omega)$ is alternatively denoted as $H^h(\Omega)$, accompanied by the norm $\|\cdot\|_h$ and semi-norm $|\cdot|_h$. In cases where h=0, the space $W^{0,2}(\Omega)$ is equivalent to $L^2(\Omega)$. The inner product within the $L^2(\Omega)$ space, spanning over the domain Ω, is indicated by $(\cdot,\cdot)$ and possesses the norm $\|\cdot\|_0 = |\cdot|_0$. Additionally, the notation $<\cdot,\cdot>$ is used to denote the duality pairing between the dual space $\left(H^1(\Omega)\right)'$ and the space $H^1(\Omega)$, where $\left(H^1(\Omega)\right)'$ represents the dual of $H^1(\Omega)$. The norm on the dual space $\left(H^1(\Omega)\right)'$ is defined as follows [13,14]:

$$\|\emptyset_1\|_{\left(H^1(\Omega)\right)'} := \sup_{\gamma\neq 0}\frac{|<\emptyset_1,\gamma>|}{\|\gamma\|_1} \equiv \sup_{\|\gamma\|_1=1}\|<\emptyset_1,\gamma>\|, \gamma\in H^1(\Omega). \tag{2.1}$$

We define specific function spaces that are contingent on both temporal and spatial parameters, denoted as $L^{\beta}(0,T;X)$ for $1\leq\beta\leq\infty$, where X represents a Banach space. These spaces encompass all functions, denoted here as $\emptyset_1$, satisfying the condition that for almost every t in the interval $(0,T)$, $\emptyset_1$ is an element of X and adheres to the criterion that the ensuing norm is bounded and finite:

$$\|\emptyset_1(t)\|_{L^{\beta}(0,T;X)} = \left(\int_0^T |\emptyset_1(t)|_X^{\beta}\,dt\right)^{\frac{1}{\beta}},$$
$$\|\emptyset_1(t)\|_{L^{\infty}(0,T;X)} = ess\sup_{t\in(0,T)}\|\emptyset_1(t)\|_X.$$

Also, we introduce the function space $L^{\beta}(\Omega_T)$ defined as $L^{\beta}\left(0,T;L^{\beta}(\Omega)\right)$, where β ranges from 1 to infinity, denoted as $\beta\in[1,\infty]$. Additionally, we present the space $C([0,T];X)$, which is the set of all continuous functions mapping the interval $[0,T]$ into the Banach space X. This space includes functions $\emptyset_1(t)$ that map from $[0,T]$ to X and satisfy the condition that $\emptyset_1(t)$ converges to $\emptyset_1(t_0)$ in X as t approaches t_0. It is important to note that $C([0,T];X)$ is itself a Banach space, equipped with a specific norm as detailed in [35].

The Sobolev theory is also required in addition to these established results:

$$H^1(\Omega)\overset{c}{\hookrightarrow} L^B(\Omega)\hookrightarrow\left(H^1(\Omega)\right)' \text{ hold for } \beta \in \begin{cases}[1,\infty] & if\ \eta=1,\\ [1,\infty) & if\ \eta=2,\\ [1,6] & if\ \eta=3,\end{cases} \tag{2.2}$$

where $\hookrightarrow$ signifies continuous embedding. Additionally, according to the Rellich-Kondrachov Theorem (refer to [36] p.114 and [37] p.8), the embedding described in (2.2) is compact. This is particularly applicable when $\beta \in [1,6]$ is substituted by $\beta \in [1,6)$ in the context of $\eta = 3$. We denote this compact embedding using the symbol $\overset{c}{\hookrightarrow}$.

The following Hölders inequality is also required frequently: for $1 \leq q_1, q_2 \leq \infty$ such that $\frac{1}{q_1} + \frac{1}{q_2} = 1$ if $\emptyset_1 \in L^{q_1}(\Omega)$ and $\emptyset_2 \in L^{q_2}(\Omega)$ then $\emptyset_1\emptyset_2 \in L^1(\Omega)$ and

$$\|\emptyset_1\emptyset_2\|_{0,1} = \int_\Omega |\emptyset_1\emptyset_2|\,dx \leq \left(\int_\Omega |\emptyset_1|^{q_1}\,dx\right)^{\frac{1}{q_1}}\left(\int_\Omega |\emptyset_2|^{q_2}\,dx\right)^{\frac{1}{q_2}} = \|\emptyset_1\|_{0,q_1}\|\emptyset_2\|_{0,q_2}. \tag{2.3}$$

The aforementioned inequality can be extended by applying it iteratively, resulting in the following generalization:

$$\|\emptyset_1\emptyset_2\emptyset_3\|_{0,1} = \int_\Omega |\emptyset_1\emptyset_2\emptyset_3|\,dx \leq \left(\int_\Omega |\emptyset_1|^{q_1}\,dx\right)^{\frac{1}{q_1}}\left(\int_\Omega |\emptyset_2|^{q_2}\,dx\right)^{\frac{1}{q_2}}\left(\int_\Omega |\emptyset_3|^{q_3}\,dx\right)^{\frac{1}{q_3}} = \|\emptyset_1\|_{0,q_1}\|\emptyset_2\|_{0,q_2}\|\emptyset_3\|_{0,q_3}, \tag{2.4}$$

for $1 \leq q_1, q_2, q_3 \leq \infty,$ such that $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$.

The subsequent inequality, commonly known as Young's inequality, is frequently utilized:

$$h_1h_2 \leq \beta^{\frac{\beta_1}{\beta_2}}\frac{h_1^{\beta_1}}{\beta_1} + \beta^{-1}\frac{h_2^{\beta_2}}{\beta_2}, \qquad \frac{1}{\beta_1} + \frac{1}{\beta_2} = 1, \tag{2.5}$$

valid for any $h_1, h_2 \geq 0, \beta > 0$ and $\beta_1, \beta_2 > 1$. The following is another useful consequence of Young's inequality:

$$h_1h_2 \geq -\beta\frac{h_1^2}{2} - \beta^{-1}\frac{h_2^2}{2}, \qquad \forall h_1, h_2 \in \mathbb{R}^\eta, \forall\beta > 0. \tag{2.6}$$

The differential version of the Grönwall Lemma is also essential:

Lemma 2.1(Grönwall Lemma): Consider $\Psi_1(t) \in W^{1,1}(0,T)$ and $\Psi_2(t), \Psi_3(t), \Psi_4(t) \in L^1(0,T)$, all being non-negative functions. This is deduced from

$$\frac{d\Psi_1(t)}{dt} + \Psi_2(t) \leq \Psi_3(t)\Psi_1(t) + \Psi_4(t)\ a.e.\ \ t \in [0,T],$$

that

$$\Psi_1(T) + \int_0^T \Psi_2(\zeta)d\zeta \leq exp\left(\int_0^T \Psi_3(\zeta)d\zeta\right)\Psi_1(0) + \exp\left(\int_0^T \Psi_3(\zeta)d\zeta\right)\int_0^T \Psi_4(\zeta)d\zeta. \tag{2.7}$$

3. Weak solutions

We present a weak formulation for system $(1.1)-(1.6)$.

$(\mathcal{M})$Find $u_1(.,t),\ v_1(.,t),\ u_2(.,t),\ v_2(.,t)\in H^1(\Omega)$ such that $u_1(.,0)=u_{1,0}(.),\ v_1(.,0)=v_{1,0}(.),\ u_2(.,0)=u_{2,0}(.),\ v_2(.,0)=v_{2,0}(.)$, and for almost every $t\in(0,T)$,

$$\varepsilon\left(\frac{\partial u_1}{\partial t},\eta\right)+d_1(\nabla u_1,\nabla\eta)=a(f(u_1),\eta)-(v_1,\eta)+\alpha(u_2-u_1,\eta),\qquad \forall\eta\in H^1(\Omega),\quad (3.1)$$

$$\left(\frac{\partial v_1}{\partial t},\eta\right)+d_2(\nabla v_1,\nabla\eta)=(u_1,\eta)-\delta(v_1,\eta)+\beta(v_2-v_1,\eta),\qquad \forall\eta\in H^1(\Omega),\quad (3.2)$$

$$\varepsilon\left(\frac{\partial u_2}{\partial t},\eta\right)+d_1(\nabla u_2,\nabla\eta)=a(f(u_2),\eta)-(v_2,\eta)-\alpha(u_2-u_1,\eta),\forall\eta\in H^1(\Omega),\quad (3.3)$$

$$\left(\frac{\partial v_2}{\partial t},\eta\right)+d_2(\nabla v_2,\nabla\eta)=(u_2,\eta)-\delta(v_2,\eta)-\beta(v_2-v_1,\eta),\ \ \forall\eta\in H^1(\Omega).\quad (3.4)$$

Theorem 3.1. Suppose $\Omega\subset\mathbb{R}^\eta$, where $\eta=1,2,3$, is an open bounded convex domain. Assume $u_{1,0}(.),\ v_{1,0}(.),\ u_{2,0}(.),\ v_{2,0}(.)\in L^2(\Omega)$. Then, the system $(1.1)-(1.6)$ possesses at least one weak solution u_1,v_1,u_2,v_2 that satisfies:

$$u_1(x,t),u_2(x,t)\in L^4(\Omega_T)\cap L^2\big(0,T;H^1(\Omega)\big)\cap L^\infty\big(0,T;L^2(\Omega)\big)\cap C\big([0,T];L^2(\Omega)\big)\quad (3.5)$$

$$v_1(x,t),v_2(x,t)\in L^2(\Omega_T)\cap L^2\big(0,T;H^1(\Omega)\big)\cap L^\infty\big(0,T;L^2(\Omega)\big)\cap C\big([0,T];L^2(\Omega)\big)\quad (3.6)$$

and the equations $(1.1)-(1.6)$ hold as equalities in $L^{\frac{4}{3}}\left(0,T;\left(H^1(\Omega)\right)'\right)$, $L^2\left(0,T;\left(H^1(\Omega)\right)'\right)$, $L^{\frac{4}{3}}\left(0,T;\left(H^1(\Omega)\right)'\right)$ and $L^2\left(0,T;\left(H^1(\Omega)\right)'\right)$, respectively.

Proof: The proof is structured into four sections, outlined as follows: Sub section 3.1 outlines the Local existence, while Sub section 3.2 delves into the Global existence, Sub section 3.3 delves into the limit, and Sub section 3.4 uniqueness of solutions.

3.1 Local existence

We employ the Faedo-Galerkin method [13], to establish existence. Let $\{z_i\}_{i=1}^\infty$ denote an orthogonal basis for $H^1(\Omega)$ and an orthonormal basis for $L^2(\Omega)$, comprising eigenfunctions for

$$-\Delta z_i+z_i=\mu_i z_i,\qquad in\ \Omega,\qquad \frac{\partial z_i}{\partial v}=0\ \ on\ \ \partial\Omega,\quad (3.7)$$

where

$$1\le\mu_1\le\ \mu_2\le\mu_3\le\cdots\le\mu_k\le\cdots\ \text{with}\ \lim_{i\to\infty}\mu_i=\infty,\quad (3.8)$$

is an infinite set of associated eigenvalues. Note, $\left(z_i,z_j\right)_{H^1(\Omega)}=\ \mu_i\delta_{ij}$ and $\left(z_i,z_j\right)_{L^2(\Omega)}=\delta_{ij}$. Now, set $V^k\coloneqq span\{z_i\}_{i=1}^k\subset H^1(\Omega)$, and seek a finite-dimensional weak form corresponding to $(\mathcal{M})$:

$(\mathcal{M}^k)$ find $u_1^k(.,t),\ v_1^k(.,t),\ u_2^k(.,t),\ v_2^k(.,t)\in V^k$ such that $u_1^k(.,0)=u_{1,0}^k,\ v_1^k(.,0)=v_{1,0}^k,\ u_2^k(.,0)=u_{2,0}^k,\ v_2^k(.,0)=v_{2,0}^k$ and for almost every $t\in(0,T)$,

$$\varepsilon\left(\frac{\partial u_1^k}{\partial t},\eta^k\right)+d_1\big(\nabla u_1^k,\nabla\eta^k\big)=a\big(f\big(u_1^k\big),\eta^k\big)-\big(v_1^k,\eta^k\big)+\alpha\big(u_2^k-u_1^k,\eta^k\big),\quad (3.9)$$

$$\left(\frac{\partial v_1^k}{\partial t},\eta^k\right)+d_2\big(\nabla v_1^k,\nabla\eta^k\big)=\big(u_1^k,\eta^k\big)-\delta\big(v_1^k,\eta^k\big)+\beta\big(v_2^k-v_1^k,\eta^k\big),\quad (3.10)$$

$$\varepsilon\left(\frac{\partial u_2^k}{\partial t},\eta^k\right)+d_1\left(\nabla u_2^k,\nabla\eta^k\right)$$
$$=a\left(f\left(u_2^k\right),\eta^k\right)-\left(v_2^k,\eta^k\right)-\alpha\left(u_2^k-u_1^k,\eta^k\right), \qquad (3.11)$$
$$\left(\frac{\partial v_2^k}{\partial t},\eta^k\right)+d_2\left(\nabla v_2^k,\nabla\eta^k\right)$$
$$=\left(u_2^k,\eta^k\right)-\delta\left(v_2^k,\eta^k\right)-\beta\left(v_2^k-v_1^k,\eta^k\right). \qquad (3.12)$$

Now, $u_1^k,\ v_1^k,\ u_2^k,\ v_2^k$ expressed as Galerkin approximations in the subsequent format

$$u_1^k(.,t)=\sum_{i=1}^{k}a_{ik}(t)z_i(.),\quad v_1^k(.,t)=\sum_{i=1}^{k}b_{ik}(t)z_i(.), \qquad (3.13)$$
$$u_2^k(.,t)=\sum_{i=1}^{k}c_{ik}(t)z_i(.),\quad v_2^k(.,t)=\sum_{i=1}^{k}d_{ik}(t)z_i(.), \qquad (3.14)$$

for $i=1,\dots,k$, let $\eta^k=z_i$. The coefficients $a_{ik}(t)$, $b_{ik}(t)$, $c_{ik}(t)$, and $d_{ik}(t)$ are not yet known. The orthogonal projection from $L^2(\Omega)$ onto V^k is introduced as $P^k\colon L^2(\Omega)\to V^k$. This projection ensures that $(P^k v,\eta^k)=(v,\eta^k)$ for all $\eta^k\in V^k$. For elements in $H^1(\Omega)\subset L^2(\Omega)$, this definition is valid.

Lemma 3.1. For any $\chi\in H^1(\Omega)$ we have
$$(\nabla(P^k\chi),\nabla\eta^k)=(\nabla\chi,\nabla\eta^k),$$
$$\forall\eta^k\in V^k. \qquad (3.15)$$

Upon direct computation, it becomes evident that this projection operator fulfils the subsequent properties:
$$\|\nabla P^k\chi\|_0\le\|\nabla\chi\|_0,\qquad \forall\chi\in H^1(\Omega). \qquad (3.16)$$
The initial values are selected in the following manner:

$$u_1^k(.,0)\coloneqq P^k u_{1,0}^k,\qquad v_1^k(.,0)\coloneqq P^k v_{1,0}^k, \qquad (3.17)$$
$$u_2^k(.,0)\coloneqq P^k u_{2,0}^k,\qquad v_2^k(.,0)\coloneqq P^k v_{2,0}^k, \qquad (3.18)$$

where the following property holds:

$$\left\{u_{1,0}^k,v_{1,0}^k,u_{2,0}^k,v_{2,0}^k\right\}\longmapsto\left\{u_{1,0},v_{1,0},u_{2,0},v_{2,0}\right\}\ in\ L^2(\Omega)\ as\ k\longmapsto\infty. \qquad (3.19)$$

The system of equations $(3.9)-(3.12)$ can be represented as a set of ordinary differential equations involving the variables $a_{ik}(t)$, $b_{ik}(t)$, $c_{ik}(t)$, and $d_{ik}(t)$. We express this system in a composite form that is equivalent to the original.

$$\varepsilon\frac{du_1^k}{dt}=d_1\Delta u_1^k+aP^kf\left(u_1^k\right)-v_1^k+\alpha\left(u_2^k-u_1^k\right),\qquad u_1^k(.,0)\coloneqq P^k u_{1,0}^k, \qquad (3.20)$$
$$\frac{dv_1^k}{dt}=d_2\Delta v_1^k+u_1^k-\delta v_1^k+\beta\left(v_2^k-v_1^k\right),\qquad v_1^k(.,0)\coloneqq P^k v_{1,0}^k, \qquad (3.21)$$
$$\varepsilon\frac{du_2^k}{dt}=d_1\Delta u_2^k+aP^kf\left(u_2^k\right)-v_2^k-\alpha\left(u_2^k-u_1^k\right),\qquad u_2^k(.,0)\coloneqq P^k u_{2,0}^k, \qquad (3.22)$$
$$\frac{dv_2^k}{dt}=d_2\Delta v_2^k+u_2^k-\delta v_2^k-\beta\left(v_2^k-v_1^k\right),\qquad v_2^k(.,0)\coloneqq P^k v_{2,0}^k. \qquad (3.23)$$

Our next task is to demonstrate that the nonlinearity in the system of ordinary differential equations is locally Lipschitz.

Lemma 3.2. Let $u \in C^{\infty}(\Omega)$ and Ω is an open bounded convex domain in $\mathbb{R}^{\eta}(\eta = 1,2,3)$. Then the nonlinear $F(u) = 3u - u^3$ in system $(\mathcal{M})$ satisfies the inequality $|F(u_1) - F(u_2)| \leq L|u_1 - u_2|$, where L is Lipschitz constant.

Proof: Based on the assumption, we obtain:

$\min\{u: u \in C^{\infty}(\Omega)\} \leq u \leq \max\{u: u \in C^{\infty}(\Omega)\}$, implying the existence of a positive integer C such that:

$$\max_{u \in C^{\infty}(\Omega)} |u| \leq C . \qquad (3.24)$$

Now, we have

$$\begin{aligned} |f(u_1) - f(u_2)| &= |((u_2)^3 - (u_1)^3) - 3(u_2 - u_1)| \\ &= |(u_2 - u_1)((u_2)^2 + u_2 u_1 + (u_1)^2) - 3(u_2 - u_1)| \\ &= |u_2 - u_1||(u_2)^2 + u_2 u_1 + (u_1)^2 - 3| \\ &\leq |u_2 - u_1|(|u_2|^2 + |u_2||u_1| + |u_1|^2 + 3| \leq (3C^2 + 3)|u_2 - u_1| \\ &\leq L|u_2 - u_1|. \qquad (3.25) \end{aligned}$$

Which completes the proof. ■

As a result, f is locally Lipschitz. According to local existence theorems, such as Picard's Theorem (see, for example, Hartman [38], p. 9), it can be concluded that the system of ordinary differential equations has a unique solution $u_1^k, v_1^k, u_2^k, v_2^k$ on a finite time interval $(0, t_k)$.

3.2 Global existence

In order to illustrate the worldwide presence of the Galerkin approximations, we establish preliminary estimates on u_1^k, v_1^k, u_2^k, and v_2^k limits that are not dependent on k and appropriate for specific function spaces. Using these boundaries as leverage, we can conclude that the global existence of the Galerkin approximations holds true for any given time value of $t_k = T$, regardless of its dependence on k.

Estimate I: On choosing $\eta^k = u_1^k$, $\eta^k = v_1^k$, $\eta^k = u_2^k$ and $\eta^k = v_2^k$ in (3.9) − (3.12), in a similar vein, summing the resulting equations yields:

$$\begin{aligned} &\epsilon \frac{1}{2}\frac{d}{dt}\int_{\Omega} |u_1^k|^2 dx + \epsilon \frac{1}{2}\frac{d}{dt}\int_{\Omega} |u_2^k|^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\Omega} |v_1^k|^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\Omega} |v_2^k|^2 dx \\ &\quad + d_1 \int_{\Omega} |\nabla u_1^k|^2 dx + d_1 \int_{\Omega} |\nabla u_2^k|^2 dx + d_2 \int_{\Omega} |\nabla v_1^k|^2 dx + d_2 \int_{\Omega} |\nabla v_2^k|^2 dx \\ &\quad + a \int_{\Omega} |u_1^k|^4 dx + \alpha \int_{\Omega} |u_2^k - u_1^k|^2 dx + \delta \int_{\Omega} |v_1^k|^2 dx + \beta \int_{\Omega} |v_2^k - v_1^k|^2 dx \\ &\quad + a \int_{\Omega} |u_2^k|^4 dx + \delta \int_{\Omega} |v_2^k|^2 dx \\ &\quad = 3a \int_{\Omega} |u_1^k|^2 dx + 3a \int_{\Omega} |u_2^k|^2 dx. \qquad (3.26) \end{aligned}$$

Add $\left(\int_\Omega |v_1^k|^2 dx + \int_\Omega |v_2^k|^2 dx\right)$, on the right-hand side, multiplying by 2,

$$\varepsilon \frac{d}{dt}\int_\Omega |u_1^k|^2 dx + \varepsilon \frac{d}{dt}\int_\Omega |u_2^k|^2 dx + \frac{d}{dt}\int_\Omega |v_1^k|^2 dx + \frac{d}{dt}\int_\Omega |v_2^k|^2 dx + 2d_1 \int_\Omega |\nabla u_1^k|^2 dx$$
$$+ 2d_1 \int_\Omega |\nabla u_2^k|^2 dx + 2d_2 \int_\Omega |\nabla v_1^k|^2 dx + 2d_2 \int_\Omega |\nabla v_2^k|^2 dx + 2a \int_\Omega |u_1^k|^4 dx$$
$$+ 2\alpha \int_\Omega |u_2^k - u_1^k|^2 dx + 2\delta \int_\Omega |v_1^k|^2 dx + 2\beta \int_\Omega |v_2^k - v_1^k|^2 dx + 2a \int_\Omega |u_2^k|^4 dx$$
$$+ 2\delta \int_\Omega |v_2^k|^2 dx$$
$$\leq 6a\left(\epsilon \int_\Omega |u_1^k|^2 dx + \epsilon \int_\Omega |u_2^k|^2 dx + \int_\Omega |v_1^k|^2 dx + \int_\Omega |v_2^k|^2 dx\right). \quad (3.27)$$

Application of Grönwall Lemma 2.1 gives

$$\varepsilon\|u_1^k(T)\|_0^2 + +\varepsilon\|u_2^k(T)\|_0^2 + \|v_1^k(T)\|_0^2 + \|v_2^k(T)\|_0^2 + 2d_1\|u_1^k\|^2_{L^2(0,T,H^1)}$$
$$+ 2d_1\|u_2^k\|^2_{L^2(0,T,H^1)} + 2d_2\|v_1^k\|^2_{L^2(0,T,H^1)} + 2d_2\|v_2^k\|^2_{L^2(0,T,H^1)}$$
$$+ 2a\|u_1^k\|^4_{L^4(\Omega T)} + 2\alpha\|u_2^k - u_1^k\|^2_{L^2(\Omega T)}$$
$$+2\delta\|v_1^k\|^2_{L^2(\Omega T)} + 2\beta\|v_2^k - v_1^k\|^2_{L^2(\Omega T)} + 2a\|u_2^k\|^4_{L^4(\Omega T)} + 2\delta\|v_2^k\|^2_{L^2(\Omega T)}$$
$$\leq \left(\epsilon\|u_1^k(0)\|_0^2 + \epsilon\|u_2^k(0)\|_0^2 + \|v_1^k(0)\|_0^2\right.$$
$$\left. + \|v_2^k(0)\|_0^2\right) e^{6aT}. \quad (3.28)$$

Recalling $u_{1,0}^k, v_{1,0}^k, u_{2,0}^k$ and $v_{2,0}^k \in L^2(\Omega)$ we have u_1^k, u_2^k is uniformly bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^4(\Omega_T)$ and v_1^k, v_2^k is uniformly bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^2(\Omega_T)$, $(u_2^k - u_1^k)$ and $(v_2^k - v_1^k) \in L^2(\Omega_T)$.

3.3 The limit

Using standard compactness methods (see [38], Theorems 4 and 5), we utilize the functions that are uniformly bounded $\{u_1^k\}_{k=1}^\infty$, $\{v_1^k\}_{k=1}^\infty$, $\{u_2^k\}_{k=1}^\infty$ and $\{v_2^k\}_{k=1}^\infty$. We extract convergent subsequences from these sequences, which we refer to as $\{u_1^k\}$, $\{v_1^k\}$, $\{u_2^k\}$, and $\{v_2^k\}$, such that

$\{u_1^k, u_2^k\} \rightharpoonup \{u_1, u_2\}\ in\ L^2(0,T;H^1(\Omega)) \cap L^4(\Omega_T)\ as\ K \to \infty,$ (3.29)

$\{v_1^k, v_2^k\} \rightharpoonup \{v_1, v_2\}\ in\ L^2(0,T;H^1(\Omega)) \cap L^2(\Omega_T)\ as\ K \to \infty,$ (3.30)

and

$\{u_1^k, u_2^k\} \rightharpoonup^* \{u_1, u_2\}\ in\ L^\infty(0,T;L^2(\Omega))\ as\ K \to \infty,$ (3.31)

$\{v_1^k, v_2^k\} \rightharpoonup^* \{v_1, v_2\}\ in\ L^\infty(0,T;L^2(\Omega))\ as\ K \to \infty,$ (3.32)

" $\rightharpoonup$ " and " $\rightharpoonup^*$ ", respectively, denote weak and weak-star convergence. For the elements in the first composite Galerkin approximation (3.20), we show how to approach the limit. Consider the expression:

$$f_1(u_1^k, v_1^k, u_2^k) = 3au_1^k - a(u_1^k)^3 - v_1^k + \alpha(u_2^k - u_1^k),$$

and it can be readily demonstrated that

$$|f_1| \leq C\left(\left|u_1^k\right|^3 + \left|u_1^k\right| + \left|v_1^k\right| + \left|u_2^k\right|\right). \qquad (3.33)$$

Then we find that

$$\int_0^T \int_\Omega |f_1|^{\frac{4}{3}} dxdt \leq C \int_0^T \int_\Omega \left(\left|u_1^k\right|^4 + \left|u_1^k\right|^{\frac{4}{3}} + \left|v_1^k\right|^{\frac{4}{3}} + \left|u_2^k\right|^{\frac{4}{3}}\right) dxdt\,. \qquad (3.34)$$

Taking into account the bounds (3.29) and (3.30), along with the embeddings $L^4(\Omega_T) \hookrightarrow L^{\frac{4}{3}}(\Omega_T)$ and $L^2(\Omega_T) \hookrightarrow L^{\frac{4}{3}}(\Omega_T)$, we observe that f_1 is uniformly bounded in $L^{\frac{4}{3}}(\Omega_T)$. Therefore, employing weak compactness arguments, we can assert the existence of $\rho \in L^{\frac{4}{3}}(\Omega_T)$ such that

$$f_1 \rightharpoonup \rho \;\; in\; L^{\frac{4}{3}}(\Omega_T) \;\; as\; k \to \infty. \qquad (3.35)$$

We further demonstrate that in $L^{\frac{4}{3}}(\Omega_T)$, $P^k f_1$ likewise tends weakly to ρ. Define the orthogonal projection to $P^k, G^k := I - p^k$. Remember that $(P^k v, \rho^k)_{H^1} = (v, \rho^k)_{H^1}$ for all ρ^kin V^k, $v \in H^1(\Omega)$. This implies that $\|P^k v - v\|_1 \leq \|\rho^k - v\|_1$ for all $\rho^k \in V^k$, $v \in H^1(\Omega)$. Consequently, since V^k is dense in $H^1(\Omega)$, for every $v \in H^1(\Omega)$, we have $P^k v \longrightarrow v$ in $H^1(\Omega)$; that is, $G^k u \longrightarrow 0$ in $H^1(\Omega)$ as $k \longrightarrow \infty$. Additionally, for any $v \in L^4(\Omega)$, we get $H^1 \hookrightarrow L^4(\Omega_T)$ and, consequently, $G^k u \longrightarrow 0$ in $L^4(\Omega)$. Let $\vartheta \in L^4(\Omega_T)$ be arbitrarily chosen. Then, using the orthogonal property of G^k and Hölder's inequality,

$$\left|\int_0^T (P^k f_1 - \rho, \vartheta) dt\right| = \left|\int_0^T [(f_1 - \rho, \vartheta) - (f_1, G^k \vartheta)] dt\right|, \qquad (3.36)$$

$$\leq \left|\int_0^T (f_1 - \rho, \vartheta) dt\right| + \int_0^T \|f_1\|_{0,\frac{4}{3}} \|G^k \vartheta\|_{0,4} dt \longrightarrow 0, as\; k \to \infty, \qquad (3.37)$$

Acknowledging the strong convergence of $G^k \vartheta$ to 0 in $L^4(\Omega)$ and relation (3.35), we obtain

$$P^k f_1 \rightharpoonup \rho \; in\; L^{\frac{4}{3}}(\Omega_T) \; as\; k \to \infty. \qquad (3.38)$$

Similarly, consider

$$f_2(u_1^k, v_1^k, v_2^k) = u_1^k - \delta v_1^k + \beta(v_2^k - v_1^k),$$

we have

$$|f_2| \leq C\left(\left|u_1^k\right| + \left|v_1^k\right| + \left|v_2^k\right|\right), \qquad (3.39)$$

Then, it follows that

$$\int_0^T \int_\Omega |f_2|^2 dxdt \leq C \int_0^T \int_\Omega \left(\left|u_1^k\right|^2 + \left|v_1^k\right|^2 + \left|v_2^k\right|^2\right) dxdt. \qquad (3.40)$$

Taking into account the bounds (3.29) and (3.30), along with the injection $L^4(\Omega_T) \hookrightarrow L^2(\Omega_T)$, we deduce that f_2 is uniformly bounded in $L^2(\Omega_T)$. Consequently, through weak compactness arguments, there exists $\xi \in L^2(\Omega_T)$ such that

$$f_2 \rightharpoonup \xi \; in\; L^2(\Omega_T) \; as\; k \to \infty. \qquad (3.41)$$

We show that $P^k f_2$ also tends weakly to ξ in $L^2(\Omega_T)$. We have $G^k v \longrightarrow 0$ in $H^1(\Omega)$ as $k \longrightarrow \infty$. Moreover, it follows form the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ that $G^k v \longrightarrow 0$ in $L^2(\Omega)$ for all $v \in L^2(\Omega)$. Let $\vartheta \in L^2(\Omega_T)$, then by utilising Hölders inequality and the orthogonality of G^k, we arrive at

$$\left|\int_0^T (P^k f_2) - \xi, \vartheta)\, dt\right|$$
$$\leq \left|\int_0^T (f_2 - \xi, \vartheta)\, dt\right| + \left|\int_0^T (f_2, G^k \vartheta) dt\right|$$
$$\leq \left|\int_0^T (f_2 - \xi, \vartheta)\, dt\right| + \int_0^T \|f_2\|_{L^2(\Omega)} \|G^k \vartheta\|_{L^2(\Omega)} dt \longrightarrow 0 \ as\ k \to \infty. \quad (3.42)$$

Then, it follows that

$$P^k f_2 \rightharpoonup \xi \ in\ L^2(\Omega_T)\ as\ k \to \infty. \quad (3.43)$$

In the same way, consider

$$f_3(u_1^k, u_2^k, v_2^k) = 3au_2^k - a(u_2^k)^3 - v_2^k - \alpha(u_2^k - u_1^k),$$

we have

$$|f_3| \leq C\left(\left|(u_2^k)^3\right| + |u_2^k| + |v_2^k| + |u_1^k|\right). \quad (3.44)$$

Then, we find that

$$\int_0^T \int_\Omega |f_3|^{\frac{4}{3}} dxdt \leq C \int_0^T \int_\Omega \left(|u_2^k|^4 + |u_2^k|^{\frac{4}{3}} + |v_2^k|^{\frac{4}{3}} + |u_1^k|^{\frac{4}{3}}\right) dxdt. \quad (3.45)$$

Taking into account the bounds (3.29) and (3.30), as well as the embedding's $L^4(\Omega_T) \hookrightarrow L^{\frac{4}{3}}(\Omega_T)$ and $L^2(\Omega_T) \hookrightarrow L^{\frac{4}{3}}(\Omega_T)$, we observe that f_1 is uniformly bounded in $L^{\frac{4}{3}}(\Omega_T)$. Consequently, through weak compactness arguments, we establish the existence of $\rho \in L^{\frac{4}{3}}(\Omega_T)$ such that

$$f_3 \rightharpoonup \rho \ in\ L^{\frac{4}{3}}(\Omega_T)\ as\ k \to \infty. \quad (3.46)$$

We show that $P^k f_3$ also tends weakly to ρ in $L^{\frac{4}{3}}(\Omega_T)$. Define $G^k := I - p^k$, the projection orthogonal to P^k. Now recall that $(P^k v, \rho^k)_{H^1} = (v, \rho^k)_{H^1}$ for all $\rho^k \in V^k, v \in H^1(\Omega)$, which implies $\|P^k v - v\|_1 \leq \|\rho^k - v\|_1$ for all $\rho^k \in V^k$, $v \in H^1(\Omega)$. Thus, as V^k is dense in $H^1(\Omega)$ we have $P^k v \longrightarrow v$ in $H^1(\Omega)$ for all $v \in H^1(\Omega)$, i.e. $G^k u \longrightarrow 0$ in $H^1(\Omega)$ as $k \longrightarrow \infty$. We also have $H^1 \hookrightarrow L^4(\Omega_T)$ and so $G^k u \longrightarrow 0$ in $L^4(\Omega)$ for all $v \in L^4(\Omega)$. Consider an arbitrary $\vartheta \in L^4(\Omega_T)$, then using Hölder's inequality and the orthogonality of G^k

$$\left|\int_0^T (P^k f_3 - \rho, \vartheta) dt\right| = \left|\int_0^T [(f_3 - \rho, \vartheta) - (f_3, G^k \vartheta)] dt\right|$$
$$\leq \left|\int_0^T (f_3 - \rho, \vartheta) dt\right| + \int_0^T \|f_3\|_{0,\frac{4}{3}} \|G^k \vartheta\|_{0,4} dt \longrightarrow 0, as\ k \to \infty, \quad (3.47)$$

Taking into account the strong convergence of $G^k \vartheta$ to 0 in $L^4(\Omega)$ and relation (3.46), we have

$$P^k f_3 \rightharpoonup \rho \ \ in\ L^{\frac{4}{3}}(\Omega_T)\ as\ k \to \infty. \qquad (3.48)$$

Similarly, consider

$$f_4\left(v_1^k, u_2^k, v_2^k\right) = u_2^k - \delta v_2^k - \beta\left(v_2^k - v_1^k\right),$$

we have

$$|f_4| \le C\left(\left|u_2^k\right| + \left|v_2^k\right| + \left|v_1^k\right|\right), \qquad (3.49)$$

Then, it follows that

$$\int_0^T \int_\Omega |f_4|^2 dxdt \le C \int_0^T \int_\Omega \left(\left|u_2^k\right|^2 + \left|v_2^k\right|^2 + \left|v_1^k\right|^2\right) dxdt. \qquad (3.50)$$

Upon observing the inequalities (3.29) and (3.30), as well as the inclusion $L^4(\Omega_T) \hookrightarrow L^2(\Omega_T)$, it can be deduced that f_4 is uniformly bounded in $L^2(\Omega_T)$. Consequently, employing weak compactness arguments, there exists an element $\xi \in L^2(\Omega_T)$ such that

$$f_4 \rightharpoonup \xi\ in\ L^2(\Omega_T)\ as\ k \to \infty. \qquad (3.51)$$

We show that $P^k f_4$ also tends weakly to ξ in $L^2(\Omega_T)$. We have $G^k v \longrightarrow 0$ in $H^1(\Omega)$ as $k \longrightarrow \infty$. Furthermore, it follows form the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ that $G^k v \longrightarrow 0$ in $L^2(\Omega)$ for all $v \in L^2(\Omega)$. Let $\vartheta \in L^2(\Omega_T)$, by utilizing H$\ddot{o}$lders inequality and the orthogonality of G^k, we arrive at

$$\left|\int_0^T (P^k f_4) - \xi, \vartheta)\, dt\right|$$
$$\le \left|\int_0^T (f_4 - \xi, \vartheta)\, dt\right| + \left|\int_0^T (f_4, G^k \vartheta) dt\right|$$
$$\le \left|\int_0^T (f_4 - \xi, \vartheta)\, dt\right| + \int_0^T \|f_4\|_{L^2(\Omega)} \|G^k \vartheta\|_{L^2(\Omega)} dt \longrightarrow 0\ \ as\ k \to \infty. \qquad (3.52)$$

Then, it follows that

$$P^k f_4 \rightharpoonup \xi\ in\ L^2(\Omega_T)\ \ as\ k \to \infty. \qquad (3.53)$$

Considering that $\Delta u_1^k \in L^2\left(0, T; \left(H^1(\Omega)\right)'\right)$ and $P^k f_1 \in L^{\frac{4}{3}}(\Omega_T)$, we can infer from (3.20) that $\frac{du_1^k}{dt}$ is uniformly bounded in $L^2\left(0, T; \left(H^1(\Omega)\right)'\right) + L^{\frac{4}{3}}(\Omega_T)$. Through weak compactness arguments, it follows that $\frac{du_1^k}{dt}$ converges weakly to some $\dot{\eta}$ in $L^2\left(0, T; \left(H^1(\Omega)\right)'\right) + L^{\frac{4}{3}}(\Omega_T)$. By the uniqueness of weak convergence, we conclude that $\dot{\eta} = \frac{du_1}{dt}$, implying

$$\frac{du_1^k}{dt} \rightharpoonup \frac{du_1}{dt}\ in\ L^2(0, T; (H^1(\Omega)') + L^{\frac{4}{3}}(\Omega_T)\ as\ k \to \infty. \qquad (3.54)$$

Initially, recall from (3.29) that $u_1^k \rightharpoonup u_1$ in the space $L^2\big(0,T;H^1(\Omega)\big) \cap L^4(\Omega_T)$ with the dual space $L^2\left(0,T;\big(H^1(\Omega)\big)'\right) + L^{\frac{4}{3}}(\Omega_T)$. Additionally, using the Sobolev Embedding Theorem and the density of $H^1(\Omega)$ in $L^2(\Omega)$, we have the dense inclusion $H^1(\Omega) \hookrightarrow L^4(\Omega)$, implying $L^{\frac{4}{3}}(\Omega) \hookrightarrow \big(H^1(\Omega)\big)'$ and consequently $L^2\left(0,T;\big(H^1(\Omega)\big)'\right) + L^{\frac{4}{3}}(\Omega_T) \subset L^{\frac{4}{3}}\left(0,T;\big(H^1(\Omega)\big)'\right)$. Consider an arbitrary function $\vartheta(t) \in C_0^\infty\big(0,T;H^1(\Omega)\big)$ within the space $L^{\frac{4}{3}}\big(0,T;H^1(\Omega)\big)$. By performing integration by parts and utilizing the weak convergence of u_1^k to u_1 in

$L^2\left(0,T;\big(H^1(\Omega)\big)'\right) + L^{\frac{4}{3}}(\Omega_T)$ and $L^{\frac{4}{3}}\left(0,T;\big(H^1(\Omega)\big)'\right)$, we derive

$$\int_0^T \left(\frac{du_1^k}{dt},\vartheta\right)dt = -\int_0^T \left(u_1^k,\frac{d\vartheta}{dt}\right)dt \to -\int_0^T \left(u_1,\frac{d\vartheta}{dt}\right)dt = \int_0^T \left(\frac{du_1}{dt},\vartheta\right)dt.$$

After recognizing that $\frac{d\varphi}{dt} \in C_0^\infty\big(0,T;H^1(\Omega)\big)$, and due to the weak convergence of $\frac{du_1^k}{dt}$ to $\dot{\eta}$ in $L^{\frac{4}{3}}\left(0,T;\big(H^1(\Omega)\big)'\right)$, we obtain:

$\int_0^T \left(\frac{du_1^k}{dt},\vartheta\right)dt \to \int_0^T (\dot{\eta},\vartheta)dt.$

Thus, due to the distinctiveness of weak boundaries, we can conclude that $\frac{du_1}{dt} = \dot{\eta}$, i.e., $\frac{du_1^k}{dt} \rightharpoonup \frac{du_1}{dt}$, in $L^{\frac{4}{3}}\left(0,T;\big(H^1(\Omega)\big)'\right)$ as $k \to \infty$.

Now as $u_1^k \rightharpoonup u_1$ in $L^2\big(0,T;H^1(\Omega)\big)$ we have (see [39] , p. 204) $\Delta u_1^k \rightharpoonup \Delta u_1$ in the space $L^2\left(0,T;\big(H^1(\Omega)\big)'\right) \subset L^{\frac{4}{3}}\left(0,T;\big(H^1(\Omega)\big)'\right)$. Hence, we achieve the necessary limit transition for all terms in $L^{\frac{4}{3}}\left(0,T;\big(H^1(\Omega)\big)'\right)$. To establish the equality $\rho = f_1$ in equation (3.35), we utilize several well-known theorems. Employing the Lions-Aubin Theorem [23]

$$\omega = \left\{\eta{:}\, \eta \in L^2\big(0,T;H^1(\Omega)\big);\ \frac{d\eta}{dt} \in L^{\frac{4}{3}}\left(0,T;\big(H^1(\Omega)\big)'\right)\right\} \overset{c}{\hookrightarrow} L^2(\Omega_T).$$

Since $u_1^k \in \omega$, a subsequence, still denoted as, can be extracted u_1^k, such that $u_1^k \to u_1$ in $L^2(\Omega_T)$. Consequently, $u_1^k \to u_1$ (pointwise) almost every where in Ω_T. Given that f_1 is locally Lipschitz in Ω_T, This continuity entails that $f_1\big(u_1^k,v_1^k,u_2^k,v_2^k\big) \to f_1(u_1,v_1,u_2,v_2)$ (pointwise) almost everywhere in Ω_T. The application of Lemma 1.3 from Lions, yields

$$f_1\big(u_1^k,v_1^k,u_2^k,v_2^k\big) \rightharpoonup f_1(u_1,v_1,u_2,v_2) \in L^{\frac{4}{3}}(\Omega_T), \tag{3.55}$$

given the uniqueness of weak limits, we can infer that ρ is equal to f_1 as it was originally stated. In the same way $\frac{du_2^k}{dt} \rightharpoonup \frac{du_2}{dt}$, in $L^{\frac{4}{3}}\left(0,T;\big(H^1(\Omega)\big)'\right)$ as $k \to \infty$, $\Delta u_2^k \rightharpoonup \Delta u_2$ in the space $L^2\left(0,T;\big(H^1(\Omega)\big)'\right)$, and we can show that $\frac{dv_1^k}{dt} \rightharpoonup \frac{dv_1}{dt}, \frac{dv_2^k}{dt} \rightharpoonup \frac{dv_2}{dt}$ in $L^2\left(0,T;\big(H^1(\Omega)\big)'\right) + L^2(\Omega_T)$, $f_2\big(u_1^k,v_1^k,u_2^k,v_2^k\big) \to f_2(u_1,v_1,\ u_2,v_2)$ in $L^2(\Omega_T)$ and $\Delta v_1^k \rightharpoonup \Delta v_1, \Delta v_2^k \rightharpoonup \Delta v_2$ in $L^2\left(0,T;\big(H^1(\Omega)\big)'\right)$. Finally, it remains to show that u_1, v_1, u_2, and $v_2 \in C\big([0,T];L^2(\Omega)\big)$. To obtain that u_1, v_1, u_2, and $v_2 \in C([0,T];L^2\Omega)$, we employ a revised edition of another well-established outcome from [40]. We have shown $u_1, u_2 \in L^2\big(0,T;H^1(\Omega)\big) \cap L^4(\Omega_T)$ and

$\frac{du_1}{dt}, \frac{du_2}{dt} \in L^2\left(0,T;\left(H^1(\Omega)\right)'\right) + L^{\frac{4}{3}}(\Omega_T)$. Moreover, it has been proved that $v_1, v_2 \in L^2\left(0,T;H^1(\Omega)\right) \cap L^2(\Omega_T)$ and $\frac{dv_1}{dt}, \frac{dv_2}{dt} \in L^2\left(0,T;\left(H^1(\Omega)\right)'\right) + L^2(\Omega_T)$. Since $L^2\left(0,T;\left(H^1(\Omega)\right)'\right) + L^{\frac{4}{3}}(\Omega_T)$ and $L^2\left(0,T;\left(H^1(\Omega)\right)'\right) + L^2(\Omega_T)$ are the dual spaces of $L^2\left(0,T;H^1(\Omega)\right) \cap L^4(\Omega_T)$ and $L^2\left(0,T;H^1(\Omega)\right) \cap L^2(\Omega_T)$, respectively, then we deduce that u_1, u_2, v_1 and $v_2 \in C\left([0,T];L^2(\Omega)\right)$.

3.4 Uniqueness

To demonstrate uniqueness, let's suppose there exist two solutions $u_1^1, u_1^2, v_1^1, v_1^2, u_2^1, u_2^2$, and v_2^1, v_2^2 of the weak form $(3.1)-(3.4)$, with initial conditions $u_1^1(.,0) = u_{1,0}^1(.)$, $u_1^2(.,0) = u_{1,0}^2(.)$,, $v_1^1(.,0) = v_{1,0}^1(.)$, $v_1^2(.,0) = v_{1,0}^2(.)$, $u_2^1(.,0) = u_{2,0}^1(.)$, $u_2^2(.,0) = u_{2,0}^2(.)$ and $v_2^1(.,0) = v_{2,0}^1(.), v_2^2(.,0) = v_{2,0}^2(.)$, respectively. Setting $\omega_1 = u_1^1 - u_1^2$, $\omega_2 = v_1^1 - v_1^2$, $\omega_3 = u_2^1 - u_2^2$ and $\omega_4 = v_2^1 - v_2^2$, and setting $\eta = \omega_1,\ \eta = \omega_2,\ \eta = \omega_3,\ \eta = \omega_4$ in $(3.1)-(3.4)$, sub tracting the weak forms and employing this subtraction in the process yields

$$\begin{aligned}
&\varepsilon \frac{1}{2}\frac{d}{dt}\int_\Omega |\omega_1|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\omega_2|^2 dx + \varepsilon \frac{1}{2}\frac{d}{dt}\int_\Omega |\omega_3|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\omega_4|^2 dx \\
&\quad + d_1 \int_\Omega |\nabla\omega_1|^2 dx + d_2 \int_\Omega |\nabla\omega_2|^2 dx + d_1 \int_\Omega |\nabla\omega_3|^2 dx + d_2 \int_\Omega |\nabla\omega_4|^2 dx \\
&\quad + \delta \int_\Omega |\omega_2|^2 dx + \delta \int_\Omega |\omega_4|^2 dx + a \int_\Omega ((u_1^1)^3 - (u_1^2)^3)\omega_1 dx \\
&\quad + a \int_\Omega ((u_2^1)^3 - (u_2^2)^3)\omega_3 dx \\
&\quad = 3a \int_\Omega |\omega_1|^2 dx + \alpha(\omega_3 - \omega_1, \omega_1) + \beta(\omega_4 - \omega_2, \omega_2) + 3a \int_\Omega |\omega_3|^2 dx \\
&\quad - \alpha(\omega_3 - \omega_1, \omega_3) \\
&\quad - \beta(\omega_4 - \omega_2, \omega_4). \qquad (3.56)
\end{aligned}$$

Applying Young's inequalities (2.6), yields that

$$\begin{aligned}
&a \int_\Omega ((u_1^1)^3 - (u_1^2)^3)\omega_1 dx = a \int_\Omega ((u_1^1)^2 + u_1^1 u_1^2 + (u_1^2)^2)\omega_1^2 dx \\
&\geq a \int_\Omega \left(\frac{(u_1^1)^2}{2} + \frac{(u_1^2)^2}{2}\right)|\omega_1|^2 dx.
\end{aligned} \qquad (3.57)$$

$$\begin{aligned}
&a \int_\Omega ((u_2^1)^3 - (u_2^2)^3)\omega_1 dx = a \int_\Omega ((u_2^1)^2 + u_2^1 u_2^2 + (u_2^2)^2)\omega_3^2 \\
&\geq a \int_\Omega \left(\frac{(u_2^1)^2}{2} + \frac{(u_2^2)^2}{2}\right)|\omega_3|^2 dx.
\end{aligned} \qquad (3.58)$$

Replacing (3.57) and (3.58) in (3.56) results in

$$\varepsilon\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\omega_1|^2dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\omega_2|^2dx+\varepsilon\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\omega_3|^2dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\omega_4|^2dx$$
$$+d_1\int_{\Omega}|\nabla\omega_1|^2dx+d_2\int_{\Omega}|\nabla\omega_2|^2dx+d_1\int_{\Omega}|\nabla\omega_3|^2dx+d_2\int_{\Omega}|\nabla\omega_4|^2dx$$
$$+\delta\int_{\Omega}|\omega_2|^2dx+\delta\int_{\Omega}|\omega_4|^2dx+a\int_{\Omega}(\frac{(u_1^1)^2}{2}+\frac{(u_1^2)^2}{2}|\omega_1|^2dx$$
$$+a\int_{\Omega}(\frac{(u_2^1)^2}{2}+\frac{(u_2^2)^2}{2}|\omega_3|^2dx$$
$$+\alpha\int_{\Omega}|\omega_3-\omega_1|^2dx+\beta\int_{\Omega}|\omega_4-\omega_2|^2dx$$
$$\leq 3a\int_{\Omega}|\omega_1|^2dx+3a\int_{\Omega}|\omega_3|^2dx. \tag{3.59}$$

By neglecting final ten terms on the left side of inequality (3.59) and multiplying the results by 2, while taking into account the supplementary term $\left(\int_{\Omega}|\omega_2|^2dx+\int_{\Omega}|\omega_4|^2dx\right)$ on the right side, we derive

$$\varepsilon\frac{d}{dt}\int_{\Omega}|\omega_1|^2dx+\frac{d}{dt}\int_{\Omega}|\omega_2|^2dx+\varepsilon\frac{d}{dt}\int_{\Omega}|\omega_3|^2dx+\frac{d}{dt}\int_{\Omega}|\omega_4|^2dx$$
$$\leq 6a\left[\int_{\Omega}\varepsilon|\omega_1|^2dx+\int_{\Omega}|\omega_2|^2dx+\int_{\Omega}\varepsilon|\omega_3|^2dx+\int_{\Omega}|\omega_4|^2dx\right]. \tag{3.60}$$

Application of Gr$\ddot{o}n$wall Lemma 2.1 gives

$$\varepsilon\|\omega_1\|_0^2+\|\omega_2\|_0^2+\varepsilon\|\omega_3\|_0^2+\|\omega_4\|_0^2$$
$$\leq \exp(6aT)\,(\varepsilon\|\omega_1(0)\|_0^2+\|\omega_2(0)\|_0^2+\varepsilon\|\omega_3(0)\|_0^2+\|\omega_4(0)\|_0^2). \tag{3.61}$$

Thus, if $u_1^1(0)=u_1^2(0), v_1^1(0)=v_1^2(0), u_2^1(0)=u_2^2(0)$ and $v_2^1(0)=v_2^2(0)$, we deduce uniqueness $u_1^1(t)=u_1^2(t), v_1^1(t)=v_1^2(t), u_2^1(t)=u_2^2(t)$ and $v_2^1(t)=v_2^2(t)$ for all t. However, if $u_1^1(0)\neq u_1^2(0), v_1^1(0)\neq v_1^2(0), u_2^1(0)\neq u_2^2(0)$ and $v_2^1(0)\neq v_2^2(0)$, then we have continuous dependence in $L^2(\Omega)$.

4. Strong solutions

We introduce a weak formulation of the system (1.1) -(1.6). $(\mathcal{M})$ find $u_1(.,t)$, $v_1(.,t)$, $u_2(.,t)$, $v_2(.,t)\in H^1(\Omega)$ such that $u_1(.,0)=u_{1,0}(.)$, $v_1(.,0)=v_{1,0}(.)$, $u_2(.,0)=u_{2,0}(.)$, and $v_2(.,0)=v_{2,0}(.)$, for almost every $t\in(0,T)$,

$$\varepsilon\left(\frac{\partial u_1^k}{\partial t},\eta^k\right)+d_1(\nabla u_1^k,\nabla\eta^k)=a(f(u_1^k),\eta^k)-(v_1^k,\eta^k)+\alpha(u_2^k-u_1^k,\eta^k), \tag{4.1}$$
$$\left(\frac{\partial v_1^k}{\partial t},\eta^k\right)+d_2(\nabla v_1^k,\nabla\eta^k)=(u_1^k,\eta^k)-\delta(v_1^k,\eta^k)+\beta(v_2^k-v_1^k,\eta^k), \tag{4.2}$$
$$\varepsilon\left(\frac{\partial u_2^k}{\partial t},\eta^k\right)+d_1(\nabla u_2^k,\nabla\eta^k)=a(f(u_2^k),\eta^k)-(v_2^k,\eta^k)-\alpha(u_2^k-u_1^k,\eta^k), \tag{4.3}$$

$$\left(\frac{\partial v_2^k}{\partial t},\eta^k\right)+d_2\left(\nabla v_2^k,\nabla\eta^k\right)=\left(u_2^k,\eta^k\right)-\delta\left(v_2^k,\eta^k\right)-\beta\left(v_2^k-v_1^k,\eta^k\right). \quad (4.4)$$

Theorem 4.1. Assume that $u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0} \in H^1(\Omega)$, then the system (1.1) - (1.6) possesses a unique, strong solution $\{u_1, v_1, u_2, v_2\}$ satisfying

$$u_1(x,t), v_1(x,t), u_2(x,t), v_2(x,t) \in L^2\left(0,T;H^2(\Omega)\right)\cap C\left([0,T],H^1(\Omega)\right), \quad (4.5)$$

and the equations (1.1) - (1.6) hold as equalities in $L^2(\Omega_T)$.
Furthermore, the

$$\left(u_{1,0}(x),v_{1,0}(x)\right)\mapsto\left(u_1\left(x,t;u_{1,0},u_{2,0},v_{1,0}\right),v_1\left(x,t;u_{1,0},v_{1,0},v_{2,0}\right)\right),$$
$$\left(u_{2,0}(x),v_{2,0}(x)\right)\mapsto\left(u_2\left(x,t;u_{1,0},u_{2,0},v_{2,0}\right),v_2\left(x,t;u_{2,0},v_{1,0},v_{2,0}\right)\right),$$

is continuous in $H^1(\Omega)$.

Proof: In order to establish both the existence and uniqueness of robust solutions, additional regularity outcomes are imperative. These can be attained through the application of more a priori estimates.

4.1 Existence

In this context, we will establish the subsequent estimates, which play a crucial role in this section.

Estimate I: Choosing $\eta^k=-\Delta u_1^k$, $\eta^k=-\Delta v_1^k$, $\eta^k=-\Delta u_2^k$, $\eta^k=-\Delta v_2^k$ in the weak forms (4.1) - (4.4), respectively. Combining the results and integrating by parts leads to

$$\varepsilon\frac{1}{2}\frac{d}{dt}\int_\Omega\left|\nabla u_1^k\right|^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega\left|\nabla v_1^k\right|^2dx+\varepsilon\frac{1}{2}\frac{d}{dt}\int_\Omega\left|\nabla u_2^k\right|^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega\left|\nabla v_2^k\right|^2dx+$$
$$d_1\int_\Omega\left|\Delta u_1^k\right|^2dx+d_1\int_\Omega\left|\Delta u_2^k\right|^2dx+d_2\int_\Omega\left|\Delta v_1^k\right|^2dx+d_2\int_\Omega\left|\Delta v_2^k\right|^2dx+\alpha\int_\Omega\left|\nabla u_2^k-\right.$$
$$\left.\nabla u_1^k\right|^2dx+\beta\int_\Omega\left|\nabla v_2^k-\nabla v_1^k\right|^2dx+3a\int_\Omega(\left|u_1^k\nabla u_1^k\right|^2dx+3a\int_\Omega\left|u_2^k\nabla u_2^k\right|^2dx+$$
$$\delta\int_\Omega\left|\nabla v_1^k\right|^2dx+\delta\int_\Omega\left|\nabla v_2^k\right|^2dx=3a\int_\Omega\left|\nabla u_1^k\right|^2dx+$$
$$3a\int_\Omega\left|\nabla u_2^k\right|^2dx. \quad (4.6)$$

Add $\left(\int_\Omega\left|\nabla v_1^k\right|^2dx+\int_\Omega\left|\nabla v_2^k\right|^2dx\right)$, on the right-hand side, multiplying by 2,

$$\varepsilon\frac{d}{dt}\int_\Omega\left|\nabla u_1^k\right|^2dx+\frac{d}{dt}\int_\Omega\left|\nabla v_1^k\right|^2dx+\varepsilon\frac{d}{dt}\int_\Omega\left|\nabla u_2^k\right|^2dx+\frac{d}{dt}\int_\Omega\left|\nabla v_2^k\right|^2dx+$$
$$2d_1\int_\Omega\left|\Delta u_1^k\right|^2dx+2d_1\int_\Omega\left|\Delta u_2^k\right|^2dx+2d_2\int_\Omega\left|\Delta v_1^k\right|^2dx+2d_2\int_\Omega\left|\Delta v_2^k\right|^2dx+$$
$$2\alpha\int_\Omega\left|\nabla u_2^k-\nabla u_1^k\right|^2dx+2\beta\int_\Omega\left|\nabla v_2^k-\nabla v_1^k\right|^2dx+6a\int_\Omega(\left|u_1^k\nabla u_1^k\right|^2dx+$$
$$6a\int_\Omega\left|u_2^k\nabla u_2^k\right|^2dx+2\delta\int_\Omega\left|\nabla v_1^k\right|^2dx+2\delta\int_\Omega\left|\nabla v_2^k\right|^2dx\le 6a\,(\varepsilon\int_\Omega\left|\nabla u_1^k\right|^2dx+$$
$$\varepsilon\int_\Omega\left|\nabla u_2^k\right|^2dx+\int_\Omega\left|\nabla v_1^k\right|^2dx+\int_\Omega\left|\nabla v_2^k\right|^2dx. \quad (4.7)$$

Application of Grönwall Lemma 2.1 gives

$$\varepsilon\left|u_1^k(T)\right|_1^2+\left|v_1^k(T)\right|_1^2+\varepsilon\left|u_2^k(T)\right|_1^2+\left|v_2^k(T)\right|_1^2+2d_1\left\|u_1^k\right\|_{L^2(0,T,H^2)}^2+2d_1\left\|u_2^k\right\|_{L^2(0,T,H^2)}^2+$$
$$2d_2\left\|v_1^k\right\|_{L^2(0,T,H^2)}^2+2d_2\left\|v_2^k\right\|_{L^2(0,T,H^2)}^2+2\alpha\left\|u_2^k-u_1^k\right\|_{L^2(0,T,\mathrm{H}^1)}^2+2\beta\left\|v_2^k-v_1^k\right\|_{L^2(0,T,H^1)}^2+$$
$$6a\left\|u_1^k\nabla u_1^k\right\|_{L^2(\Omega_T)}^2+6a\left\|u_2^k\nabla u_2^k\right\|_{L^2(\Omega_T)}^2+2\delta\left\|v_1^k\right\|_{L^2(0,T,H^1)}^2+2\delta\left\|v_2^k\right\|_{L^2(0,T,H^1)}^2\le$$

$e^{6aT}\left(\epsilon\left|u_1^k(0)\right|_1^2+\left|v_1^k(0)\right|_1^2+\varepsilon\left|u_2^k(0)\right|_1^2+\right.$
$\left.\left|v_2^k(0)\right|_1^2\right).$ (4.8)

Then, we deduce that $u_1^k, v_1^k, u_2^k, v_2^k$, are uniformly bounded in $L^\infty\left(0,T;H^1(\Omega)\right)$, see Theorem 3.1. We now recall that $L^1(0,T;H^1(\Omega)')$, which is the pre-dual of $L^\infty\left(0,T;H^1(\Omega)\right)$, is a separable Banach space but not reflexive. Therefore, we conclude from the first and second bounds in (4.8) that

$$\{u_1^k, v_1^k, u_2^k, v_2^k\} \rightharpoonup^* \{u_1, v_1, u_2, v_2\}\ in\ L^\infty\left(0,T;H^1(\Omega)\right), \tag{4.9}$$

Then, we have u_1, v_1, u_2,and $v_2 \in L^\infty\left(0,T\,;H^1(\Omega)\right)$. We utilize established elliptic regularity results for bounded, convex, open domains. This involves considering the eigenvalue equations (3.7) and (3.8), as detailed in [38], Theorem 3.2.1.3, and Remark 3.2.1.4, we have for fixed (finite) k that $z_i \in H^2(\Omega)(i=1,\dots,k)$, and hence $u_1^k(.,t)$, $v_1^k(.,t), u_2^k(.,t)$ and $v_2^k(.,t) \in L^2(\Omega)$ for a.e. $t\in(0,T)$. Thus, by [38] Theorem 3.1.3.3, we have $\left\|u_1^k\right\|_2 \le C\left\|\Delta u_1^k\right\|_0$, $\left\|u_2^k\right\|_2 \le C\left\|\Delta u_2^k\right\|_0$, for some positive constant C and a.e. $t\in(0,T)$. Therefore, from the fifth to eighth bounds in (4.8), we conclude that u_1^k, v_1^k, u_2^k and v_2^k are uniformly bounded in $L^2\left(0,T;H^2(\Omega)\right)$. Since $L^2\left(0,T;H^2(\Omega)\right)$ is a reflexive Banach space (see [40] page 40), then, by compactness arguments (see [38] page 289), we deduce the existence of subsequence's u_1, v_1, u_2,and $v_2 \in L^2\left(0,T;H^2(\Omega)\right)$ such that

$$\{u_1^k, v_1^k, u_2^k, v_2^k\} \rightharpoonup \{u_1, v_1, u_2, v_2\}\ in\ L^2\left(0,T;H^2(\Omega)\right). \tag{4.10}$$

Thus, we arrive at u_1, v_1, u_2, and $v_2 \in L^2\left(0,T;H^2(\Omega)\right)$, furthermore as $\frac{\partial u_1^k}{\partial v}=0$, $\frac{\partial v_1^k}{\partial v}=0$, $\frac{\partial u_2^k}{\partial v}=0$ and $\frac{\partial v_2^k}{\partial v}=0$ on $\partial\Omega$, it follow by the weak convergence of $u_1^k\to u_1$, $v_1^k\to v_1$, $u_2^k\to u_2$ and $v_2^k\to v_2$ in $H^2(\Omega)$, that $\frac{\partial u_1}{\partial v}=0$, $\frac{\partial v_1}{\partial v}=0$, $\frac{\partial u_2}{\partial v}=0$ and $\frac{\partial v_2}{\partial v}=0$ on $L^2(\partial\Omega)$.

Estimate II: Set $\eta^k=\frac{\partial u_1^k}{\partial t}$, $\eta^k=\frac{\partial v_1^k}{\partial t}$, $\eta=\frac{\partial u_2^k}{\partial t}$, $\eta^k=\frac{\partial v_2^k}{\partial t}$ in the weak form (4.1)-(4.4) respectively, combine the results, yields

$$\begin{aligned}&\varepsilon\int_\Omega\left|\frac{\partial u_1^k}{\partial t}\right|^2dx+\int_\Omega\left|\frac{\partial v_1^k}{\partial t}\right|^2dx+\varepsilon\int_\Omega\left|\frac{\partial u_2^k}{\partial t}\right|^2dx+\int_\Omega\left|\frac{\partial v_2^k}{\partial t}\right|^2dx+\frac{d_1}{2}\frac{d}{dt}\int_\Omega\left|\nabla u_1^k\right|^2dx+\\&\frac{d_1}{2}\frac{d}{dt}\int_\Omega\left|\nabla u_2^k\right|^2dx+\frac{d_2}{2}\frac{d}{dt}\int_\Omega\left|\nabla v_1^k\right|^2dx+\frac{d_2}{2}\frac{d}{dt}\int_\Omega\left|\nabla v_2^k\right|^2dx+\frac{\alpha}{4}\frac{d}{dt}\int_\Omega\left|u_1^k\right|^4dx+\\&\frac{a}{4}\frac{d}{dt}\int_\Omega\left|u_2^k\right|^4dx\frac{a}{2}\frac{d}{dt}\int_\Omega\left|u_2^k-u_1^k\right|^2dx+\frac{\beta}{2}\frac{d}{dt}\int_\Omega\left|v_2^k-v_1^k\right|^2dx+\delta\frac{1}{2}\frac{d}{dt}\int_\Omega\left|v_1^k\right|^2dx+\\&\delta\frac{1}{2}\frac{d}{dt}\int_\Omega\left|v_2^k\right|^2dx=\frac{3a}{2}\frac{d}{dt}\int_\Omega\left|u_1^k\right|^2dx+\frac{3a}{2}\frac{d}{dt}\int_\Omega\left|u_2^k\right|^2dx+\int_\Omega u_1^k\frac{\partial v_1^k}{\partial t}dx+\int_\Omega u_2^k\frac{\partial v_2^k}{\partial t}dx-\\&\int_\Omega v_1^k\frac{\partial u_1^k}{\partial t}dx-\int_\Omega v_2^k\frac{\partial u_2^k}{\partial t}dx.\end{aligned} \tag{4.11}$$

By applying Young's inequality (2.5) on the final four terms in the right-hand side, we have that

$$\int_\Omega u_1^k\frac{\partial v_1^k}{\partial t}dx\le\frac{1}{2}\int_\Omega\left|u_1^k\right|^2dx+\frac{1}{2}\int_\Omega\left|\frac{\partial v_1^k}{\partial t}\right|^2dx \tag{4.12}$$

$$\int_{\Omega} u_2^k \frac{\partial v_2^k}{\partial t} dx \leq \frac{1}{2}\int_{\Omega} \left|u_2^k\right|^2 dx + \frac{1}{2}\int_{\Omega} \left|\frac{\partial v_2^k}{\partial t}\right|^2 dx \tag{4.13}$$

$$-\int_{\Omega} v_1^k \frac{\partial u_1^k}{\partial t} dx \leq \frac{1}{2\varepsilon}\int_{\Omega} \left|v_1^k\right|^2 dx + \frac{\varepsilon}{2}\int_{\Omega} \left|\frac{\partial u_1^k}{\partial t}\right|^2 dx \tag{4.14}$$

$$-\int_{\Omega} v_2^k \frac{\partial u_2^k}{\partial t} d \leq \frac{1}{2\varepsilon}\int_{\Omega} \left|v_2^k\right|^2 dx + \frac{\varepsilon}{2}\int_{\Omega} \left|\frac{\partial u_2^k}{\partial t}\right|^2 dx. \tag{4.15}$$

Combining (4.12) - (4.15) in (4.11) and multiplying through by 2, gives

$$\begin{aligned}
&\varepsilon\int_{\Omega} \left|\frac{\partial u_1^k}{\partial t}\right|^2 dx + \int_{\Omega} \left|\frac{\partial v_1^k}{\partial t}\right|^2 dx + \varepsilon\int_{\Omega} \left|\frac{\partial u_2^k}{\partial t}\right|^2 dx + \int_{\Omega} \left|\frac{\partial v_2^k}{\partial t}\right|^2 dx + d_1\frac{d}{dt}\int_{\Omega} \left|\nabla u_1^k\right|^2 dx \\
&+ d_1\frac{d}{dt}\int_{\Omega} \left|\nabla u_2^k\right|^2 dx + d_2\frac{d}{dt}\int_{\Omega} \left|\nabla v_1^k\right|^2 dx + d_2\frac{d}{dt}\int_{\Omega} \left|\nabla v_2^k\right|^2 dx \\
&+ \frac{\alpha}{2}\frac{d}{dt}\int_{\Omega} \left|u_1^k\right|^4 dx + \frac{a}{2}\frac{d}{dt}\int_{\Omega} \left|u_2^k\right|^4 dx + \alpha\frac{d}{dt}\int_{\Omega} \left|u_2^k - u_1^k\right|^2 dx \\
&+ \beta\frac{d}{dt}\int_{\Omega} \left|v_2^k - v_1^k\right|^2 dx + \delta\frac{d}{dt}\int_{\Omega} \left|v_1^k\right|^2 dx + \delta\frac{d}{dt}\int_{\Omega} \left|v_2^k\right|^2 dx \\
&\leq 3a\frac{d}{dt}\int_{\Omega} \left|u_1^k\right|^2 dx + 3a\frac{d}{dt}\int_{\Omega} \left|u_2^k\right|^2 dx \\
&+ \int_{\Omega} \left|u_1^k\right|^2 dx + \int_{\Omega} \left|u_2^k\right|^2 dx + \frac{1}{\varepsilon}\int_{\Omega} \left|v_1^k\right|^2 dx \\
&+ \frac{1}{\varepsilon}\int_{\Omega} \left|v_2^k\right|^2 dx. \qquad (4.16)
\end{aligned}$$

Integrating over time $(0, t)$, lead to

$$\begin{aligned}
&\varepsilon\left\|\frac{\partial u_1^k}{\partial t}\right\|^2_{L^2(\Omega_T)} + \left\|\frac{\partial v_1^k}{\partial t}\right\|^2_{L^2(\Omega_T)} + \varepsilon\left\|\frac{\partial u_2^k}{\partial t}\right\|^2_{L^2(\Omega_T)} + \left\|\frac{\partial v_2^k}{\partial t}\right\|^2_{L^2(\Omega_T)} + d_1\left|u_1^k(T)\right|_1^2 + d_1\left|u_2^k(T)\right|_1^2 + \\
&d_2\left|v_1^k(T)\right|_1^2 + d_2\left|v_2^k(T)\right|_1^2 + \frac{a}{2}\left|u_1^k(T)\right|_{0,4}^2 + \frac{a}{2}\left|u_2^k(T)\right|_{0,4}^2 + \alpha\left\|u_2^k(T) - u_1^k(T)\right\|_0^2 \\
&+\beta\left\|v_2^k(T) - v_1^k(T)\right\|_0^2 + \delta\left\|v_1^k(T)\right\|_0^2 + \delta\left\|v_2^k(T)\right\|_0^2 + 3a\left\|u_1^k(0)\right\|_0^2 + 3a\left\|u_2^k(0)\right\|_0^2 \\
&\leq 3a\left\|u_1^k(T)\right\|_0^2 + 3a\left\|u_2^k(T)\right\|_0^2 + \frac{1}{\varepsilon}\left\|v_1^k\right\|^2_{L^2(\Omega_T)} + \frac{1}{\varepsilon}\left\|v_2^k\right\|^2_{L^2(\Omega_T)} + \left\|u_1^k\right\|^2_{L^2(\Omega_T)} + \left\|u_2^k\right\|^2_{L^2(\Omega_T)} + \\
&d_1\left|u_1^k(0)\right|_1^2 + d_1\left|u_2^k(0)\right|_1^2 + d_2\left|v_1^k(0)\right|_1^2 + d_2\left|v_1^k(0)\right|_1^2 + \frac{a}{2}\left\|u_1^k(0)\right\|_{0,4}^2 + \frac{a}{2}\left\|u_2^k(0)\right\|_{0,4}^2 + \\
&\alpha\left\|u_2^k(0) - u_1^k(0)\right\|_0^2 + \beta\left\|v_2^k(0) - v_1^k(0)\right\|_0^2 + \delta\left\|v_1^k(0)\right\|_0^2 + \\
&\delta\left\|v_2^k(0)\right\|_0^2. \qquad (4.17)
\end{aligned}$$

Considering the bounds in Estimates I, specifically, $L^4(\Omega_T) \hookrightarrow L^2(\Omega_T), H^1(\Omega_T) \hookrightarrow L^4(\Omega_T), H^1(\Omega_T) \hookrightarrow L^2(\Omega_T)$, and given that the initial condition $u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0} \in H^1$, it follows that the right-hand side of (4.17) is bounded by a positive constant. Consequently, $\frac{\partial u_1^k}{\partial t}$, $\frac{\partial v_1^k}{\partial t}$, $\frac{\partial u_2^k}{\partial t}$ and $\frac{\partial v_2^k}{\partial t}$ are uniformly bounded in $L^2(\Omega_T)$. Since $L^2(\Omega_T)$ is a reflexive Banach space, by compactness arguments, we deduce the existence of subsequence's $u_1^k, v_1^k, u_2^k, v_2^k \in L^2(\Omega_T)$ such that

$$\left\{\frac{\partial u_1^k}{\partial t},\frac{\partial v_1^k}{\partial t},\frac{\partial u_2^k}{\partial t},\frac{\partial v_2^k}{\partial t}\right\} \rightharpoonup \left\{\frac{\partial u_1}{\partial t},\frac{\partial v_1}{\partial t},\frac{\partial u_2}{\partial t},\frac{\partial v_2}{\partial t}\right\} \ in\ L^2(\Omega_T). \quad (4.18)$$

Thus, we have that $\frac{\partial u_1}{\partial t},\frac{\partial v_1}{\partial t},\frac{\partial u_2}{\partial t}$ and $\frac{\partial v_2}{\partial t} \in L^2(\Omega_T), u_1, v_1, u_2$ and $v_2 \in L^\infty\big(0,T;H^1(\Omega)\big), u_1$ and $u_2 \in L^\infty\big(0,T;L^4(\Omega)\big), v_1$ and $v_2 \in L^\infty\big(0,T;L^2(\Omega)\big)$ and $(u_2-u_1),(v_2-v_1) \in L^\infty\big(0,T;L^2(\Omega)\big)$.

Lemma 4.1. For some $\eta \geq 0$, suppose that

$$\varsigma \in L^2\big(0,T;H^{\eta+1}(\Omega)\big), \frac{\partial \varsigma}{\partial t} \in L^2\big(0,T;H^{\eta-1}(\Omega)\big).$$

It follows that $\varsigma \in C\big([0,T];H^1(\Omega)\big)$.

Proof: (See [37], pages 191-194).

Here, in our case, $\eta = 1, H^{\eta+1}(\Omega) = H^2(\Omega),\ H^\eta(\Omega) = H^1(\Omega), H^{\eta-1}(\Omega) = L^2(\Omega)$. Thus, from Lemma 4.1 we have that $u_1^k, v_1^k, u_2^k, and\ v_2^k \in C\big([0,T];H^1(\Omega)\big)$.

4.2 Continuous dependence

Assume $u_1^1, u_1^2, v_1^1, v_1^2, u_2^1, u_2^2$ and v_2^1, v_2^2 satisfy the weak form (4.1) - (4.4), with initial conditions $u_1^1(.,0) = u_{1,0}^1(.)$, $u_1^2(.,0) = u_{1,0}^2(.)$, $v_1^1(.,0) = v_{1,0}^1(.)$, $v_1^2(.,0) = v_{1,0}^2(.)$, $u_2^1(.,0) = u_{2,0}^1(.)$, $u_2^2(.,0) = u_{2,0}^2(.)$ and $v_2^1(.,0) = v_{2,0}^1(.)$, $v_2^2(.,0) = v_{2,0}^2(.)$, respectively, such that $u_{1,0}^1(.) \neq u_{1,0}^2(.)$, $v_{1,0}^1(.) \neq v_{1,0}^2(.)$, $u_{2,0}^1(.) \neq u_{2,0}^2(.)$ and $v_{2,0}^1(.) \neq v_{2,0}^2(.)$. Setting $\omega_1 = u_1^1 - u_1^2$, $\omega_2 = v_1^1 - v_1^2$, $\omega_3 = u_2^1 - u_2^2$ and $\omega_4 = v_2^1 - v_2^2$ and setting $\eta = -\Delta\omega_1 + \omega_1$ and $\eta = -\Delta\omega_2 + \omega_2$, $\eta = -\Delta\omega_3 + \omega_3$, $\eta = -\Delta\omega_4 + \omega_4$ in (4.1) - (4.4), Subtracting weak forms results, after integrating by parts, in

$$\varepsilon\frac{1}{2}\frac{d}{dt}\int_\Omega (|\omega_1|^2 + |\nabla\omega_1|^2)dx + \frac{1}{2}\frac{d}{dt}\int_\Omega (|\omega_2|^2 + |\nabla\omega_2|^2)dx + \varepsilon\frac{1}{2}\frac{d}{dt}\int_\Omega (|\omega_3|^2 + |\nabla\omega_3|^2)dx + \frac{1}{2}\frac{d}{dt}\int_\Omega (|\omega_4|^2 + |\nabla\omega_4|^2)dx + d_1\int_\Omega (|\nabla\omega_1|^2 + |\Delta\omega_1|^2)dx + d_2\int_\Omega (|\nabla\omega_2|^2 + |\Delta\omega_2|^2)dx + d_1\int_\Omega (|\nabla\omega_3|^2 + |\Delta\omega_3|^2)dx + d_2\int_\Omega (|\nabla\omega_4|^2 + |\Delta\omega_4|^2)dx + a\int_\Omega ((u_1^1)^3 - (u_1^2)^3)(-\Delta\omega_1 + \omega_1)dx + a\int_\Omega ((u_2^1)^3 - (u_2^2)^3)(-\Delta\omega_3 + \omega_3)dx + \alpha\ \int_\Omega |\omega_3 - \omega_1|^2 + |\nabla\omega_3 - \nabla\omega_1|^2 dx + \beta\int_\Omega |\omega_4 - \omega_2|^2 + |\nabla\omega_4 - \nabla\omega_2|^2 dx + \delta\int_\Omega |\omega_2|^2 dx + \delta\int_\Omega |\nabla\omega_2|^2 dx + \delta\int_\Omega |\omega_4|^2 dx + \delta\int_\Omega |\nabla\omega_4|^2 dx = 3a\int_\Omega (|\omega_1|^2 + |\nabla\omega_1|^2)dx + 3a\int_\Omega (|\omega_3|^2 + |\nabla\omega_3|^2)dx. \quad (4.19)$$

Applying Young's inequalities (2.6), yields that

$$a\int_\Omega ((u_1^1)^3 - (u_1^2)^3)(-\Delta\omega_1 + \omega_1)dx = a\int_\Omega (u_1^1 - u_1^2)((u_1^1)^2 + u_1^1u_1^2 + (u_1^2)^2)(-\Delta\omega_1 + \omega_1)dx \geq a\int_\Omega \left(\frac{(u_1^1)^2}{2} + \frac{(u_1^2)^2}{2}\right)(|\nabla\omega_1|^2 + |\omega_1|^2)dx. \quad (4.20)$$

$$a\int_\Omega ((u_2^1)^3 - (u_2^2)^3)(-\Delta\omega_3 + \omega_3)dx = a\int_\Omega (u_2^1 - u_2^2)((u_2^1)^2 + u_2^1u_2^2 + (u_2^2)^2)(-\Delta\omega_3 + \omega_3)dx \geq a\int_\Omega \left(\frac{(u_2^1)^2}{2} + \frac{(u_2^2)^2}{2}\right)(|\nabla\omega_3|^2 + |\omega_3|^2)dx. \quad (4.21)$$

Substitute (4.20) and (4.21) in to (4.19), add $(\|\omega_2\|_1^2 + \|\omega_4\|_1^2)$, on the right-hand side, multiplying by 2, leads to

$$\varepsilon\frac{d}{dt}\|\omega_1\|_1^2+\frac{d}{dt}\|\omega_2\|_1^2+\varepsilon\frac{d}{dt}\|\omega_3\|_1^2+\frac{d}{dt}\|\omega_4\|_1^2+2d_1\int_\Omega(|\nabla\omega_1|^2+|\Delta\omega_1|^2)dx+ 2d_2\int_\Omega(|\nabla\omega_2|^2+|\Delta\omega_2|^2)dx+2d_1\int_\Omega(|\nabla\omega_3|^2+|\Delta\omega_3|^2)dx+2d_2\int_\Omega(|\nabla\omega_4|^2+ |\Delta\omega_4|^2)dx+2a\int_\Omega\left(\frac{(u_1^1)^2}{2}+\frac{(u_1^2)^2}{2}\right)(|\nabla\omega_1|^2+|\omega_1|)dx+2a\int_\Omega\left(\frac{(u_2^1)^2}{2}+\frac{(u_2^2)^2}{2}\right)(|\nabla\omega_3|^2+ |\omega_3|)dx+2\alpha\int_\Omega(|\omega_3-\omega_1|^2+|\nabla\omega_3-\nabla\omega_1|^2)dx+2\beta\int_\Omega(|\omega_4-\omega_2|^2+|\nabla\omega_4- \nabla\omega_2|^2)dx+2\delta\int_\Omega|\omega_2|^2dx+2\delta\int_\Omega|\nabla\omega_2|^2dx+2\delta\int_\Omega|\omega_4|^2dx+2\delta\int_\Omega|\nabla\omega_4|^2dx\le 6a(\|\omega_1\|_1^2+\|\omega_3\|_1^2+\|\omega_2\|_1^2+\|\omega_4\|_1^2). \quad (4.22)$$

By neglecting final (twelve) terms on the left side of inequality (4.22), application of Grönwall Lemma 2.1 we have

$$\varepsilon\|\omega_1(T)\|_1^2+\|\omega_2(T)\|_1^2+\varepsilon\|\omega_3(T)\|_1^2+\|\omega_4(T)\|_1^2 \le (\varepsilon\|\omega_1(0)\|_1^2+\|\omega_2(0)\|_1^2+\varepsilon\|\omega_3(0)\|_1^2 +\|\omega_4(T)\|_1^2\exp(6aT). \quad (4.23)$$

Thus, if $\left(u_1^1(0),v_1^1(0),u_2^1(0),v_2^1(0)\right)=\left(u_1^2(0),v_1^2(0),u_2^2(0),v_2^2(0)\right)$ then $\left(\omega_1(0),\omega_2(0),\omega_3(0),\omega_4(0)\right)=(0,0,0,0)$ and hence it follows from (4.23) that $\left(\omega_1(T),\omega_2(T),\omega_3(T),\omega_4(T)\right)=(0,0,0,0)$ and hence $u_1^1(T)=u_1^2(T)$, $v_1^1(T)=v_1^2(T)$, $u_2^1(T)=u_2^2(T)$ and $v_2^1(T)=v_2^2(T)$ for all *t*, we deduce uniqueness of solution. However, if $\left(u_1^1(0),v_1^1(0),u_2^1(0),v_2^1(0)\right)\neq\left(u_1^2(0),v_1^2(0),u_2^2(0),v_2^2(0)\right)$, Thus, we establish continuous dependence in $H^1(\Omega)$. This concludes the proof of Theorem 4.1.

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