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Generalized Amply Cofinitely Supplemented Modules

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Abstract
Let R be an associative ring with identity. An R-module M is called generalized amply cofinitely supplemented module if every cofinite submodule of M has an ample generalized supplement in M. In this paper we proved some new results about this conc- ept.
Keyword: Amply, cofinitely supplemented, R-module

المقاسات المكملة المنتهية المضاده باسهاب معمم

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الخلاصة
لتكن R حلقة تجميعية تمتلك عنصر محايد. المقاس M من النمط R يدعى مقاس مكمل منتهي مضاد باسهاب معمم اذا كان لكل مقاس جزئي منتهي مضاد من M يمتلك تعميم مكمل باسهاب في M. في هذا لبحث قدمنا نتائج جديدة حول هذا النوع من المقاسات.

1. Introduction

In this paper, R will denote on arbitrary ring with unity and M is a unitary left R-module.

Let M be an R-module and, recall that a submodule N of M is called small, denoted by $N<<M$, if $N+K\neq M$ for every proper submodule K of M (See [1], 5. 1.1). K is supplemented of N in M if and only if $N+K=M$ and $N\cup K<<K$ (See [2]) where K and N are submodules of M.

M is called supplemented if every submodule of M has a supplemt in M (See [2]). On the other hand, the module M is called amply supplemented if, for every submodules A, B with $A+B=M$, there exists a supplement C of A in M such that $C\subseteq B$ (See [2]).

A submodule N of M is said to be cofinite if $\frac{M}{N}$ is finitely generated (See [3]).

An R-module M is called a cofinitely supplemented module if every cofinite submodule of M has a supplement in M (See [3]). Clearly every supplemented module is cofinitely supplemented module.

An R-module M is called a cofinitely amply supplemented module if for every cofinite submodule of M has an ample supplement in M (See [4]).

Submodules A and B of an R-module M with $A+B=M$, B is called a generalized supplement of A in M in case $A\cup B\subseteq Rad(B)$, where Rad(B) is the Jacobson radical of B, (See [5])

M is called generalized supplemented module or briefly a GS-module if every submodule N of M has a generalized supplemented K in M (See[5]).

Following [5], M is called a generalized amply supplemented module or briefly a GAS-module in case $M=N+K$ implies that N has a generalized supplement $N\subseteq K$. it is clear that every GAS-module is GS-module.

In [6] M is called generalized cofinitely supplemented if every cofinite submodule of M has a generalized supplement and denoted by GCS.

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Clearly every generalized supplemented module is generalized cofinitely supplemented.

It is shown in [6] that a generalized cofinitely supplemented module need not to be generalized supplemented, also in [8] there is one more example about such modules.

An R-module M is called generalized amply cofinitely supplemented if every cofinite submodule of M has an ample generalized supplement [7]. Clearly that every generalized amply cofinitely supplemented is generalized cofinitely supplemented, but the converse is not true [8].

2- Generalized amply cofinitely supplemented modules

An R-module M is called generalized amply cofinitely supplemented if every cofinite submodule of M has an ample generalized supplement [7].

In this section we give a new characterization for generalized amply cofinitely supplemented module when every submodule of this module is generalized cofinitely supplemented as the following lemma shows.

Theorem 2.1

If every submodule of an R-module M is generalized cofinitely supplemented, then M is generalized amply cofinitely supplemented module.

Proof:

Let N be a cofinite submodule of an R-module M. By assumption N has generalized supplement K in M such that N+K=M and N∪K⊆ Rad (K).

Now, N∪K⊆ K also N∪K is a cofinite submodule since N is a cofinite thus $\frac{M}{N}$ is finitely generated and $\frac{M}{N}=\frac{N+K}{N}\cong\frac{K}{N\cap K}$.

This means that $\frac{K}{N\cap K}$ is finitely generated thus N∪K is a cofinite submodule of K, thus by assumption, there is a submodule H of K such that K= N∪K+H and (N∪K) ∪H= N∪H ⊆ Rad (H). So, M=N+K=N+ N∪K+H=N+H.

Thus H is a generalized amply supplement to a cofinite submodule N of M.■

An R-module is called π-projective module if for every two submodules A and B of M such that A+B=M, there exists a homomorphism f∈End (M) such that f(M) ⊆A and (I-f)(M) ⊆B,[2].

In [8] proved that every weakly supplemented and π-projective module is amply weak supplemented module. In [9], proved that if M is cofinitely weak supplemented and π-projective module; then M is cofinitely amply weakly supplemented module. Also,[10] proved that every generalized supplemented and π-projective module is a generalized amply supplemented module.

The following theorem is proved in [11]. Here we will give another proof with more details.

Theorem 2.2[11] Let M be a generalized cofinitely and π-projective module. Then M is generalized amply cofinitely supplemented module.

Proof:

Let U be a cofinite submodule of an R-module M and M=U+V where V is a submodule of M. Since M is generalized cofinitely supplemented, then U has generalized supplement X in M, i.e U+X=M and U∪X⊆Rad (X).But M is π-projective, thus there exists a homorphism f ∈ End(M) such that f(M) ⊆ V and (I-f)(M) ⊆ U .

We claim that M=f(X)+U; to prove this, let m∈ M⇒m=u+x,u∈U,x∈X, hence m=u + x – f(x) + f(x) = u+(I-f)(x)+f(x) ∈U+f(X).

Therefore M⊆ U+f(X)/Also we have U+f(X) ⊆ M,thus M=U+f(X).

We claim that U∪f(X) ⊆f(U∪X).To prove this, let y∈ U∪f(X) this implies that y=f(x),where x∈X. Now, x-y=x-f(x) = (I-f)(x) ∈ U and since y ∈ U, hence y = f(x) ∈ f(U∪X).But U∪X⊆Rad (X), therefore f(U∪X) ⊆f(Rad (X))[1]. .Also,by [1], f(Rad (X)) ⊆Rad f(X)). Thus U∪f(X)⊆Rad(f(X)). Therefore f(X) is a generalized supplement of U in M where M=U+V. ■

Corollary2.3

Every projective and generalized amply cofinitely supplement module is generalized amply cofinitely supplemented module

Proof:

Since every projective is π-projective module[2], then by theorem 2.2 we get the result. ■

3-Supplement submodule of a generalized amply cofinitely supplemented modules

In this section we will prove that the supplement of a generalized amply cofinitely supplement module is amply cofinitely supplemented module

Lemma 3.1

Every supplement submodule of a generalized amply cofinitely supplemented modules is amply cofinitely supplemented modules.

Proof:

Let M be generalized amply cofinitely supplemented module and V be any supplement submodule of M. Let V be a supplement of U in M.

Let $K \subseteq V$ be a cofinite submodule of V such that K+T=V if we find $T' \subseteq T$ such that $K+T'=V$ and $K \cup T' \subseteq Rad(T')$, then we get the result.

Since K is a cofinite supplement of V then $\frac{V}{K}$ is finitely generated.

Now, $\frac{M}{U+K} = \frac{U+V}{U+K} \cong \frac{V}{V \cap (U+K)}$ thus U+K is a cofinite submodule of M.

Let K+T=V, for every $T \subseteq V$. Since M=U+V then U+K+T=M, but M is a generalized amply cofinitely supplemented module and U+K is a cofinite submodule of M, U+K has a generalized supplement submodule T' in M such that $T' \subseteq T$, $(U+K)+T'=M$ and $(U+K) \cup T' \subseteq Rad(T')$. Now, $K+T' \subseteq V$ and V is a supplement of U in M, $U+K+T'=M$, hence $V \subseteq K+T'$, thus $K+T'=V$, also $K \cup T' \subseteq Rad(T')$. Therefore K has an ample generalized supplement. █

Corollary3.2

Every direct summand of a generalized amply cofinitely supplemented module is generalized amply cofinitely supplemented

Proof:

Let M be a generalized amply cofinitely supplemented module. Since every direct summand of M is supplement in M, then by lemma 3.1, every direct summand of M is a generalized amply cofinitely supplemented. █

Proposition3.3

Let M be a module, if every submodule of M is a cofinitely generalized supplemented module, then M is amply generalized cofinitely supplemented.

Proof:

Let N be a cofinite submodule of M and suppose that N+K=M, where K is submodule of M.

Now, notice that $\frac{K}{N \cap K} \cong \frac{N+K}{N} = \frac{M}{N}$, since N is a cofinite submodule of M, $\frac{M}{N}$ is finitely generated module. Thus $\frac{K}{N \cap K}$ is finitely generated, here $N \cap K$ is cofinite submodule of K, by assumption $N \cap K$ has a generalized supplement $H \leq K$ such that $(N \cap K) + H = K$ and $(N \cap K) \cap H = N \cap H \subseteq Rad(H)$, also $N + H \geq N \cap K + H = K$ thus $N + H \geq K + N = M$. Hence $N + H = M$. █

Corollary3.4

Let R be any ring. Then the following statements are equivalent:-

1- Every R-module is an amply generalized cofinitely supplemented module.

2- Every R-module is a generalized amply supplemented module. █

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