Local Bifurcation and Persistence of an Ecological System Consisting of a Predator and Stage Structured Prey

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Abstract
In this paper, the conditions of persistence of a mathematical model, consists from a predator interacting with stage structured prey are established. The occurrence of local bifurcation and Hopf bifurcation are investigated. Finally, in order to confirm our obtained analytical results, numerical simulations have been done for a hypothetical set of parameter values .
Keyword: equilibrium point, persistence, bifurcation, sotomayor theorem

التفرع المحلي والإصرار لنظام بيئي يتكون من مفترس وفريسة ذات مراحل عمرية مركبة

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الخلاصة
في هذا البحث،شروط الاصرار لنموذج رياضي يتكون من مفترس يتفاعل مع فريسة ذات مراحل عمرية مركبة وجدت. امكانية حدوث التفرع المحلي وتفرع هوبف بحثت، وأخيراً، لتأكيد النتائج التحليلية التي تم الحصول عليها وتحديد المعلمات الاساسية، استخدمنا المحاكاة العددية لمجموعة من قيم المعلمات الافتراضية.

1. Introduction:

Over the last few decades there has been a considerable interest in the study of population dynamics with stage structure. Such studies are important, since the life cycle of the most of the animals and insects in nature have two stages: immature and mature. The species in the first stage can't interact or reproduce with the other species rather than that; it depends completely on its relative from mature species, see for example [1-4] and the references therein. Most of these studies were focused on prey-predator interactions involving a stage structured predators with or without time delay.

Later on Cui and Song [5] proposed and analyzed a prey-predator model with stage structure for prey. It is assumed that the predator consumed the immature prey according to Lotka-Volterra type of functional response. They obtained a set of sufficient and necessary conditions which guarantee the permanence of the system. However, Chen [6] studied the permanence of periodic predator–prey system with stage structure for prey. He obtained sufficient and necessary conditions which guarantee the predator and the prey species to be permanent. Recently, Chen and You [7] studied the permanence, extinction and periodic solution of the periodic predator–prey system with Beddington–DeAngelis functional response and stage structure for prey. They obtained a set of sufficient and necessary conditions which guarantee the permanent of the system in this paper however, we will propose and analyze Holling type-II prey-predator having stage structure for prey. The intraspecific competition for immature prey and predator is also included in the system.

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2. Mathematical model:[8]

An ecological model consists of prey-predator system with stage structure for prey is proposed. In order to formulate the dynamic equations for such a model the following assumptions are made.

A1) The prey population is divided into two classes, immature prey population, whose population density at time T is denoted by $x_1(T)$, and mature prey population, whose population density at time T is denoted by $x_2(T)$.

A2) It is assumed that in the absence of predation only the mature prey population has the ability for reproduction logistically with carrying capacity k ($k>0$) and intrinsic growth rate α ($\alpha>0$). However, the immature prey population depends completely in his reproduction on the food supplied by mature prey. In addition to the above , the immature prey individuals still compete between each other for food and space with intraspecific rate constant η ($\eta>0$).

A3) The immature prey population transfer to mature prey population at a rate βx_1, where β ($\beta>0$) represents the conversion rate coefficient. Finally, both the immature and mature prey populations decreases due to the natural death rates r_1 ($r_1>0$) and r_2 ($r_2>0$) respectively. Thus, depending on the above assumptions the evolution equations for prey can be written as:

$$\frac{dx_1}{dT}=\alpha x_2\left(1-\frac{x_2}{k}\right)-r_1x_1-\beta x_1-\eta x_1^2 \qquad (1a)$$

$$\frac{dx_2}{dT}=\beta x_1-r_2x_2 \qquad (1b)$$

A4) In case of existence of predator, whose population density denoted by $x_3(T)$, it is assumed that the predator consumes the immature prey only (the immature prey is more vulnerable to predation than the mature prey) according to Holling type-II functional response $\frac{\beta_1x_1}{\gamma_1+x_1}$, where β_1 ($\beta_1>0$) and γ_1 ($\gamma_1>0$) represent respectively, the maximum attack rate and half saturation constants. However, the predator's contribution from the prey species is assumed to be $\frac{c\beta_1x_1}{\gamma_1+x_1}$ where c ($c>0$) denotes to conversion rate constant.

A5) Finally, it is assumed that , the predator individuals still compete with each other for food and spaces with intraspecific rate constant η_1 ($\eta_1>0$), and decrease due to natural death rate r ($r>0$).

Consequently, in the existence of predator species, the evolution equation (1) of a stage structure prey species becomes:

$$\frac{dx_1}{dT}=\alpha x_2\left(1-\frac{x_2}{k}\right)-r_1x_1-\beta x_1-\eta x_1^2-\frac{\beta_1x_1x_3}{\gamma_1+x_1} \qquad ..(2a)$$

$$\frac{dx_2}{dT}=\beta x_1-r_2x_2 \qquad (2b)$$

$$\frac{dx_3}{dT}=-rx_3+c\frac{\beta_1x_1x_3}{\gamma_1+x_1}-\eta_1x_3^2 \qquad (2c)$$

Now, for further simplifying the system (2), the following dimensionless variables are used

$$y_1=\frac{c\beta_1}{\alpha\gamma_1}x_1,\quad y_2=\frac{c\beta_1}{\alpha\gamma_1}x_2,\quad y_3=\frac{\beta_1}{\alpha\gamma_1}x_3,\quad t=\alpha T.$$

Thus, system (2) can be turned into the following dimensionless form:

$$\frac{dy_1}{dt}=y_2(1-w_1y_2)-w_2y_1-w_3y_1-w_4y_1^2-\frac{y_1y_3}{1+w_5y_1}=f_1(y_1,y_2,y_3) \qquad (3a)$$

$$\frac{dy_2}{dt}=w_3y_1-w_6y_2=f_2(y_1,y_2,y_3) \qquad (3b)$$

$$\frac{dy_3}{dt}=y_3\left(-w_7+\frac{y_1}{1+w_5y_1}-w_8y_3\right)=f_3(y_1,y_2,y_3) \qquad (3c)$$

where $w_1=\frac{\alpha\gamma_1}{c\beta_1k}$; $w_2=\frac{r_1}{\alpha}$; $w_3=\frac{\beta}{\alpha}$; $w_4=\frac{\eta\gamma_1}{c\beta_1}$; $w_5=\frac{\alpha}{c\beta_1}$; $w_6=\frac{r_2}{\alpha}$; $w_7=\frac{r}{\alpha}$ and $w_8=\frac{\eta_1\gamma_1}{\beta}$ are the dimensionless parameters.

System (3) needs to analyzed with a specific initial condition, which may be taken as any point in the region $R_+^3=\{(y_1,y_2,y_3)\in R^3: y_i\geq 0; i=1,2,3\}$.

Theorem 1:[8]- All solutions of system (3), which are initiate in R_+^3 are uniformly bounded.

3. Existence and stability analysis of system (3):[8]

The stage structured prey-predator model given by system (3) has at most three nonnegative equilibrium points, namely

$E_0 = (0,0,0)$, $E_1 = (\bar{y}_1, \bar{y}_2, 0)$, and $E_2 = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$.

The equilibrium point E_0 always exists, however the equilibrium point E_1 exists in the $Int.R_+^2$ of $y_1 y_2$ −plane where

$$\bar{y}_1 = \frac{-(w_2 + w_3)w_6^2 + w_3 w_6}{(w_4 w_6^2 + w_1 w_3^2)} \tag{4a}$$

$$\bar{y}_2 = \frac{w_3}{w_6}\bar{y}_1 \tag{4b}$$

provided that:

$$w_3 > (w_2 + w_3)w_6 \tag{5}$$

Finally the positive equilibrium point $E_2 = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ where

$$\hat{y}_2 = \frac{w_3}{w_6}\hat{y}_1 \tag{6a}$$

$$\hat{y}_3 = \frac{-w_7}{w_8} + \frac{\hat{y}_1}{w_8(1 + w_5 \hat{y}_1)} \tag{6b}$$

while $\hat{y}_1$ is a positive root of the following third order equation:

$$A_1 y_1^3 + A_2 y_1^2 + A_3 y_1 + A_4 = 0.$$

Here

$$A_1 = \frac{w_1 w_3^2 w_8 w_5^2}{w_6^2} + w_4 w_8 w_5^2 > 0,$$

$$A_2 = \frac{-w_3 w_8 w_5^2}{w_6} + \frac{2 w_1 w_3^2 w_5 w_8}{w_6^2} + w_2 w_8 w_5^2 + w_3 w_8 w_5^2 + 2 w_4 w_8 w_5,$$

$$A_3 = \frac{-2 w_3 w_5 w_8}{w_6} + \frac{w_1 w_3^2 w_8}{w_6^2} + 2 w_2 w_8 w_5 + 2 w_3 w_8 w_5 + w_4 w_8 - w_5 w_7 + 1,$$

$$A_4 = \frac{-w_3 w_8}{w_6} + w_2 w_8 + w_3 w_8 - w_7,$$

Obviously E_2 exists uniquely in the $Int.R_+^2$ if and only if the following conditions hold

$$w_2 w_6 < w_3 < \frac{w_7}{w_8} \tag{7a}$$

$$-w_8 w_5^2 \left[\frac{w_3}{w_6} - w_2\right] + R > 0; \text{ with} \tag{7b}$$

$$R = \frac{2 w_1 w_3^2 w_5 w_8}{w_6^2} + w_3 w_8 w_5^2 + 2 w_4 w_8 w_5, \text{ and}$$

$$\hat{y}_1 > w_7(1 + w_5 \hat{y}_1) \tag{7c}$$

In the following, the local dynamical behavior of system (3) around each of the above equilibrium points is discussed. First the Jacobian matrix of system (3) at each point is determined and then the eigenvalues for the resulting matrix are computed. The jacobian matrix of system (3) at the equilibrium point $E_0 = (0,0,0)$ can be written by

$$J(E_0) = \begin{bmatrix} -(w_2 + w_3) & 1 & 0 \\ w_3 & -w_6 & 0 \\ 0 & 0 & -w_7 \end{bmatrix}.$$

Therefore, it is easy to verify that, the eigenvalues of $J(E_0)$, say $\lambda_{01}, \lambda_{02}$ and λ_{03} that describe the dynamics in the y_1, y_2 and y_3-direction respectively satisfy the following relations :

$$\lambda_{01} + \lambda_{02} = -(w_2 + w_3) - w_6 < 0 \tag{8a}$$

$$\lambda_{01}.\lambda_{02} = (w_2 + w_3)w_6 - w_3 \tag{8b}$$

$$\lambda_{03} = -w_7 < 0 \tag{8c}$$

Note that, according to Eq. (9b), the eigenvalues λ_{01} and λ_{02} have opposite sign provided that

$$w_3 > (w_2 + w_3)w_6. \tag{9a}$$

Hence E_0 is a saddle point in the R_+^2 of $y_1 y_2$-plane and since the eigenvalue λ_{03} that describes the dynamics in y_3-direction is negative , hence E_0 is a saddle point in R_+^3 with locally stable manifold of dimension two and with locally unstable manifold of dimension one However, λ_{01} and λ_{02} are negative provided that

$$w_3 < (w_2 + w_3)w_6 \tag{9b}$$

and then E_0 is a locally asymptotically stable in the R_+^3.

The Jacobian matrix of system (3) at the equilibrium point $E_1 = (\bar{y}_1, \bar{y}_2, 0)$ is given by: $J(E_1) = (b_{ij})_{3\times 3}$, where

$b_{11} = -w_2 - w_3 - 2 w_4 \bar{y}_1$, $b_{12} = 1 - 2 w_1 \bar{y}_2$, $b_{13} = \frac{\bar{y}_1}{1 + w_5 \bar{y}_1}$, $b_{21} = w_3$, $b_{22} = -w_6$, $b_{23} = b_{31} = b_{32} = 0$, $b_{33} = -w_7 + \frac{\bar{y}_1}{1 + w_5 \bar{y}_1}$.

Now, straightforward computation shows that , the eigenvalues of the Jacobian matrix $J(E_1)$, say λ_{11} , λ_{12} and λ_{13} which describe the dynamics in the directions y_1, y_2 and y_3 respectively , satisfy the following relations:

$$\lambda_{11} + \lambda_{12} = -(w_2 + w_3) - 2 w_4 \bar{y}_1 - w_6 < 0, \tag{10a}$$

$$\lambda_{11}.\lambda_{12} = -M_1 + M_2\bar{y}_1, \quad (10b)$$

$$\lambda_{13} = -w_7 + \frac{\bar{y}_1}{1+w_5\bar{y}_1} \quad (10c)$$

where $M_1 = w_3 - (w_2 + w_3)w_6$, which is positive under the existence condition of E_1, and $M_2 = 2(w_4 w_6 + \frac{w_1 w_3^2}{w_6})$.

Note that, according to the Eqs. (10a)-(10c) we have the following two cases :

Case 1:- If the following condition holds

$$\lambda_{13} < 0 \Leftrightarrow \frac{\bar{y}_1}{1+w_5\bar{y}_1} < w_7. \quad (11a)$$

Then E_1 is locally asymptotically stable in y_3-direction, and hence we have the following two subcases:

1- E_1 is locally asymptotically stable in R_+^3 provided that the following condition holds

$$M_1 < M_2\bar{y}_1 \quad (11b)$$

2- E_1 is a saddle point in R_+^3 with locally stable manifold of dimension two and locally unstable manifold of dimension one provided that the following condition holds :

$$M_1 > M_2\bar{y}_1 \quad (11c)$$

Case 2:- If the following condition holds

$$\lambda_{13} > 0 \Leftrightarrow \frac{\bar{y}_1}{1+w_5\bar{y}_1} > w_7 \quad (11d)$$

Then E_1 is a saddle point in R_+^3.

Finally, the Jacobian matrix of the system (3) at the positive equilibrium point $E_2 = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ can be written as:

$$J(E_2) = (a_{ij})_{3\times 3} \quad (12)$$

here $a_{11} = -(w_2 + w_3) - 2w_4\hat{y}_1 - \frac{\hat{y}_3}{N_0^2}$;

$a_{12} = 1 - 2w_1\hat{y}_2$; $a_{13} = \frac{\hat{y}_1}{N_0}$; $a_{21} = w_3$;

$a_{22} = -w_6$; $a_{23} = 0$; $a_{31} = \frac{\hat{y}_1}{N_0^2}$; $a_{32} = 0$;

$a_{33} = -w_7 + \frac{\hat{y}_1}{N_0} - 2w_8\hat{y}_3$; $N_0 = 1 + w_5\hat{y}_1$.

Accordingly the characteristic equation of $J(E_2)$ is given by

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \quad (13a)$$

where

$$A_1 = \frac{1}{N_0^2}\left[N_1 + N_0N_2 + w_6N_0^2\right] \quad (13b)$$

$$A_2 = \frac{w_6}{N_0^2}\left[N_1 + N_0N_2\right] - \frac{1}{N_0^3}\left[\hat{y}_1^2 - N_1N_2\right] - w_3N_3 \quad (13c)$$

$$A_3 = \frac{-w_6}{N_0^3}\left[\hat{y}_1^2 - N_1N_2\right] - \frac{w_3N_2N_3}{N_0} \quad (13d)$$

with

$N_1 = (w_2 + w_3 + 2w_4\hat{y}_1)N_0^2 + \hat{y}_3 > 0$,

$N_2 = (w_7 + 2w_8\hat{y}_3)N_0 - \hat{y}_1$, $N_3 = 1 - 2w_1\hat{y}_2$.

Note that, due to Routh-Hurwitz criterion, the necessary and sufficient conditions for E_2 to be locally asymptotically stable in the $Int.R_+^3$, are $A_1 > 0$, $A_3 > 0$ and $\Delta = A_1A_2 - A_3 > 0$.

Straightforward computation shows that, if the following condition holds

$$N_2 > 0 \Leftrightarrow \hat{y}_1 < (w_7 + 2w_8\hat{y}_3)N_0 \quad (14a)$$

Then we obtain $A_1 > 0$. In addition to condition (15a), if the following conditions hold

$$N_3 < 0 \Leftrightarrow \hat{y}_2 > \frac{1}{2w_1}, \text{ and} \quad (14b)$$

$$\hat{y}_1^2 < N_1N_2, \quad (14c)$$

Then we get that $A_3 > 0$.

Finally, substituting the values of A_i for $i = 1,2,3$ in $\Delta = A_1A_2 - A_3$ and then simplifying the resulting term we get that

$$\Delta = \frac{(N_1 + N_0N_2)}{N_0^2}\left[\frac{w_6}{N_0^2}(N_1 + N_0N_2) - \frac{1}{N_0^3}(\hat{y}_1^2 - N_1N_2) - w_3N_3 + w_6^2\right] \ldots\ldots\ldots(14d') + w_3N_3\left(\frac{N_2}{N_0} - w_6\right)$$

$$= N_4 + w_3N_3N_5,$$

where

$$N_4 = \frac{(N_1 + N_0N_2)}{N_0^2}\left[\frac{w_6}{N_0^2}(N_1 + N_0N_2) - \frac{1}{N_0^3}(\hat{y}_1^2 - N_1N_2) - w_3N_3 + w_6^2\right] > 0$$

$$N_5 = \frac{N_2}{N_0} - w_6,$$

obviously $\Delta > 0$ if and only if in addition to conditions (14a)-(14c) one of the following two conditions holds:

$N_5 \le 0 \Leftrightarrow \frac{N_2}{N_0} \le w_6$ (14d)

or

$N_5 > 0$ with $N_4 + w_3 N_3 N_5 > 0$ (14e)

Consequently the following theorem for locally stability of E_2 can be proved easily.

Theorem 2[8]:- Assume that the positive equilibrium point E_2 of system (3) exists. Then E_2 is locally asymptotically stable in the $Int.R_+^3$ if the conditions (14a)-(14c) with (14d) or (14e) are satisfied.

4.Persistence

In general persistence is a global property of a dynamical system, it is not dependence upon interior solution space structure but is dependent upon solution behavior near extinction boundaries (boundary planes).From the biological point of view, persistence of a system means the survival of all population of the system in future time. However, mathematically it means that strictly positive solutions do not have omega limit set on the boundary of the non-negative cone [9].Accordingly, if the dynamical system does not persists then the solution have omega limit set on the boundary of the nonnegative cone, and hence the dynamical system faces extinction. Now, before examine the persistence of stage structure model given by system (3) by using the method of average Lyapunov function as given in [10], we need to study the global dynamics in the boundary plane $y_1 y_2$ as shown in the following theorem.

Theorem 3:Suppose that the equilibrium point $E_1 = (\bar{y}_1, \bar{y}_2, 0)$ is locally asymptotically stable in the $Int.R_+^2$ then it is a globally asymptotically stable in the $Int.R_+^2$ of the $y_1 y_2$-plane provided that

$w_1 y_2 < 1$ … .(15)

Proof:-We will proof the theorem in the $Int.R_+^2$.Clearly for any initial value in the $Int.R_+^2$ of $y_1 y_2$-plane, system (3) reduces to the following subsystem

$$\left.\begin{aligned} \frac{dy_1}{dt} &= y_2(1 - w_1 y_2) - w_2 y_1 - w_3 y_1 \\ &\quad - w_4 y_1^2 = h_1(y_1, y_2) \quad \ldots\ldots\ldots \\ \frac{dy_2}{dt} &= w_3 y_1 - w_6 y_2 = h_2(y_1, y_2) \end{aligned}\right\} \quad (16)$$

Obviously E_1 represents the positive equilibrium point of subsystem (16) in the $Int.R_+^2$ of $y_1 y_2$-plane. Assume that $H(y_1, y_2) = \frac{1}{y_1 y_2}$**.** Clearly $H(y_1, y_2)$ is a C^1 function and is a positive for all $(y_1, y_2) \in Int.R_+^2$.Further

$$\Delta(y_1, y_2) = \frac{\partial}{\partial y_1}(Hh_1) + \frac{\partial}{\partial y_2}(Hh_2)$$
$$= -\frac{1}{y_1^2}(1 - w_1 y_2) - \frac{w_4}{y_2} - \frac{w_3}{y_2^2}.$$

Note that $\Delta(y_1, y_2)$ does not change sign under condition (15) and is not identically zero in the $Int.R_+^2$ of the $y_1 y_2$-plane. Then according to Bendixson –Dualic criterion subsystem (16) has no periodic dynamic in the interior of positive quadrant of $y_1 y_2$-plane. Further, since E_1 is the only positive equilibrium point of subsystem (16) in the interior of positive quadrant of $y_1 y_2$-plane. Hence according to Poincare-Bendixson theorem E_1 is a globally asymptotically stable in the interior of positive quadrant.

Theorem 4: Assume that there are no periodic dynamics in the boundary plane $y_1 y_2$. Further, if in addition to conditions (5),(5a) and (15) the following conditions are holds:

$w_2 \bar{y}_2 < 1$ ....(16 a)

and

$(w_2 + w_3 + w_4 \bar{y}_1)\bar{y}_1 < \bar{y}_2(1 - w_2 \bar{y}_2)$ ....(16b)

Proof**:** Consider the following average Lyapunov function $\delta(y_1, y_2, y_3) = y_1^{p_1+1} y_2^{p_2+1} y_3^{p_3+1}$, where each $p_i, i = 1,2,3$ is assumed to be positive, obviously $\delta(y_1, y_2, y_3)$ is continuously differentiable positive function defined in

R^3_+.Now, since

$$\Psi(y_1,y_2,y_3)=\frac{\delta'(y_1,y_2,y_3)}{\delta(y_1,y_2,y_3)}$$
$$=(p_1+1)\left[y_2(1-w_1y_2)-w_2y_1-w_3y_1-w_4y_1^2-\frac{y_1y_3}{1+w_5y_1}\right]$$
$$+(p_2+1)\left[w_3y_1-w_6y_2\right]$$
$$+(p_3+1)\left[y_3(-w_7+\frac{y_1}{1+w_5y_1}-w_8y_3)\right].$$

Now, since it is assumed that there are no periodic attractors in the boundary plane, and the vanishing equilibrium point E_0 is unstable saddle point under condition (9a) with locally unstable manifold in the y_1-direction or in the y_2-direction, and hence E_0 does not belong to the possible omega limit set of system (3), then the only possible omega limit set of system (3) is the equilibrium point E_1.So the proof is follows and the system is uniformly persists if we can proof that $\Psi(\cdot)>0$ at each of these equilibrium points. Note that, for $E_1=(\bar{y}_1,\bar{y}_2,0)$ we have

$$\Psi(E_1)=(p_1+1)\left[\bar{y}_2(1-w_1\bar{y}_2)-w_2\bar{y}_1-w_3\bar{y}_1-w_4\bar{y}_1^2\right]+(p_2+1)\left[w_3\bar{y}_1-w_6\bar{y}_2\right]$$
$$=(p_1+1)\left[\bar{y}_2(1-w_1\bar{y}_2)-(w_2+w_3+w_4\bar{y}_1)\bar{y}_1\right]+(p_2+1)\left[w_3\bar{y}_1-w_6\bar{y}_2\right].$$

But $\bar{y}_2=\frac{w_3}{w_6}\bar{y}_1$, hence the last term of $\Psi(E_1)$ equal zero, hence $\Psi(E_1)=(p_1+1)\left[\bar{y}_2(1-w_1\bar{y}_2)-(w_2+w_3+w_4\bar{y}_1)\bar{y}_1\right]$Thus $\Psi(E_1)>0$ for any $p_1>0$ provided conditions (16a) and (16b) holds. Hence system (3) is uniformly persists if E_1 exist that is condition (5) and condition (16a) with (16b) hold.

5.The local Bifurcation.

In this section an investigation for dynamical behavior of system (3) under the effect of varying one parameter at each time is carried out. The occurrence of local bifurcation in the neighborhood of the equilibrium point of system (3) are studied in the following theorem.

Theorem 5: If the parameter w_3 passes through the value $\tilde{w}_3=\frac{w_2w_6}{1-w_6}, where\ w_6<1$, then the vanishing equilibrium point E_0 transforms into nonhyperpolic equilibrium point and if

$$w_1\neq\frac{w_4}{b_{11}^2}\ \ldots\ldots \qquad \ldots(17)$$

then system (3) possesses transcritical bifurcation, but no saddle-node bifurcation nor pitch-fork bifurcation can occur.

Proof: According to the jacobian matrix of system (3) at E_0 that is given by $J(E_0)$ it is easy to verify that as $w_3=\tilde{w}_3$, the $J(E_0,\tilde{w}_3)$ has the following eigenvalues**:**

$$\lambda_{01}=\frac{-w_2-(1-w_6)w_6}{1-w_6}<0,\qquad \lambda_{02}=0$$
$$\lambda_{03}=-w_7<0.$$

Let $v=(\theta_1,\theta_2,\theta_3)^T$ be the eigenvector of $J(E_0,\tilde{w}_3)$ corresponding to the eigenvalue of $\lambda_{02}=0$.Then it is easy to check that $v=(-\frac{1}{b_{11}}\theta_2,\theta_2,0)^T$,where $b_{11}=-(w_2+\tilde{w}_3)$ and θ_2 represents any nonzero real value. Also, let $y=(h_1,h_2,h_3)^T$ represents the eigenvector of $J^T(E_0,\tilde{w}_3)$ that corresponding to the eigenvalue $\lambda_{02}=0$. Straight forward calculation shows that

$y=(-\frac{b_{21}}{b_{11}}h_2,h_2,0)^T$, where $b_{21}=\tilde{w}_3$ and h_2 represents any nonzero real number.

Now, since $\frac{\partial F}{\partial w_3}=F_{w_3}(X,w_3)=[-y_1,y_1,0]^T$, *where* $X=(y_1,y_2,y_3)^T$ *and* $F=(f_1,f_2,f_3)^T$

With $f_i\ ;i=1,2,3$ represent the right hand side of system (3). Then we get $F_{w_3}(E_0,\tilde{w}_3)=(0,0,0)^T$ and the following is obtained:

$y^T[F_{w_3}(E_0,\tilde{w}_3)]=(-\frac{b_{21}}{b_{11}}h_2,h_2,0)(0,0,0)^T=0.$Thus system (3) at E_0 does not experience any saddle-node bifurcation in view of sotomayor theorem [11]. Also, since

$$y^T[DF_{w_3}(E_0,\tilde{w}_3)v]=(-\frac{b_{21}}{b_{11}}h_2,h_2,0)(-\theta_1,\theta_1,0)^T$$
$$=(\frac{b_{21}}{b_{11}}+1)h_2\theta_1\neq 0$$

here, $DF_{w_3}(E_0,\tilde{w}_3)=\frac{\partial}{\partial X}F_{w_3}(X,w_3)\mid_{X=E_0,w_3=\tilde{w}_3}$.
Moreover, we have

$y^T[D^2F_{w_3}(E_0,\tilde{w}_3)(v,v)]=\frac{-2b_{21}}{b_{11}}h_2\theta_2^2(\frac{w_4}{b_{11}^2}-w_1)\neq 0$ by condition (17).
Here,
$D^2F_{w_3}(E_0,\tilde{w}_3)=DJ(X,w_3)\mid_{X=E_0,w_3=\tilde{w}_3}$.
Then by sotomayor theorem, system (3) possesses a transcritical bifurcation but not pitch-fork bifurcation near E_0 where
$w_3=\tilde{w}_3$.
However, violate condition (17) gives that $y^T[D^2F_{w_3}(E_0,\tilde{w}_3)(v,v)]=0$, and hence further computation shows

$$y^T[D^3F_{w_3}(E_0,\tilde{w}_3)(v,v,v)]=(\frac{-b_{21}}{b_{11}}h_2,h_2,0)(0,0,-2w_5\theta_1^3)^T = 0.$$

Therefore according to Sotomayor theorem, there is no pitch-fork bifurcation.

Theorem 6: Assume that condition (5) holds and the parameter w_7 passes through the value $\overline{w}_7=\frac{\overline{y}_1}{1+w_5\overline{y}_1}$, *where* $\overline{y}_1$ given in (4a), then the equilibrium point $E_1=(\overline{y}_1,\overline{y}_2,0)$ transforms into nonhyperpolic equilibrium point and if $w_8\neq\frac{(\overline{\theta}_1^2+\overline{\theta}_1\overline{\theta}_3)}{2(1+w_5\overline{y}_1)^2\overline{\theta}_3^2}$ ....(18)
Where,

$$\overline{\theta}_1=-\frac{\overline{b}_{22}}{\overline{b}_{21}}\overline{\theta}_2 \quad and$$

$$\overline{\theta}_3=\frac{\overline{b}_{11}\overline{b}_{22}-\overline{b}_{12}\overline{b}_{21}}{\overline{b}_{13}\overline{b}_{21}}\theta_2$$

then system (3) possesses transcritical bifurcation, but no saddle-node bifurcation nor pitch-fork bifurcation can occur under condition (18), violate condition (18) gives pitch-fork bifurcation.

Proof: According to the jacobian matrix of system (3) at $E_1=(\overline{y}_1,\overline{y}_2,0)$ that is given by $J(E_1)$ it is easy to verify that as $w_7=\overline{w}_7$, then the eigenvalues of $J(E_1,\overline{w}_7)$ satisfy the following relations**:**

$$\lambda_{11}+\lambda_{12}=-(w_2+w_3)-2w_4\overline{y}_1-w_6<0$$
$$\lambda_{11}\cdot\lambda_{12}=-M_1+M_2\overline{y}_1 \;,\; where$$
$$M_1=w_3-(w_2+w_3)w_6,\; and$$
$$M_2=2(w_4w_6+\frac{w_1w_3^2}{w_6}),$$
$$\lambda_{13}=0.$$

Let $\overline{v}=(\overline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3)^T$ be the eigenvector of $J(E_1,\overline{w}_7)$ corresponding to the eigenvalue of $\lambda_{13}=0$. Then it is easy to check that $\overline{v}=(-\frac{\overline{b}_{22}}{\overline{b}_{21}}\overline{\theta}_2,\overline{\theta}_2,(\frac{\overline{b}_{11}\overline{b}_{22}-\overline{b}_{12}\overline{b}_{21}}{\overline{b}_{13}\overline{b}_{21}})\overline{\theta}_2)^T$,where θ_2 represents any nonzero real value. Also, let $\overline{y}=(\overline{h}_1,\overline{h}_2,\overline{h}_3)^T$ represents the eigenvector of $J^T(E_1,\overline{w}_7)$ that

corresponding to the eigenvalue $\lambda_{13}=0$. Straight forward calculation shows that $\overline{y}=(0,0,\overline{h}_3)^T$, where h_3 represents any nonzero real number.
Now, since $\frac{\partial F}{\partial w_3}=F_{w_3}(X,w_3)=[0,0,-y_3]^T$, *where*
$X=(y_1,y_2,y_3)^T$ *and* $F=(f_1,f_2,f_3)^T$
With $f_i\;;i=1,2,3$ represent the right hand side of system (3). Then we get $F_{w_7}(E_1,\overline{w}_7)=(0,0,0)^T$ and the following is obtained:
$\overline{y}^T[F_{w_7}(E_1,\overline{w}_7)]=(0,0,\overline{h}_3)(0,0,0)^T=0$.
Thus system (3) at E_1 does not experience any saddle-node bifurcation in view of sotomayor theorem . Also, since

$$\overline{y}^T[DF_{w_7}(E_1,\overline{w}_7)\overline{v}]=(0,0,\overline{h}_3)(\;0,0,-\overline{\theta}_3)^T = -\overline{h}_3\overline{\theta}_{31}\neq 0$$

here, $DF_{w_7}(E_1,\overline{w}_7)=\frac{\partial}{\partial X}F_{w_7}(X,w_7)\mid_{X=E_1,w_7=\overline{w}_7}$.
Moreover, we have
$\overline{y}^T[D^2F_{w_7}(E_1,\overline{w}_7)(\overline{v},\overline{v})]=(\frac{\overline{\theta}_1^2+\overline{\theta}_1\overline{\theta}_3}{(1+w_5\overline{y}_1)^2}-2w_8\overline{\theta}_3^2)\overline{h}_3\neq 0$ by condition (18).
Here,
$D^2F_{w_7}(E_1,\overline{w}_7)=DJ(X,w_7)\mid_{X=E_1,w_7=\overline{w}_7}$.

Then by sotomayor theorem, system (3) possesses a transcritical bifurcation but not pitch-fork bifurcation near E_1 where $w_7 = \overline{w}_7$.

However, violate condition (18) gives that $\bar{y}^T[D^2F_{w_7}(E_1,\overline{w}_7)(\bar{v},\bar{v})]=0$, and hence further computation shows

$$\bar{y}^T[D^3F_{w_7}(E_1,\overline{w}_7)(\bar{v},\bar{v},\bar{v})] = \frac{-2w_5(\bar{\theta}_1^3+\bar{\theta}_1^2\bar{\theta}_3)\bar{h}_3}{(1+w_5y_1)^3} \neq 0.$$

Therefore system (3) possesses a pitch-fork bifurcation near E_1 where $w_7 = \overline{w}_7$.

6.Hopf bifurcation.

Finally, in order to investigate the Hopf bifurcation of the model in system (3), we will follow the Liu approach [12] as shown in the following theorem:

Theorem 7 : Assume that the coexistence equilibrium point of system (3) exist and let in addition to conditions (15a)-(15e), the following conditions hold:

$$1-L_1 > w_3\left[\frac{(N_1+N_0N_2)}{N_0^2}-N_5\right] \quad \text{.....................(19)}$$

$$2-L_1-w_3\left[\frac{(N_1+N_0N_2)}{N_0^2}-N_5\right] > \frac{1}{2w_3\hat{y}_2\left(-\frac{(N_1+N_0N_2}{N_0^2}+N_5\right)}$$

Where

$$L_1 = \frac{N_1+N_0N_2}{N_0^2}\left(\frac{w_6}{N_0^2}(N_1+N_0N_2)-\frac{1}{N_0^3}(\hat{y}_1^2-N_1N_2)+w_6^2\right).$$

Then a simple Hopf bifurcation of the model in system (3) occurs at

$$w_1 = w_1^* = \frac{1}{2w_3\hat{y}_2\left(N_5-\frac{(N_1+N_0N_2)}{N_0^2}\right)}\cdot\left(L_1-w_3\left(\frac{(N_1+N_0N_2}{N_0^2}-N_5\right)\right).$$

Proof: According to the Liu approach a simple Hopf bifurcation occurs if and only if $A_1(\mu_*)>0,\ A_3(\mu_*),\ \Delta(\mu_*)=0$ *and* $\frac{\partial\Delta}{\partial\mu}\big|_{\mu=\mu_*}\neq 0,$ *where* μ_* *is a critical value of* the key parameter and A_i *for* $i=1,3,$ *and* Δ *are given in* Equations (13b),(13d) and $(14d')$. Note that it is clear that $N_0 = 1+w_5\hat{y}_1 > 0$

$N_1>0$, $N_2>0$ *and* $\hat{y}_1^2 < N_1N_2$ under conditions (14a)-(14c) and (14e) and $N_5 \leq 0$ under condition (15d) and hence w_1^* is positive under the above conditions with condition given in (19).

Now, by substituting of the value of w_1^* in these equations we obtain

$$A_1(w_1^*) = \frac{1}{N_0^2}\left[N_1+N_0N_2+w_6N_0^2\right],$$

which is positive due to conditions (14a)-(14c).

$$A_3(w_1^*) = \frac{-w_6}{N_0^3}\left[\hat{y}_1^2-N_1N_2\right]-\frac{w_3N_2N_3}{N_0},$$ where

$N_3 = 1-2w_1\hat{y}_2 < 0$ under condition (14b), clearly, $A_3(w_1^*)>0$ under condition (14b) and (14c) with condition (19).Moreover, rewrite equation $(14d')$ gives that

$$\Delta = \frac{(N_1+N_0N_2)}{N_0^2}\left[\frac{w_6}{N_0^2}(N_1+N_0N_2) - \frac{1}{N_0^3}(\hat{y}_1^2-N_1N_2)+w_6^2\right]-w_3\left[\frac{(N_1+N_0N_2)}{N_0^2}-N_5\right] -2w_1w_3\hat{y}_2\left[\frac{-(N_1+N_0N_2)}{N_0^2}+N_5\right].$$

Hence, it is easy to verify that $\Delta(w_1^*)=0$.

Finally, since

$$\frac{\partial\Delta}{\partial w_1}\bigg|_{w_1=w_1^*} = -2w_3\hat{y}_2\left[\frac{-(N_1+N_0N_2)}{N_0^2}+\left(\frac{N_2}{N_0}-W_6\right)\right]$$

Thus, a simple Hopf bifurcation occurs in system (3) at $w_1 = w_1^*$.

7. Numerical Simulation:-

In this section, the global dynamics of system (3) is further investigated by solving it numerically. The objective is to verify our previous analytical results and understand the effect of varying the parameters values.

For the following set of parameters value:

$$w_1=0.02, w_2=0.1, w_3=0.1, w_4=0.2, w_5=0.1, w_6=0.2, w_7=0.1, w_8=0.1 \quad (20)$$

The trajectories of system (3) approach asymptotically to global stable point in the $Int.R_+^3$, as shown in Fig. (1) and (4) Clearly, for this set of data, the numerical result confirms our analytical result. Moreover it is observed that, increasing the conversion rate from immature prey to mature prey further, i.e $w_3 > 0.01$, system (3) still have a globally stable point in $Int.R_+^3$.

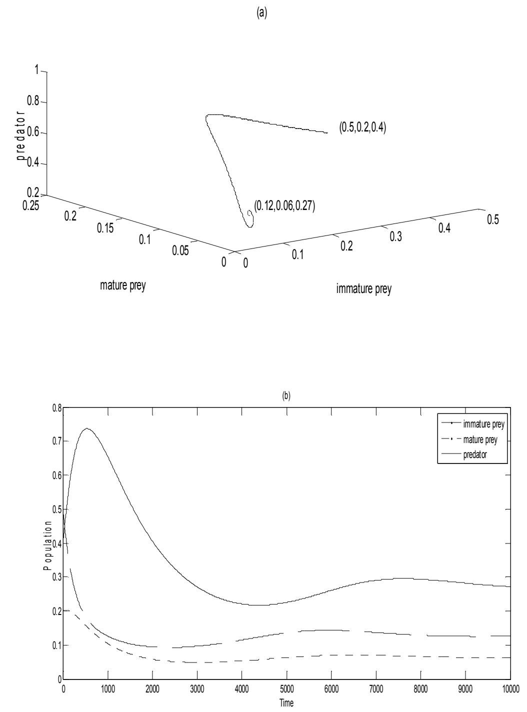

Figure 1-(a)Global stable point in $Int.R_+^3$ (b)Time series starting from (0.5,0.2,0.4) for the parameter set (20) at which the conditions for persistence hold, the trajectory of system (3) approaches to the positive point E_2,that is the system is persist.

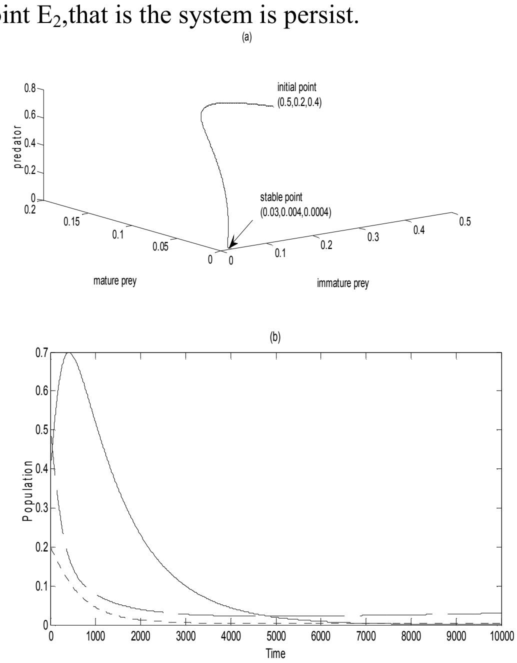

Figure 2- (a) $E_1 = (\bar{y}_1, \bar{y}_2, 0)$ is global stable point (b)Time series of starting point (0.5,0.2,0.4) for the parameter set (20) with w_3=0.03,condition(11b) holds and the trajectory of system(3) approaches asymptotically to E_1,so system(3) not persist.

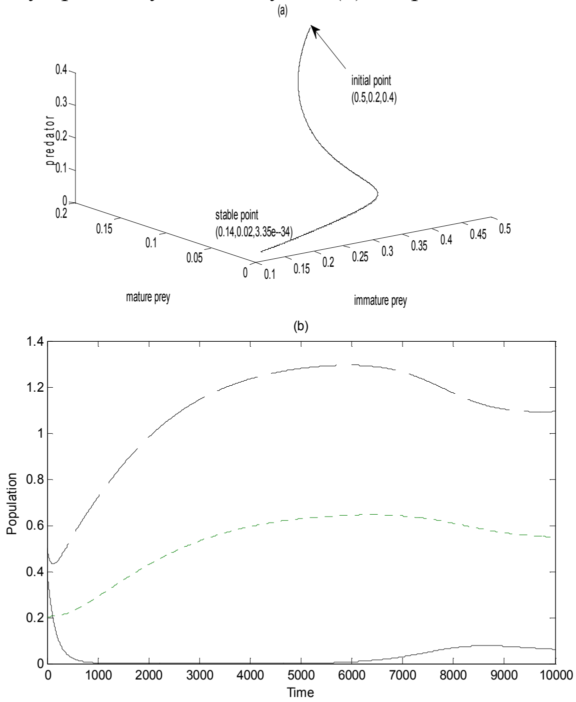

Figure 3- (a) Global stable point in $Int.R_+^3$ (b)Time series of starting point (0.5,0.2,0.4) for the parameter set (20) with w_7=1, the trajectory of system (3) approaches asymptotically to E_2,so system(3) persist.

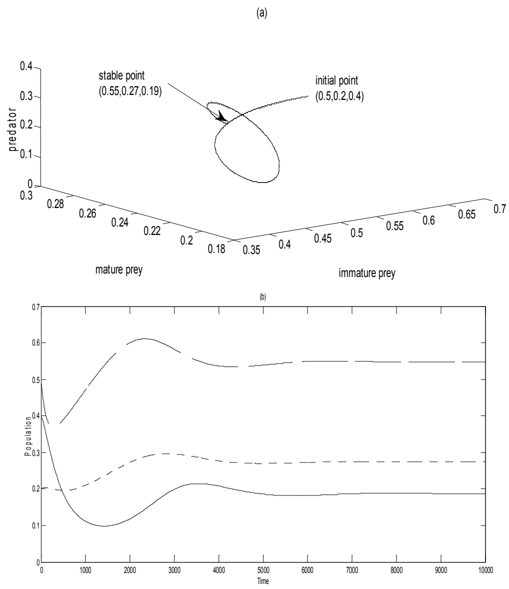

Figure 4- (a) Global stable point in $Int.R_+^3$ (b)Time series of starting point (0.5,0.2,0.4) for the parameter set (20) with w_7=0.5, the trajectory of system(3) approaches asymptotically to E_2,so system(3) persist.

According to the above, the effect of the other parameters on the dynamics of system(3) is also studied in case of varying the parameters and obtained results are summarized in the following tables.

Table 1: Numerical behaviors and persistence of system (3) as varying in some parameters keeping the rest of parameters fixed as in eq. (20).

Parameters varied in system(3)	Numerical behavior of system (3)	Persistence of system (3)
$0.1 \le w_2 \le 0.36$ $0.37 \le w_2 \le 0.47$ $w_2 \ge 0.48$	Approaches to stable point in Int. R^3_+ Approaches to stable point $E_1=(\bar{y}_1,\bar{y}_2,0)$ Approaches to stable point $E_0=(0,0,0)$	Persists Not persist Not persist
$w_3 < 0.03$ $w_3 = 0.03$ $w_3 \ge 0.04$	Approaches to stable point $E_0=(0,0,0)$ Approaches to stable point $E_1=(\bar{y}_1,\bar{y}_2,0)$ Approaches to stable point in Int. R^3_+	Not persist Not persist Persists
$0.01 \le w_4 \le 5.4$ $w_4 \ge 5.5$	Approaches to stable point in Int. R^3_+ Approaches to stable point $E_1=(\bar{y}_1,\bar{y}_2,0)$	Persists Not persist
$w_5 \ge 0.01$	Approaches to stable point in Int. R^3_+	Persists
$0.1 \le w_6 \le 0.3$ $w_6 = 0.4$ $w_6 > 0.5$	Approaches to stable point in Int. R^3_+ Approaches to stable point $E_1=(\bar{y}_1,\bar{y}_2,0)$ Approaches to stable point $E_0=(0,0,0)$	Persists Not persist Not persist
$0.01 \le w_7 \le 0.8$ $w_7 \ge 0.9$	Approaches to stable point in Int. R^3_+. Approaches to stable point $E_1=(\bar{y}_1,\bar{y}_2,0)$	Persists Not persist

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