The dynamics of two harmful phytoplankton and herbivorous zooplankton system

Raid Kamel Naji* and Asraa Amer Aaid
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.

Abstract
In this paper, a mathematical model consisting of the two harmful phytoplankton interacting with a herbivorous zooplankton is proposed and studied. The existence of all possible equilibrium points is carried out. The dynamical behaviors of the model system around biologically feasible equilibrium points are studied. Suitable Lyapunov functions are used to construct the basins of attractions of those points. Conditions for which the proposed model persists are established. The occurrence of local bifurcation and a Hopf bifurcation are investigated. Finally, to confirm our obtained analytical results and specify the vital parameters, numerical simulations are used for a hypothetical set of parameter values.

Keywords: phytoplankton-zooplankton, stability, local bifurcation, Hopf bifurcation.

ديناميكية نظام يتكون من اثنين من العوالق النباتية الضارة وعالق حيواني عاشب

رائد كامل ناجي و اسراء عامر عايد
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصة
في هذا البحث، تم اقتراح ودراسة نموذج رياضي يتكون من اثنين من العوالق النباتية الضارة و عالق حيواني عاشب. تم ايجاد جميع نقاط التوازن الممكنة ومن ثم دراسة السلوك الديناميكي حولها. استخدمنا دوال ليابانوف مناسبة لايجاد احواض التجاذب لتلك النقاط. شروط الاصرار لهذا النموذج وجدت. بحثنا امكانية حدوث التفرع المحلي وتفرع هوبف. وأخيرا، لتأكيد النتائج التحليلية التي تم الحصول عليها وتحديد المعلمات الاساسية، استخدمنا المحاكاة العددية لمجموعة من قيم المعلمات الافتراضية.

1. Introduction

Plankton is the basis of the entire aquatic food chain. Phytoplankton, in particular, occupies the first trophic level. Plankton performs services for the Earth: it serves as food for marine life, gives off oxygen and also absorbs half of the carbon dioxide from the Earth's atmosphere. The dynamics of a rapid (or massive) increase or decrease of plankton populations is an important subject in marine plankton ecology and generally termed as a 'bloom'. Harmful algal blooms (HABs) have adverse effects on human health, fishery, tourism, and the environment. In recent years, considerable scientific attention has been given to HABs, see for example [1-9]. Toxic substances released by harmful plankton play an important role in this context. Recent studies reveal that some times bloom of certain harmful species may lead to the release of both toxins and allelopathic substances [10-12]. Allelopathic substances tend to be directly targeted and may physiologically impair, stun,

Email: rknaji@scbaghdad.edu.iq

repel, induce avoidance reactions, and kill grazers. Toxin-producing plankton (TPP) release toxic chemicals in the water and reduce the grazing pressure of zooplankton. As a result, TPP may act as a biological control for the termination of planktonic blooms see [8,9,13-15].

Consequently in this paper, we will give special emphasis to the fact that the occurrence of toxin producing phytoplankton may not always be harmful but may help to maintain the stable equilibrium in trophodynamics through the coexistence of all the species. A mathematical model consisting of two harmful phytoplankton interacting with herbivorous zooplankton is proposed and studied. Two types of distributions for the released toxic substance by toxic / harmful phytoplankton species, which reduces the growth of zooplankton, are considered.

2. Mathematical model formulation

Consider the simple phytoplankton-zooplankton system with Holling type-II functional response which can be written as:

$$\begin{aligned}\frac{dP_1}{dT} &= r_1P_1\left(1-\frac{P_1}{K}\right)-\frac{mP_1Z}{\gamma+P_1}\\ \frac{dZ}{dT} &= \frac{m_1P_1Z}{\gamma+P_1}-\mu Z\end{aligned} \tag{1}$$

Here $P_1(T)$ and $Z(T)$ represent the densities of phytoplankton and zooplankton at time T respectively. While the parameters r_1, K, m, γ, m_1 and μ are assumed to be positive parameters and can be described as the following: r_1 represents the intrinsic growth rate of phytoplankton; K is the carrying capacity; m represents the maximum attack rate of zooplankton to the phytoplankton P_1; γ is the half saturation constant; m_1 represents the zooplankton conversion rate from phytoplankton P_1; μ is the natural death rate of zooplankton.

Assume that, the phytoplankton P_1 produces a toxin, as a defensive strategy against the predation from zooplankton, which effect negatively on the growth of the zooplankton. Therefore, the above system can be reformulated as:

$$\begin{aligned}\frac{dP_1}{dT} &= r_1P_1\left(1-\frac{P_1}{K}\right)-\frac{mP_1Z}{\gamma+P_1}\\ \frac{dZ}{dT} &= \frac{m_1P_1Z}{\gamma+P_1}-\mu Z-\theta_1 f(P_1)Z\end{aligned} \tag{2}$$

here $\theta_1 > 0$ represents the liberation rate of toxic substance by the harmful phytoplankton P_1; While $f(P_1)$ represents the distribution of toxic substance which is assumed to be follows either Holling type-I form (called case 1) or Holling type-II form (called case 2) that means:

$$f(P_1)=\begin{cases} aP_1 & \text{for case 1}\\ \frac{a_1P_1}{\gamma_1+P_1} & \text{for case 2}\end{cases}$$

(3)

here $a > 0$ and $a_1 > 0$ represent the maximum zooplankton ingestion rates for the toxic substance produced by phytoplankton P_1, while $\gamma_1 > 0$ is the half saturation constant of the zooplankton by the toxic substance.

Now, if we imposed the following additional assumptions on system (2):

1- There exists another harmful phytoplankton, denoted by $P_2(T)$, within the environment.

2- It is assumed that, the second phytoplankton P_2 growth logistically with intrinsic growth rate $r_2 > 0$ and carrying capacity $L > 0$, while there is a competition interaction between P_1 and P_2 for light and space with competition rates $\alpha > 0$ and $\beta > 0$.

3- The second phytoplankton produces a toxic substance that effects on the zooplankton too and the distribution of this toxic follows:

$$g(P_2)=\begin{cases} dP_2 & \text{for case 1}\\ \frac{d_1P_2}{\gamma_2+P_2} & \text{for case 2}\end{cases} \tag{4}$$

here d, d_1 and γ_2 have the same meaning as those in $f(P_1)$.

4- The zooplankton consumes the food from phytoplankton P_1 and phytoplankton P_2 as well as according to Holling type-II.

Therefore, the above two species system (2) can be extended to three species system and reformulated as:

$$\begin{aligned}\frac{dP_1}{dT} &= r_1P_1\left(1-\frac{P_1}{K}\right)-\alpha P_1P_2-\frac{mP_1Z}{\gamma+P_1+cP_2}\\ \frac{dP_2}{dT} &= r_2P_2\left(1-\frac{P_2}{L}\right)-\beta P_1P_2-\frac{nP_2Z}{\gamma+P_1+cP_2}\\ \frac{dZ}{dT} &= \left(\frac{m_1P_1}{\gamma+P_1+cP_2}+\frac{n_1P_2}{\gamma+P_1+cP_2}\right)Z\\ &\quad -\mu Z-\theta_1 f(P_1)Z-\theta_2 g(P_2)Z\end{aligned} \tag{5}$$

Clearly, the positive parameters n, n_1 and c can be described as follows: n represents the maximum attack rate of the zooplankton to the second phytoplankton P_2; n_1 represents the zooplankton conversion rate from phytoplankton and c the preference rate between P_1 and P_2 respectively. While the parameter $\theta_2 > 0$

represents the liberation rate of toxic substance by the harmful phytoplankton P_2.

Note that system (5) has 17 parameters for case 1 and 19 parameters for case 2, which makes the analysis difficult. Therefore, to reduce the number of parameters and then simplifying our system the following dimensionless variables are used

$$t = r_1 T, x = \frac{P_1}{K}, y = \frac{\alpha P_2}{r_1} \text{ and } z = \frac{mZ}{r_1 K}$$

Therefore, substituting these new variables in system (5) and then simplifying the resulting terms. We obtain the following dimensionless system:

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x) - xy - \frac{xz}{w_1 + x + w_2 y} \\
\frac{dy}{dt} &= w_3 y(1 - w_4 y) - w_5 xy - \frac{w_6 yz}{w_1 + x + w_2 y} \\
\frac{dz}{dt} &= \frac{w_7 xz}{w_1 + x + w_2 y} + \frac{w_8 yz}{w_1 + x + w_2 y} \\
&\quad - w_9 z - w_{10} f(x) z - w_{11} g(y) z
\end{aligned} \tag{6}$$

where:

$$f(x) = \begin{cases} w_{12} x & \text{for case 1} \\ \frac{a_1 x}{\overline{w}_{12} + x} & \text{for case 2} \end{cases} \tag{7a}$$

$$g(y) = \begin{cases} w_{13} y & \text{for case 1} \\ \frac{d_1 y}{\overline{w}_{13} + y} & \text{for case 2} \end{cases} \tag{7b}$$

with $w_1 = \frac{\gamma}{K}$, $w_2 = \frac{c r_1}{\alpha K}$, $w_3 = \frac{r_2}{r_1}$, $w_4 = \frac{r_1}{\alpha L}$, $w_5 = \frac{\beta K}{r_1}$, $w_6 = \frac{n}{m}$, $w_7 = \frac{m_1}{r_1}$, $w_8 = \frac{n_1}{\alpha K}$, $w_9 = \frac{\mu}{r_1}$, $w_{10} = \frac{\theta_1}{r_1}$, $w_{11} = \frac{\theta_2}{r_1}$, $w_{12} = aK$, $w_{13} = \frac{d r_1}{\alpha}$, $\overline{w}_{12} = \frac{\gamma_1}{K}$, $\overline{w}_{13} = \frac{\alpha \gamma_2}{r_1}$ represent the dimensionless parameters. Clearly, system (6) contains 13 parameters in case 1 and 15 parameters for case 2, which may make the analysis of system (6) easier. Further, the initial condition for system (6) may be taken as any point in the region $R_+^3 = \{(x, y, z): \ x \ge 0, y \ge 0, z \ge 0\}$.

Obviously, the interaction functions in the right hand side of system (6) are continuously differentiable functions on R_+^3, hence they are Lipschitizian. Therefore the solution of system (6) exists and is unique. Further, all the solutions of system (6) with non-negative initial condition are uniformly bounded as shown in the following theorem.

Theorem 1. All the solutions of the system (6), which initiate in R_+^3 are uniformly bounded.

Proof. Let $(x(t), y(t), z(t))$ be any solution of the system (6). Since

$$\frac{dx}{dt} \le x(1-x), \ \frac{dy}{dt} \le w_3 y(1 - w_4 y)$$

Thus by solving these differential inequalities we get $x(t) \le 1, \forall t > 0$ and $y(t) \le \frac{1}{w_4}, \forall t > 0$. Now, consider the function $W(x, y, z) = w_7 x + \frac{w_8}{w_6} y + z$, then the time derivative of $W(\cdot)$ along the solution of the system (6) is:

$$\frac{dW}{dt} + w_9 W \le M$$

where $M = w_7(1 + w_9) + \frac{w_8}{w_4 w_6}(w_3 + w_9)$. By comparing the above differential inequality with the associated linear differential equation, we obtain:

$$0 < W \le \frac{M(1 - e^{-w_9 t})}{w_9} + w_0 e^{-w_9 t}$$

Therefore $0 < W \le \frac{M}{w_9}$, as $t \to \infty$. Hence, all the solutions of system (6) are uniformly bounded, and then the proof is complete. ■

According to the above theorem system (6) is dissipative system.

3. Existence of equilibrium points and stability Analysis

The system (6) have at most seven non-negative equilibrium points, three of them namely $E_0 = (0,0,0)$, $E_x = (1,0,0)$, $E_y = \left(0, \frac{1}{w_4}, 0\right)$ always exist. While the existence of other equilibrium points are shown in the following:

The zooplankton free equilibrium point $E_{xy} = (\hat{x}, \hat{y}, 0)$, where

$$\hat{x} = \frac{w_3(w_4 - 1)}{w_3 w_4 - w_5}, \ \hat{y} = \frac{w_3 - w_5}{w_3 w_4 - w_5} \tag{8}$$

exists uniquely in the $Int.R_+^2$ of $xy-$plane provided that:

$$w_4 > 1 \text{ and } w_3 > w_5 \tag{9a}$$

Or,

$$w_4 < 1 \text{ and } w_3 < w_5 \tag{9b}$$

The second phytoplankton free equilibrium point $E_{xz} = (\overline{x}, 0, \overline{z})$ exists in $Int.R_+^2$ of xz - plane, where

$$\overline{z} = (1 - \overline{x})(w_1 + \overline{x}) \tag{10}$$

while $\overline{x}$ in **case 1**, represents the positive root to the following equation:

$$b_1 x^2 + b_2 x + b_3 = 0 \tag{11}$$

where $b_1 = w_{10} w_{12} > 0$, $b_3 = w_1 w_9 > 0$ and $b_2 = w_9 - w_7 + w_1 w_{10} w_{12}$. So by using Descartes rule of signs, Eq. (11) has either no positive root and hence there is no equilibrium point or two positive roots given by:

$$\overline{x}_1, \overline{x}_2 = \frac{-b_2}{2b_1} \pm \frac{\sqrt{b_2^2 - 4b_1 b_3}}{2b_1} \tag{12a}$$

Clearly $\bar{x}_1$ and $\bar{x}_2$ are positive provided that

$$b_2 < 0 \Rightarrow w_9 + w_1 w_{10} w_{12} < w_7 \quad (12b)$$

$$b_2^2 > 4b_1 b_3 \quad (12c)$$

and then, by substituting $\bar{x}_i,\ i = 1,2$ in Eq. (10), there exist two second phytoplankton free equilibrium points in the $Int.R_+^2$ of xz − plane namely $E_{x_1 z_1}$ and $E_{x_2 z_2}$, provided that

$$\bar{x}_i < 1 \text{ for } i = 1,2 . \quad (13)$$

Now for **case 2**, $\bar{x}$ represents the positive root to the following equation:

$$b_4 x^2 + b_5 x + b_6 = 0 \quad (14)$$

here $b_4 = w_7 - w_9 - a_1 w_{10}$, $b_6 = -w_1 w_9 \overline{w}_{12} < 0$ and $b_5 = w_7 \overline{w}_{12} - w_1 w_9 - w_9 \overline{w}_{12} - a_1 w_1 w_{10}$. So by using Descartes rule of signs, Eq. (14) has a positive root given by:

$$\bar{x} = \frac{-b_5}{2b_4} + \frac{\sqrt{b_5^2 - 4b_4 b_6}}{2b_4} \quad (15a)$$

provided that the following condition holds

$$b_4 > 0 \Rightarrow w_7 > w_9 + a_1 w_{10} \quad (15b)$$

Therefore, by substituting $\bar{x}$ in Eq. (10), system (6) has a unique second phytoplankton free equilibrium point in the $Int.R_+^2$ of xz − plane denoted by E_{xz}, provided that

$$\bar{x} < 1 \quad (16)$$

The first phytoplankton free equilibrium point $E_{yz} = (0, \tilde{y}, \tilde{z})$ exists in $Int.R_+^2$ of yz - plane, where

$$\tilde{z} = \left(\frac{w_1 + w_2 \tilde{y}}{w_6}\right)[w_3(1 - w_4 \tilde{y})] \quad (17)$$

while $\tilde{y}$ in **case 1**, represents the positive root to the following equation:

$$b_7 y^2 + b_8 y + b_9 = 0 \quad (18)$$

here $b_7 = w_2 w_{11} w_{13} > 0$, $b_9 = w_1 w_9 > 0$ and $b_8 = w_2 w_9 - w_8 + w_1 w_{11} w_{13}$. So by using Descartes rule of signs, Eq. (18) has either no positive root and hence there is no equilibrium point or two positive roots given by:

$$\tilde{y}_1, \tilde{y}_2 = \frac{-b_8}{2b_7} \pm \frac{\sqrt{b_8^2 - 4b_7 b_9}}{2b_7} \quad (19a)$$

Clearly $\tilde{y}_1$ and $\tilde{y}_2$ are positive provided that

$$b_8 < 0 \Rightarrow w_2 w_9 + w_1 w_{11} w_{13} < w_8 \quad (19b)$$

$$b_8^2 > 4b_7 b_9 \quad (19c)$$

and then, by substituting $\tilde{y}_i,\ i = 1,2$ in Eq. (17), there exist two first phytoplankton free equilibrium points in the $Int.R_+^2$ of yz − plane namely $E_{y_1 z_1}$ and $E_{y_2 z_2}$, provided that

$$\tilde{y}_i < \frac{1}{w_4} \quad for\, i = 1,2 \quad (20)$$

Now for **case 2**, $\tilde{y}$ represents the positive root to the following equation:

$$b_{10} y^2 + b_{11} y + b_{12} = 0 \quad (21)$$

here $b_{10} = w_8 - w_2 w_9 - d_1 w_2 w_{11}$, $b_{12} = -w_1 w_9 \overline{w}_{13} < 0$ and $b_{11} = w_8 \overline{w}_{13} - w_1 w_9 - w_2 w_9 \overline{w}_{13} - d_1 w_1 w_{11}$. So by using Descartes rule of signs, Eq. (21) has a positive root given by:

$$\tilde{y} = \frac{-b_{11}}{2b_{10}} + \frac{\sqrt{b_{11}^2 - 4b_{10} b_{12}}}{2b_{10}} \quad (22a)$$

provided that the following condition holds

$$b_{10} > 0 \Rightarrow w_8 > w_2 w_9 + d_1 w_2 w_{11} \quad (22b)$$

Therefore, by substituting $\tilde{y}$ in Eq. (17), system (6) has a unique first phytoplankton free equilibrium point in the $Int.R_+^2$ of yz − plane given by $E_{yz} = (0, \tilde{y}, \tilde{z})$, provided that

$$\tilde{y} < \frac{1}{w_4} \quad (23)$$

Finally the coexistence equilibrium point $E_{xyz} = (x^*, y^*, z^*)$ exists in $Int.R_+^3$, where

$$y^* = A + Bx^* \quad (24)$$

$$z^* = \frac{b^*\left[w_3 - \left(w_3 w_4 A + (w_3 w_4 B + w_5) x^*\right)\right]}{w_6} \quad (25)$$

with $A = \frac{w_3 - w_6}{w_3 w_4 - w_6}$, $B = \frac{w_6 - w_5}{w_3 w_4 - w_6}$ and

$$b^* = w_1 + x^* + w_2 y^* \quad (26)$$

While, x^* represents the positive root of each of the following two equations for **case 1** and **case 2** respectively.

$$S_1 x^2 + S_2 x + S_3 = 0 \quad (27)$$

$$Q_1 x^3 + Q_2 x^2 + Q_3 x + Q_4 = 0 \quad (28)$$

where:

$$S_1 = (1 + w_2 B)(w_{10} w_{12} + w_{11} w_{13} B)$$

$$S_2 = (1 + w_2 B)(w_9 + w_{11} w_{13} A) + (w_1 + w_2 A)(w_{10} w_{12} + w_{11} w_{13} B) - w_7 - w_8 B$$

$$S_3 = (w_1 + w_2 A)(w_9 + w_{11} w_{13} A) - w_8 A$$

$$Q_1 = B[w_7 + w_8 B - (w_9 + a_1 w_{10} + d_1 w_{11})(1 + w_2 B)]$$

$$Q_2 = w_7 \overline{w}_{12} B + (\overline{w}_{13} + A)(w_7 + w_8 B - w_9 - w_2 w_9 B) + (\overline{w}_{12} B + A)(w_8 B - w_2 w_9 B - d_1 w_2 w_{11} B) - (w_1 + \overline{w}_{12})(w_9 + d_1 w_{11}) B - (1 + w_2 B)(a_1 w_{10} \overline{w}_{13} + a_1 w_{10} A + d_1 w_{11} A) - a_1 w_{10} B(w_1 + w_2 A)$$

$$Q_3 = w_8 A^2 + (\overline{w}_{13} + A)(w_7\overline{w}_{12} + w_8\overline{w}_{12}B - w_2 w_9 A) + (\overline{w}_{12}B + \overline{w}_{13})(w_8 A - w_1 w_9) - w_9 A(w_1 + w_2\overline{w}_{12}B) - (1 + w_2 B)(w_9\overline{w}_{12}\overline{w}_{13} + w_9\overline{w}_{12}A + d_1 w_{11}\overline{w}_{12}A) - (w_1 + w_2 A)(a_1 w_{10}\overline{w}_{13} + a_1 w_{10}A + d_1 w_{11}\overline{w}_{12}B + d_1 w_{11}A)$$

$$Q_4 = (\overline{w}_{13} + A)(w_8\overline{w}_{12}A - w_1 w_9\overline{w}_{12} - w_2 w_9\overline{w}_{12}A) - d_1 w_{11}\overline{w}_{12}A(w_1 + w_2 A)$$

Obviously, equation (27) has a unique positive root say x^* provided that one set of the following sets of conditions hold.

$$S_1 > 0 \ and \ S_3 < 0 \qquad (29a)$$

$$S_1 < 0 \ and \ S_3 > 0 \qquad (29b)$$

while equation (28) has a unique positive root say x^* provided that one set of the following sets of conditions hold:

$$Q_1 > 0, Q_2 > 0 \text{ and } Q_4 < 0 \ldots\ldots\ldots\ldots(30a)$$

$$Q_1 > 0, Q_3 < 0 \text{ and } Q_4 < 0 \ldots\ldots\ldots\ldots(30b)$$

$$Q_1 < 0, Q_2 < 0 \text{ and } Q_4 > 0 \ldots\ldots\ldots\ldots(30c)$$

$$Q_1 < 0, Q_3 > 0 \text{ and } Q_4 > 0 \ldots\ldots\ \ldots\ldots(30d)$$

Consequently, E_{xyz} exists uniquely in the $Int.R^3_+$ if and only if in addition to condition (29) in case 1 and condition (30) in case 2 one set of the following sets of conditions hold:

$$\left.\begin{array}{l} w_5 < w_6 < \min.\{w_3, w_3 w_4\}, \text{ or} \\ \max.\{w_3, w_3 w_4\} < w_6 < w_5 \end{array}\right\} \qquad (31a)$$

$$x^* < \hat{x} \qquad (31b)$$

clearly (31a) guarantees that $y^* > 0$ while (31b) guarantees that $z^* > 0$. The other set is:

$$\left.\begin{array}{l} w_6 < \min.\{w_3, w_3 w_4, w_5\}, \text{ or} \\ \max.\{w_3, w_3 w_4, w_5\} < w_6 \end{array}\right\} \qquad (32a)$$

$$x^* < \frac{w_3 - w_6}{w_5 - w_6} \qquad (32b)$$

$$\left.\begin{array}{l} x^* < \hat{x}, \frac{w_6(w_3 w_4 - w_5)}{w_3 w_4 - w_6} > 0, \text{ or} \\ \frac{w_6(w_3 w_4 - w_5)}{w_3 w_4 - w_6} < 0, \ w_4 > 1, \text{or} \\ x^* > \hat{x}, \ \frac{w_6(w_3 w_4 - w_5)}{w_3 w_4 - w_6} < 0 \end{array}\right\} \qquad (32c)$$

clearly (32a)-(32b) guarantee that $y^* > 0$ while (32c) guarantees that $z^* > 0$. Finally we have that:

$$\left.\begin{array}{l} \max.\{w_3, w_5\} < w_6 < w_3 w_4, \text{or} \\ w_3 w_4 < w_6 < \min.\{w_3, w_5\} \end{array}\right\} \qquad (33a)$$

$$x^* > \frac{w_6 - w_3}{w_6 - w_5} \qquad (33b)$$

$$x^* < \hat{x} \qquad (33c)$$

here (33a)-(33b) guarantee that $y^* > 0$, however (33c) guarantees that $z^* > 0$.

In the following, the local dynamical behavior of the system (6) around each of the above equilibrium points is investigated. First the Jacobian matrix of system (6) at each of these points is determined and then the eigenvalues for the resulting matrix are computed, finally the obtained results are summarized in the following:

The Jacobian matrix of system (6) at the equilibrium point $E_0 = (0,0,0)$ can be written as $J_0 = J(E_0) = [c_{ij}]_{3\times 3}; i, j = 1,2,3$, where $c_{11} = 1$, $c_{22} = w_3$, $c_{33} = -w_9$ and zero otherwise. Then the eigenvalues of J_0 are:

$$\lambda_{01} = 1 > 0, \ \lambda_{02} = w_3 > 0, \ \lambda_{03} = -w_9 < 0$$

Therefore, the equilibrium point E_0 is a saddle point.

The Jacobian matrix of system (6) at the equilibrium point $E_x = (1,0,0)$ can be written as $J_x = J(E_x) = [d_{ij}]_{3\times 3}; i, j = 1,2,3$, where $d_{11} = d_{12} = -1$, $d_{13} = \frac{-1}{w_1 + 1}$, $d_{22} = w_3 - w_5$, $d_{33} = \frac{w_7}{w_1 + 1} - w_9 - w_{10} f(1)$ and zero otherwise. Hence, the eigenvalues of J_x are:

$$\breve{\lambda}_1 = -1 < 0, \ \breve{\lambda}_2 = w_3 - w_5, \ \breve{\lambda}_3 = \frac{w_7}{w_1 + 1} - w_9 - w_{10} f(1)$$

where $f(1)$ is obtained from Eq. (7a) by substituting $x = 1$. Clearly, E_x is locally asymptotically stable in the R^3_+ if the following two conditions are satisfied

$$w_3 < w_5 \qquad (34a)$$

$$\frac{w_7}{w_1 + 1} < w_9 + w_{10} f(1) \qquad (34b)$$

However, E_x is a saddle point in the R^3_+ if at least one of the following two conditions are satisfied:

$$w_3 > w_5 \qquad (34c)$$

$$\frac{w_7}{w_1 + 1} > w_9 + w_{10} f(1) \qquad (34d)$$

Now, the Jacobian matrix of system (6) at the equilibrium point $E_y = \left(0, \frac{1}{w_4}, 0\right)$ can be written as $J_y = J(E_y) = [e_{ij}]_{3\times 3}; i, j = 1,2,3$, where $e_{11} = 1 - \frac{1}{w_4}$, $e_{21} = \frac{-w_5}{w_4}$, $e_{22} = -w_3$, $e_{23} = \frac{-w_6}{w_1 w_4 + w_2}$, $e_{33} = \frac{w_8}{w_1 w_4 + w_2} - w_9 - w_{11} g\left(\frac{1}{w_4}\right)$, and zero otherwise. The eigenvalues of J_y are:

$$\tilde{\lambda}_1 = 1 - \tfrac{1}{w_4},\ \tilde{\lambda}_2 = -w_3 < 0$$
$$\tilde{\lambda}_3 = \tfrac{w_8}{w_1 w_4 + w_2} - w_9 - w_{11} g\left(\tfrac{1}{w_4}\right)$$
where $g(1/w_4)$ is obtained from Eq. (7b) by substituting $y = (1/w_4)$. Hence, E_y is locally asymptotically stable in the R_+^3 if the following two conditions are satisfied

$$w_4 < 1 \quad (35a)$$
$$\tfrac{w_8}{w_1 w_4 + w_2} < w_9 + w_{11} g\left(\tfrac{1}{w_4}\right) \quad (35b)$$

While E_y is a saddle point in the R_+^3 if at least one of the following two conditions are satisfied

$$w_4 > 1 \quad (35c)$$
$$\tfrac{w_8}{w_1 w_4 + w_2} > w_9 + w_{11} g\left(\tfrac{1}{w_4}\right) \quad (35d)$$

The Jacobian matrix of system (6) at the zooplankton free equilibrium point $E_{xy} = (\hat{x}, \hat{y}, 0)$ in the $Int.R_+^2$ of $xy-$plane can be written as $J_{xy} = J(E_{xy}) = [f_{ij}]_{3\times3}; i, j = 1,2,3$, where

$$f_{11} = f_{12} = -\hat{x},\ f_{13} = \tfrac{-\hat{x}}{\hat{b}},\ f_{21} = -w_5 \hat{y},$$
$$f_{22} = -w_3 w_4 \hat{y},\ f_{23} = \tfrac{-w_6 \hat{y}}{\hat{b}},$$
$$f_{33} = \tfrac{w_7 \hat{x}}{\hat{b}} + \tfrac{w_8 \hat{y}}{\hat{b}} - w_9 - w_{10} f(\hat{x}) - w_{11} g(\hat{y})$$

and zero otherwise. Therefore, the eigenvalues of J_{xy} are given by:

$$\hat{\lambda}_{1,2} = \tfrac{-(\hat{x} + w_3 w_4 \hat{y})}{2} \pm \tfrac{\sqrt{(\hat{x} + w_3 w_4 \hat{y})^2 - 4\hat{x}\hat{y}(w_3 w_4 - w_5)}}{2}$$
$$\hat{\lambda}_3 = \tfrac{w_7 \hat{x}}{\hat{b}} + \tfrac{w_8 \hat{y}}{\hat{b}} - w_9 - w_{10} f(\hat{x}) - w_{11} g(\hat{y})$$

where $f(\hat{x})$ and $g(\hat{y})$ are obtain from Eq. (7a) and (7b) by substituting $x = \hat{x}$ and $y = \hat{y}$ respectively, while

$$\hat{b} = w_1 + \hat{x} + w_2 \hat{y} \quad (36)$$

Obviously, E_{xy} is locally asymptotically stable in the R_+^3 if in addition to condition (9a), which guarantees the local stability of E_{xy} in the $Int.R_+^2$ of the $xy-$plane, the following condition holds

$$\tfrac{w_7 \hat{x}}{\hat{b}} + \tfrac{w_8 \hat{y}}{\hat{b}} < w_9 + w_{10} f(\hat{x}) + w_{11} g(\hat{y}) \quad (37a)$$

Here condition (37a) insure the convergent of solution to E_{xy} from $z-$direction. On the other hand, E_{xy} is a saddle point in the R_+^3 if at least one of the conditions (9b) and the following condition

$$\tfrac{w_7 \hat{x}}{\hat{b}} + \tfrac{w_8 \hat{y}}{\hat{b}} > w_9 + w_{10} f(\hat{x}) + w_{11} g(\hat{y}) \quad (37b)$$

hold.

Before we go further to analyze the dynamical behavior of system (6) in the neighborhood of the second phytoplankton free equilibrium point, recall that the system have either two equilibrium points $E_{x_1 z_1}$ and $E_{x_2 z_2}$ or there is no equilibrium point in case 1.While, it has a unique equilibrium point E_{xz} in case 2. Since all these equilibrium points, whenever they exist have the same local stability conditions which depend on the form of equilibrium points, therefore we assume here E_{xz} represent any one of them that belongs to $xz-$plane.

So, the Jacobian matrix of system (6) at the second phytoplankton free equilibrium point $E_{xz} = (\bar{x}, 0, \bar{z})$ in $xz-$plane, can be written as $J_{xz} = J(E_{xz}) = [g_{ij}]_{3\times3}; i, j = 1,2,3$, with

$$g_{11} = \bar{x}\left(-1 + \tfrac{\bar{z}}{\bar{b}^2}\right),\ g_{12} = \bar{x}\left(-1 + \tfrac{w_2 \bar{z}}{\bar{b}^2}\right),\ g_{13} = \tfrac{-\bar{x}}{\bar{b}},$$
$$g_{22} = w_3 - w_5 \bar{x} - \tfrac{w_6 \bar{z}}{\bar{b}},\ g_{31} = \bar{z}\left(\tfrac{w_1 w_7}{\bar{b}^2} - w_{10} f'(\bar{x})\right),$$
$$g_{32} = \bar{z}\left(\tfrac{-w_2 w_7 \bar{x}}{\bar{b}^2} + \tfrac{w_8}{\bar{b}} - w_{11} g'(0)\right)$$

and zero otherwise. Where

$$\bar{b} = w_1 + \bar{x} \quad (38a)$$
$$\left.\begin{aligned} f'(\bar{x}) &= \tfrac{d}{dx} f(x)\big|_{x=\bar{x}} \\ g'(0) &= \tfrac{d}{dy} g(y)\big|_{y=0} \end{aligned}\right\} \quad (38b)$$

While, $f(x)$ and $g(y)$ are given in Eq. (7). Clearly, the eigenvalues of J_{xz} are given by:

$$\bar{\lambda}_{1,3} = \tfrac{\bar{x}}{2}\left(-1 + \tfrac{\bar{z}}{\bar{b}^2}\right) \pm \tfrac{\sqrt{\bar{x}^2\left(-1 + \frac{\bar{z}}{\bar{b}^2}\right)^2 - 4\frac{\bar{x}\bar{z}}{\bar{b}}\left(\frac{w_1 w_7}{\bar{b}^2} - w_{10} f'(\bar{x})\right)}}{2}$$
$$\bar{\lambda}_2 = w_3 - w_5 \bar{x} - \tfrac{w_6 \bar{z}}{\bar{b}}$$

Consequently, E_{xz} is locally asymptotically stable in the R_+^3 if the following conditions are satisfied.

$$\bar{z} < \bar{b}^2 \quad (39a)$$
$$w_1 w_7 > w_{10} \bar{b}^2 f'(\bar{x}) \quad (39b)$$
$$w_3 < w_5 \bar{x} + \tfrac{w_6 \bar{z}}{\bar{b}} \quad (39c)$$

Obviously, conditions (39a) and (39b) guarantee the local stability of E_{xz} in the $Int.R_+^2$ of the $xz-$plane while condition (39c) guarantees the convergent of the solution to E_{xz} from $y-$direction. However, E_{xz} will be unstable point in the R_+^3 if we reversed any one of the above conditions.

Similarly, it is assumed that, E_{yz} represent any one of the first phytoplankton free of the equilibrium points those may belong to $yz-$plane. Hence the Jacobian matrix of system (6) at the equilibrium point $E_{yz}=(0,\tilde{y},\tilde{z})$ in $yz-$plane, can be written as $J_{yz}=J(E_{yz})=[h_{ij}]_{3\times3}; i,j=1,2,3$, where $h_{11}=1-\tilde{y}-\frac{\tilde{z}}{\tilde{b}}$, $h_{21}=\tilde{y}\left(-w_5+\frac{w_6\tilde{z}}{\tilde{b}^2}\right)$, $h_{22}=\tilde{y}\left(-w_3w_4+\frac{w_2w_6\tilde{z}}{\tilde{b}^2}\right)$, $h_{23}=\frac{-w_6\tilde{y}}{\tilde{b}}$, $h_{31}=\tilde{z}\left(\frac{w_7}{\tilde{b}}-\frac{w_8\tilde{y}}{\tilde{b}^2}-w_{10}f'(0)\right)$, $h_{32}=\tilde{z}\left(\frac{w_1w_8}{\tilde{b}^2}-w_{11}g'(\tilde{y})\right)$ and zero otherwise, here $f'(0)$ and $g'(\tilde{y})$ result from Eq. (38b) for $x=0$ and $y=\tilde{y}$, however:

$$\tilde{b}=w_1+w_2\tilde{y} \tag{40}$$

Clearly the eigenvalues of J_{yz} are given by:

$$\tilde{\lambda}_1=1-\tilde{y}-\frac{\tilde{z}}{\tilde{b}}$$

$$\tilde{\lambda}_{2,3}=\frac{\tilde{y}}{2}\left(-w_3w_4+\frac{w_2w_6\tilde{z}}{\tilde{b}^2}\right)\pm\frac{\sqrt{\tilde{y}^2\left(-w_3w_4+\frac{w_2w_6\tilde{z}}{\tilde{b}^2}\right)^2-4\frac{w_6\tilde{y}\tilde{z}}{\tilde{b}}\left(\frac{w_1w_8}{\tilde{b}^2}-w_{11}g'(\tilde{y})\right)}}{2}$$

Consequently, E_{yz} is locally asymptotically stable in the R_+^3 if the following conditions are satisfied:

$$\tilde{y}+\frac{\tilde{z}}{\tilde{b}}>1 \tag{41a}$$

$$\tilde{z}<\frac{w_3w_4\tilde{b}^2}{w_2w_6} \tag{41b}$$

$$w_1w_8>w_{11}\tilde{b}^2g'(\tilde{y}) \tag{41c}$$

Obviously, conditions (41b) and (41c) guarantee the local stability of E_{yz} in the $Int.R_+^2$ of the $yz-$plane while condition (41a) insure the convergent of the solution to E_{yz} from $x-$direction. Moreover, E_{yz} is unstable point in the R_+^3 if we reversed any one of the above conditions.

Finally, the Jacobian matrix of the system (6) at the positive equilibrium point $E_{xyz}=(x^*,y^*,z^*)$ in the $Int.R_+^3$ can be written as:

$$J_{xyz}=J(E_{xyz})=\left[a_{ij}\right]_{3\times3}; i,j=1,2,3 \tag{42a}$$

where $a_{11}=\frac{x^*}{b^{*2}}D_1$, $a_{12}=\frac{x^*}{b^{*2}}D_2$, $a_{13}=\frac{-x^*}{b^*}$, $a_{21}=\frac{y^*}{b^{*2}}D_3$, $a_{22}=\frac{y^*}{b^{*2}}D_4$, $a_{23}=\frac{-w_6y^*}{b^*}$, $a_{31}=\frac{z^*\left(D_5-w_{10}b^{*2}f'(x^*)\right)}{b^{*2}}$, $a_{32}=\frac{z^*\left(D_6-w_{11}b^{*2}g'(y^*)\right)}{b^{*2}}$ and $a_{33}=0$. In addition, we have

$$D_1=-b^{*2}+z^*,\ D_2=-b^{*2}+w_2z^*,$$
$$D_3=-w_5b^{*2}+w_6z^*,\ D_4=-w_3w_4b^{*2}+w_2w_6z^*,$$
$$D_5=w_1w_7+(w_2w_7-w_8)y^*,$$
$$D_6=w_1w_8-(w_2w_7-w_8)x^*$$

and b^* is given in Eq. (26), while

$$\left.\begin{aligned} f'(x^*)&=\frac{d}{dx}f(x)\Big|_{x=x^*}\\ g'(y^*)&=\frac{d}{dy}g(y)\Big|_{y=y^*}\end{aligned}\right\} \tag{42b}$$

where $f(x)$ and $g(y)$ are given in Eq. (7).

Accordingly the characteristic equation of J_{xyz} can be written as:

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0 \tag{43}$$

here

$$A_1=-(a_{11}+a_{22})$$
$$A_2=a_{11}a_{22}-a_{12}a_{21}-a_{23}a_{32}-a_{13}a_{31}$$
$$A_3=a_{23}(a_{11}a_{32}-a_{12}a_{31})-a_{13}(a_{21}a_{32}-a_{22}a_{31})$$

and

$$\begin{aligned}\Delta&=A_1A_2-A_3\\&=-(a_{11}+a_{22})(a_{11}a_{22}-a_{12}a_{21})\\&\quad+a_{31}(a_{11}a_{13}+a_{12}a_{23})\\&\quad+a_{32}(a_{22}a_{23}+a_{13}a_{21})\end{aligned}$$

So, by substituting the values of a_{ij}, and then simplifying the resulting terms we obtain:

$$A_1=\frac{-1}{b^{*2}}\left(x^*D_1+y^*D_4\right)\ldots\ldots\ldots\ldots\ldots\ldots(44a)$$

$$A_3=\frac{x^*y^*z^*}{b^{*5}}\left[\left(D_6-w_{11}b^{*2}g'(y^*)\right)(-w_6D_1+D_3)+\left(D_5-w_{10}b^{*2}f'(x^*)\right)(w_6D_2-D_4)\right]$$

……. (44b)

$$\Delta=-\frac{x^*y^*}{b^{*6}}\left(x^*D_1+y^*D_4\right)\left(D_1D_4-D_2D_3\right)+\frac{x^*z^*}{b^{*5}}\left(D_5-w_{10}b^{*2}f'(x^*)\right)\left(-x^*D_1-w_6y^*D_2\right)+\frac{y^*z^*}{b^{*5}}\left(D_6-w_{11}b^{*2}g'(y^*)\right)\left(-w_6y^*D_4-x^*D_3\right)$$

…. (44c)

Therefore, in the following theorem, the local stability conditions for the positive equilibrium point E_{xyz} in the $Int.R_+^3$ are established.

Theorem 2. Assume that E_{xyz} exists in the $Int.R_+^3$ and the following conditions are satisfied;

$$z^* < \min\left(b^{*2}, \frac{b^{*2}}{w_2}, \frac{w_5}{w_6} b^{*2}, \frac{w_3 w_4}{w_2 w_6} b^{*2}\right) \quad (45a)$$

$$D_5 > w_{10} f'(x^*) b^{*2} \quad (45b)$$

$$\frac{D_3}{D_1} < w_6 < \frac{D_4}{D_2} \quad (45c)$$

$$D_6 > w_{11} g'(y^*) b^{*2} \quad (45d)$$

Then it is locally asymptotically stable.

Proof. According to the Routh-Hawirtiz criterion the characteristic equation (43) has roots with negative real parts if and only if $A_1 > 0$, $A_3 > 0$ and $\Delta > 0$.

Note that, it is easy to verify that, condition (45a) guarantees that $D_i < 0, \forall i = 1,2,3,4$ and hence $A_1 > 0$; while conditions (45a)- (45d) ensure the positivity of A_3 (i.e. $A_3 > 0$) and $\Delta > 0$. Hence, all the roots (eigenvalues) of the J_{xyz} have negative real parts. Therefore E_{xyz} is locally asymptotically stable in the $Int.R_+^3$ and hence the proof is complete. ■

Now, before go further to study the global dynamical behavior of system (6) in the $Int.R_+^3$, we will discuss the dynamical behavior of system (6) in the interior of the boundary planes as shown in the following theorems.

Theorem 3. Suppose that the zooplankton free equilibrium point E_{xy} is locally asymptotically stable in the $Int.R_+^2$ of the $xy-$plane, then it is a globally asymptotically stable in $Int.R_+^2$ of the $xy-$plane.

Proof. The proof follows directly by using Bendixson-Dulic criterion with Dulic function $1/xy$ and then using Poincare-Bendixson theorem. ■

Theorem 4. System (6) has no periodic dynamics in the $Int.R_+^2$ of $xz-$ and $yz-$ planes provided that

$$z < (w_1 + x)^2 \quad (46a)$$

$$z < \frac{w_3 w_4}{w_2 w_6}(w_1 + w_2 y)^2 \quad (46b)$$

respectively.

Proof. The proof follows directly by using Bendixson-Dulic criterion with Dulic functions $1/xz$ and $1/yz$ respectively. ■

Keeping the above in view, Since all the solutions of the system (6) are bounded and E_{xz} and E_{yz} (for case 2) are the unique positive equilibrium points in $Int.R_+^2$ of the $xz-$ and $yz-$ planes respectively, hence by using the Poincare-Bendixson theorem E_{xz} and E_{yz} are globally asymptotically stable in the $Int.R_+^2$ of $xz-$plane and $yz-$ plane respectively.

4. Global stability of the system

In this section the global stability of the equilibrium points E_x, E_y, E_{xy}, E_{xz}, E_{yz} and E_{xyz} in R_+^3 are investigated as shown in the following theorems.

Theorem 5. Assume that the equilibrium point E_x is locally asymptotically stable in R_+^3, and let the following condition holds:

$$w_9 \geq \frac{w_7}{w_1} \quad (47)$$

Then the basin of attraction of E_x can be written as $B(E_x) = \{(x,y,z) \in R_+^3 : x \geq \Omega_1, \quad y \geq 0, z \geq 0\}$ with $\Omega_1 = \frac{w_3 w_8 + w_6 w_7}{w_6 w_7 + w_5 w_8}$.

Proof. Consider the following positive definite function:

$$V_1(x,y,z) = c_1(x - 1 - \ln x) + c_2 y + c_3 z$$

Clearly $V_1 : R_+^3 \to R$, and is a C^1 positive definite function, where $c_i, (i = 1,2,3)$ are nonnegative constants to be determined. Now, since the derivative of V_1 along the trajectory of system (6) can be written as:

$$\frac{dV_1}{dt} < -c_1(x-1)^2 - [(c_1 + c_2 w_5)x - (c_2 w_3 + c_1)]y - \frac{xz}{b}(c_1 - c_3 w_7) - z\left(c_3 w_9 - \frac{c_1}{b}\right) - \frac{yz}{b}(c_2 w_6 - c_3 w_8)$$

Here

$$b = w_1 + x + w_2 y \quad (48)$$

So, by choosing the nonnegative constants as $c_1 = w_7$, $c_2 = \frac{w_8}{w_6}$ and $c_3 = 1$ gives:

$$\frac{dV_1}{dt} \leq -w_7(x-1)^2 - \left[\left(\frac{w_6 w_7 + w_5 w_8}{w_6}\right)x - \left(\frac{w_3 w_8 + w_6 w_7}{w_6}\right)\right]y - z\left(w_9 - \frac{w_7}{w_1}\right)$$

Therefore, for any initial point in the interior of $B(E_x)$, $\frac{dV_1}{dt} < 0$ under condition (47) and hence V_1 is strictly Lyapunov function. Thus, E_x is globally asymptotically stable in the $B(E_x)$ and then $B(E_x)$ is the basin of attraction of E_x. ■

Theorem 6. Assume that the equilibrium point E_y is locally asymptotically stable in R_+^3, and let the following condition holds:

$$w_9 \geq \frac{w_8}{w_1 w_4} \quad (49)$$

Then the basin of attraction of E_y can be written as $B(E_y) = \{(x,y,z) \in R_+^3 : x \geq 0, y \geq \Omega_2, z \geq 0\}$ with $\Omega_2 = \frac{w_4 w_6 w_7 + w_5 w_8}{w_4 (w_5 w_8 + w_6 w_7)}$.

Proof. Follows similarly as the proof of theorem (5) with using the candidate Lyapunov function

$$V_2(x,y,z) = c_1 x + c_2 \left(y - \hat{y} - \hat{y} \ln \frac{y}{\hat{y}}\right) + c_3 z$$ ■

Theorem 7. Assume that the zooplankton free equilibrium point E_{xy} is locally asymptotically stable in R_+^3, and let the following conditions hold:

$$\left(w_7 + \frac{w_5 w_8}{w_6}\right)^2 \leq 4 \frac{w_3 w_4 w_7 w_8}{w_6} \quad (50a)$$

$$w_9 \geq \frac{w_7 \hat{x} + w_8 \hat{y}}{w_1} \quad (50b)$$

Then E_{xy} is globally asymptotically stable in the R_+^3.

Proof. Follows directly by using the candidate Lyapunov function

$$V_3(x,y,z) = c_1 \left(x - \hat{x} - \hat{x} \ln \frac{x}{\hat{x}}\right) + c_2 \left(y - \hat{y} - \hat{y} \ln \frac{y}{\hat{y}}\right) + c_3 z$$ ■

Now, since system (6) in case 1, may have either two equilibrium points or no equilibrium points in the $Int.R_+^3$ of the $xz-$ and $yz-$ planes respectively. Therefore, in the following two theorems we will study the global dynamics of system (6) in these planes for case 2 only.

Theorem 8. Assume that the second phytoplankton free equilibrium point E_{xz} is locally asymptotically stable in R_+^3. Then the basin of attraction of E_{xz} is given by:

$$B(E_{xz}) = \{(x,y,z) \in R_+^3 : x > \bar{x}, y \geq 0, z > \bar{z}\}$$

provided that:

$$\bar{z} < w_1 \bar{b} \quad (51a)$$

$$\bar{z} \leq \frac{w_1 \bar{b}}{w_2} \quad (51b)$$

$$\frac{w_7}{\bar{b}} > \frac{a_1 w_{10} \bar{w}_{12}}{(\bar{w}_{12}+1)(\bar{w}_{12}+\bar{x})} \quad (51c)$$

$$\frac{w_2 w_4 w_7 \bar{x}}{(w_1 w_4 + w_4 + w_2)\bar{b}} + \frac{d_1 w_4 w_{11}}{w_4 \bar{w}_{13} + 1} \geq \frac{w_8}{w_1} \quad (51d)$$

Proof. Follows directly by using the candidate Lyapunov function

$$V_4(x,y,z) = c_1 \left(x - \bar{x} - \bar{x} \ln \frac{x}{\bar{x}}\right) + c_2 y + c_3 \left(z - \bar{z} - \bar{z} \ln \frac{z}{\bar{z}}\right)$$ ■

Theorem 9. Assume that the first phytoplankton free equilibrium point E_{yz} is locally asymptotically stable in R_+^3. Then the basin of attraction of E_{yz} is given by:

$$B(E_{yz}) = \{(x,y,z) \in R_+^3 : x \geq 0, y > \tilde{y}, z > \tilde{z}\}$$

Provided that:

$$\tilde{z} < \frac{w_1 w_3 w_4 \tilde{b}}{w_2 w_6} \quad (52a)$$

$$\tilde{z} \leq \frac{w_1 w_5 \tilde{b}}{w_6} \quad (52b)$$

$$\frac{w_8}{\tilde{b}} > \frac{d_1 w_4 w_{11} \bar{w}_{13}}{(w_4 \bar{w}_{13}+1)(\bar{w}_{13}+\tilde{y})} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (52c)$$

$$\frac{w_4 w_8 \tilde{y}}{(w_1 w_4 + w_4 + w_2)\tilde{b}} + \frac{a_1 w_{10}}{\bar{w}_{12}+1} \geq \frac{w_7}{w_1} \quad (52d)$$

Proof. Follows directly by using the candidate Lyapunov function

$$V_5(x,y,z) = c_1 x + c_2 \left(y - \tilde{y} - \tilde{y} \ln \frac{y}{\tilde{y}}\right) + c_3 \left(z - \tilde{z} - \tilde{z} \ln \frac{z}{\tilde{z}}\right)$$ ■

Theorem 10. Assume that the coexistence equilibrium point E_{xyz} is locally asymptotically stable in $Int.R_+^3$. Then the basin of attraction of E_{xyz} is given by:

$$B(E_{xyz}) = \{(x,y,z) : x > x^*, y > y^*, z > z^*\}$$

Provided that:

$$z^* < \min\left\{w_1 b^*, \frac{w_1 b^*}{w_2}, \frac{w_1 w_5 b^*}{w_6}, \frac{w_1 w_3 w_4 b^*}{w_2 w_6}\right\} \quad (53a)$$

$$w_1 w_7 + (w_2 w_7 - w_8) y^* > w_1 w_{10} f'(x^*) b^* \quad (53b)$$

$$w_1 w_8 - (w_2 w_7 - w_8) x^* > w_1 w_{11} g'(y^*) b^* \quad (53c)$$

$$K_{12}{}^2 \leq 4 K_{11} K_{22} \quad (53d)$$

where:

$$k_{11} = \left(\frac{w_1 w_7 + (w_2 w_7 - w_8) y^* - w_1 w_{10} w_{12} b^*}{b^*}\right)\left(1 - \frac{z^*}{w_1 b^*}\right)$$

$$K_{12} = \left(\frac{w_1 w_7 + (w_2 w_7 - w_8) y^* - w_1 w_{10} w_{12} b^*}{b^*}\right)\left(1 - \frac{w_2 z^*}{w_1 b^*}\right) + \left(\frac{w_1 w_8 - (w_2 w_7 - w_8) x^* - w_1 w_{11} w_{13} b^*}{w_6 b^*}\right)\left(w_5 - \frac{w_6 z^*}{w_1 b^*}\right)$$

$$K_{22} = \left(\frac{w_1 w_8 - (w_2 w_7 - w_8) x^* - w_1 w_{11} w_{13} b^*}{w_6 b^*} \right) \left(w_3 w_4 - \frac{w_2 w_6 z^*}{w_1 b^*} \right)$$

Proof. Follows directly by using the candidate Lyapunov function

$$V_6(x,y,z) = c_1\left(x - x^* - x^* \ln \frac{x}{x^*}\right) + c_2\left(y - y^* - y^* \ln \frac{y}{y^*}\right) + c_3\left(z - z^* - z^* \ln \frac{z}{z^*}\right) \quad \blacksquare$$

5. Persistence Analysis

In this section, the persistence of system (6) is studied. It is well known that the system is said to be persistence if and only if each species persists. Mathematically this is meaning that the solution of system (6) do not have omega limit set in the boundaries of R_+^3 [16]. Therefore, in the following theorem, the necessary and sufficient conditions for the uniform persistence of the system (6) are derived.

Theorem 11. Assume that there are no periodic dynamics in the boundary planes xy, xz and yz respectively. Further, if in addition to conditions (34c), (34d), (35c), (35d), (37b) the following conditions are hold.

$$w_3 > w_5 \bar{x} + \frac{w_6 \bar{z}}{\bar{b}} \tag{54a}$$

$$\tilde{y} + \frac{\tilde{z}}{\tilde{b}} < 1 \tag{54b}$$

Then, system (6) is uniformly persistence.

Proof. Consider the following function $\sigma(x,y,z) = x^{p_1} y^{p_2} z^{p_3}$, where $p_i; i = 1,2,3$ are an undetermined positive constants. Obviously $\sigma(x,y,z)$ is a C^1 positive function defined in R_+^3, and $\sigma(x,y,z) \to 0$ if $x \to 0$ or $y \to 0$ or $z \to 0$. Consequently we obtain

$$\Psi(x,y,z) = \frac{\sigma'(x,y,z)}{\sigma(x,y,z)} = p_1 f_1 + p_2 f_2 + p_3 f_3$$

Here $f_i; i = 1,2,3$ are given in system (6). Therefore

$$\begin{aligned}\Psi(x,y,z) = {} & p_1\left(1 - x - y - \frac{z}{w_1 + x + w_2 y}\right) \\ & + p_2\left(w_3(1 - w_4 y) - w_5 x - \frac{w_6 z}{w_1 + x + w_2 y}\right) \\ & + p_3\left(\frac{w_7 x}{w_1 + x + w_2 y} + \frac{w_8 y}{w_1 + x + w_2 y} - w_9 \right. \\ & \left. - w_{10} f(x) - w_{11} g(y)\right)\end{aligned}$$

Now, since it is assumed that there are no periodic attractors in the boundary planes, then the only possible omega limit sets of the system (6) are the equilibrium points $E_0, E_x, E_y, E_{xy}, E_{xz}$ *and* E_{yz}. Thus according to the Gard technique [16] the proof is follows and the system is uniformly persists if we can proof that $\Psi(.) > 0$ at each of these points. Since

$$\Psi(E_0) = p_1 + w_3 p_2 - w_9 p_3 \tag{55a}$$

$$\Psi(E_x) = (w_3 - w_5) p_2 + \left(\frac{w_7}{w_1 + 1} - w_9 - w_{10} f(1)\right) p_3 \tag{55b}$$

$$\Psi(E_y) = \left(1 - \frac{1}{w_4}\right) p_1 + \left(\frac{w_8}{w_1 w_4 + w_2} - w_9 - w_{11} g\left(\frac{1}{w_4}\right)\right) p_3 \tag{55c}$$

$$\Psi(E_{xy}) = \left(\frac{w_7 \hat{x}}{\hat{b}} + \frac{w_8 \hat{y}}{\hat{b}} - w_9 - w_{10} f(\hat{x}) - w_{11} g(\hat{y})\right) p_3 \tag{55d}$$

$$\Psi(E_{xz}) = \left(w_3 - w_5 \bar{x} - \frac{w_6 \bar{z}}{\bar{b}}\right) p_2 \tag{55e}$$

$$\Psi(E_{yz}) = \left(1 - \tilde{y} - \frac{\tilde{z}}{\tilde{b}}\right) p_1 \tag{55f}$$

where $f(1)$, $g(1/w_4)$, $f(\hat{x})$, $g(\hat{y})$, $\hat{b}$, $\bar{b}$ and $\tilde{b}$ are given in previous sections. Obviously, $\Psi(E_0) > 0$ for all values of $p_i; i = 1,2$ sufficiently large than $p_3 > 0$. $\Psi(E_x) > 0$ for any positive constants $p_i; i = 2,3$ provided that conditions (34c) and (34d) hold. $\Psi(E_y) > 0$ for any positive constants $p_i; i = 1,3$ if and only if conditions (35c) and (35d) are satisfied. However, $\Psi(E_{xy})$, $\Psi(E_{xz})$ and $\Psi(E_{yz})$ are positive provided that the conditions (37b), (54a) and (54b) are satisfied respectively. Then strictly positive solution of system (6) do not have omega limit set and hence, system (6) is uniformly persistence. $\blacksquare$

6. The local Bifurcation

In this section an investigation for the dynamical behavior of system (6) under the effect of varying one parameter of each time is carried out. The occurrence of local bifurcation in the neighborhood of the above equilibrium points are studied in the following theorems.

Theorem 12. Assume that condition (34b) holds and the parameter w_3 passes through the value $\breve{w}_3 = w_5$, then system (6) near the equilibrium E_x has:

1. No saddle-node bifurcation.
2. A transcritical bifurcation but no pitch-fork bifurcation can occur provided that the following condition holds:

$$w_4 \neq 1 \tag{56}$$

Otherwise there is no bifurcation.

Proof. According to the Jacobian matrix of system (6) at E_x that is given by J_x, it is easy to verify that as $w_3 = \breve{w}_3$, the $J_x(E_x, \breve{w}_3)$ has the following eigenvalues:

$$\breve{\lambda}_1 = -1,\ \breve{\lambda}_2 = 0 \text{ and } \breve{\lambda}_3 = \tfrac{w_7}{w_1+1} - w_9 - w_{10} f(1)$$

where $f(1)$ is obtain from Eq. (7a). Let $\breve{v} = (\breve{\theta}_1, \breve{\theta}_2, \breve{\theta}_3)^T$ be the eigenvector of $J_x(E_x, \breve{w}_3)$ corresponding to the eigenvalue $\breve{\lambda}_2 = 0$. Then it is easy to check that $\breve{v} = \left[\frac{-\breve{a}_{12}\breve{\theta}_2}{\breve{a}_{11}}, \breve{\theta}_2, 0\right]^T$, where $\breve{\theta}_2$ represents any nonzero real value. Also, let $\breve{w} = (\breve{h}_1, \breve{h}_2, \breve{h}_3)^T$ represents the eigenvector of $J_x^{\,T}(E_x, \breve{w}_3)$ that corresponding to the eigenvalue $\breve{\lambda}_2 = 0$. Straightforward calculation shows that $\breve{w} = (0, \breve{h}_2, 0)^T$, where $\breve{h}_2$ is any nonzero real number. Now since

$$\tfrac{\partial F}{\partial w_3} \equiv F_{w_3}(X, w_3) = [0, y(1 - w_4 y), 0]^T$$

where $X = (x, y, z)^T$ and $F = (F_1, F_2, F_3)^T$ with $F_i; i = 1,2,3$ represent the right hand side of system (6). Then we get that $F_{w_3}(E_x, \breve{w}_3) = (0,0,0)^T$ and then the following is obtained:

$$\breve{w}^T [F_{w_3}(E_x, \breve{w}_3)] = (0, \breve{h}_2, 0)(0,0,0)^T = 0$$

Thus the system (6) at E_x does not experience any saddle-node bifurcation in view of Sotomayor theorem [17].

Also, since

$$\breve{w}^T [DF_{w_3}(E_x, \breve{w}_3)\breve{v}] = (0, \breve{h}_2, 0)(0, \breve{\theta}_2, 0)^T = \breve{h}_2 \breve{\theta}_2 \neq 0$$

here $DF_{w_3}(E_x, \breve{w}_3) = \frac{\partial}{\partial X} F_{w_3}(X, w_3)\big|_{X = E_x, w_3 = \breve{w}_3}$.

Moreover, we have

$$\breve{w}^T [D^2 F(E_x, \breve{w}_3)(\breve{v}, \breve{v})] = 2 w_5 (1 - w_4) \breve{h}_2 \breve{\theta}_2^{\,2} \neq 0$$

where $D^2 F(E_x, \breve{w}_3) = DJ_x(X, w_3)\big|_{X = E_x, w_3 = \breve{w}_3}$.

Clearly, $\breve{w}^T [D^2 F(E_x, \breve{w}_3)(\breve{v}, \breve{v})] \neq 0$ provided that condition (56) holds, and then by using Sotomayor theorem again system (6) possesses a transcritical bifurcation but not pitch-fork bifurcation near E_x where $w_3 = \breve{w}_3$. However, violate condition (56) gives that $\breve{w}^T [D^2 F(E_x, \breve{w}_3)(\breve{v}, \breve{v})] = 0$, and hence further computation shows

$$\breve{w}^T [D^3 F(E_x, \breve{w}_3)(\breve{v}, \breve{v}, \breve{v})] = 0$$

Therefore according to Sotomayor theorem, there is no bifurcation. ■

Theorem 13. Assume that condition (35a) holds and the parameter w_8 passes through the value $\hat{w}_8 = \left(w_9 + w_{11} g\left(\tfrac{1}{w_4}\right)\right)\left(w_1 w_4 + w_2\right)$ where $g\left(\tfrac{1}{w_4}\right)$ is obtain fron Eq. (7b), then system (6) near the equilibrium E_y has :

1. No saddle-node bifurcation.
2. A transcritical bifurcation but nopitch-fork bifurcation can occur provided that the following condition holds:

$$\frac{w_1 w_4^{\,2}\left[w_9 + w_{11} g\left(\frac{1}{w_4}\right)\right]}{(w_1 w_4 + w_2)} \neq w_{11} g'\left(\tfrac{1}{w_4}\right) \qquad (57)$$

where:

$$g'\left(\tfrac{1}{w_4}\right) = \begin{cases} w_{13} & \text{forcasel} \\ \dfrac{d_1 w_4^2 \overline{w}_{13}}{(w_4 \overline{w}_{13} + 1)^2} & \text{forcase 2} \end{cases}$$

3. A pitch-fork bifurcation otherwise.

Proof. According to the Jacobian matrix of system (6) at E_y that is given by J_y, it is easy to verify that as $w_8 = \hat{w}_8$, the $J_y(E_y, \hat{w}_8)$ has the following eigenvalues:

$$\hat{\lambda}_1 = 1 - \frac{1}{w_4},\ \hat{\lambda}_2 = -w_3 < 0 \text{ and } \hat{\lambda}_3 = 0$$

Let $\hat{v} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)^T$ be the eigenvector of $J_y(E_y, \hat{w}_8)$ corresponding to the eigenvalue $\hat{\lambda}_3 = 0$. Then it is easy to check that $\hat{v} = \left[0, \frac{-\hat{a}_{23}\hat{\theta}_3}{\hat{a}_{22}}, \hat{\theta}_3\right]^T$, where $\hat{\theta}_3$ represents any nonzero real value. Also, let $\hat{w} = (\hat{h}_1, \hat{h}_2, \hat{h}_3)^T$ represents the eigenvector of $J_y^{\,T}(E_y, \hat{w}_8)$ that corresponding to the eigenvalue $\hat{\lambda}_3 = 0$. Straightforward calculation shows that $\hat{w} = (0, 0, \hat{h}_3)^T$, where $\hat{h}_3$ is any nonzero real number.

Now, since $\frac{\partial F}{\partial w_8} \equiv F_{w_8}(X, w_8) = [0, 0, \frac{yz}{b}]^T$, where b is given in (48), $X = (x, y, z)^T$ and $F = (F_1, F_2, F_3)^T$ with $F_i; i = 1,2,3$ represent the right hand side of system (6). Then we get $F_{w_8}(E_y, \hat{w}_8) = (0,0,0)^T$ from which we get

$$\hat{w}^T [F_{w_8}(E_y, \hat{w}_8)] = (0, 0, \hat{h}_3)(0,0,0)^T = 0.$$

Thus the system (6) at E_y does not experience any saddle-node bifurcation in view of Sotomayor theorem [17]. Also, since:

$$\hat{w}^T[DF_{w_8}(E_y,\hat{w}_8)\hat{v}] = (0,0,\hat{h}_3)\left(0,0,\frac{\hat{\theta}_3}{w_1w_4+w_2}\right)^T$$
$$= \frac{\hat{\theta}_3\hat{h}_3}{w_1w_4+w_2} \neq 0$$

here $DF_{w_8}(E_y,\hat{w}_8) = \frac{\partial}{\partial X}F_{w_8}(X,w_8)\big|_{X=E_y, w_8=\hat{w}_8}$.

Moreover, we have

$$\hat{w}^T[D^2F(E_y,\hat{w}_8)(\hat{v},\hat{v})] = -2\frac{w_6\hat{\theta}_3{}^2\hat{h}_3}{(w_1w_4+w_2)w_3} \cdot\left(\frac{w_1w_4{}^2\left[w_9+w_{11}g\left(\frac{1}{w_4}\right)\right]}{(w_1w_4+w_2)} - w_{11}g'\left(\frac{1}{w_4}\right)\right)$$

where $D^2F(E_y,\hat{w}_8) = DJ_y(X,w_8)\big|_{X=E_y, w_8=\hat{w}_8}$.

Clearly, $\hat{w}^T[D^2F(E_y,\hat{w}_8)(\hat{v},\hat{v})] \neq 0$ provided that condition (57) holds, and then by Sotomayor theorem, the system (6) possesses a transcritical bifurcation but not pitch-fork bifurcation near E_y where $w_8 = \hat{w}_8$. However, violate condition (57) gives that $\hat{w}^T[D^2F(E_y,\hat{w}_8)(\hat{v},\hat{v})] = 0$, and hence further computation shows

$$\hat{w}^T[D^3F(E_y,\hat{w}_8)(\hat{v},\hat{v},\hat{v})] = -3\left(\frac{w_6}{(w_1w_4+w_2)w_3}\right)^2\hat{\theta}_3{}^3h_3 \cdot\left(\frac{w_1w_2w_4{}^3\left[w_9+w_{11}g\left(\frac{1}{w_4}\right)\right]}{(w_1w_4+w_2)^2} + w_{11}g''\left(\frac{1}{w_4}\right)\right)$$

where:

$$g''\left(\frac{1}{w_4}\right) = \begin{cases} 0 & \text{forcase1} \\ \frac{-2d_1w_4^3\bar{w}_{13}}{(w_4\bar{w}_{13}+1)^3} & \text{forcase 2}\end{cases}$$

Therefore system (6) possesses a pitch-fork bifurcation near E_y where $w_8 = \hat{w}_8$. ■

Theorem 14. Assume that condition (9a) holds and the parameter w_7 passes through the value $\hat{w}_7 = \frac{\hat{b}}{\hat{x}}\left(w_9 + w_{10}f(\hat{x}) + w_{11}g(\hat{y}) - \frac{w_8\hat{y}}{\hat{b}}\right)$ where $f(\hat{x})$ and $g(\hat{y})$ are obtain from Eqs. (7a-7b), then system (6) near the equilibrium E_{xy} has:

1. No saddle-node bifurcation.
2. A transcritical bifurcation but no pitch-fork bifurcation can occur provided that at least one of the following conditions hold:
3.

$$\left.\begin{array}{r} \frac{\hat{b}}{\hat{x}}\left(w_9+w_{10}f(\hat{x})+w_{11}g(\hat{y})-\frac{w_8\hat{y}}{\hat{b}}\right)(w_1+w_2\hat{y}) \\ \neq w_8\hat{y}+w_{10}f'(\hat{x})\hat{b}^2 \\ w_8(w_1+\hat{x}) \neq w_2\left(w_9+w_{10}f(\hat{x})+w_{11}g(\hat{y})\right. \\ \left.-\frac{w_8\hat{y}}{\hat{b}}\right)\hat{b}+w_{11}g'(\hat{y})\hat{b}^2 \end{array}\right\} \quad \text{... (58)}$$

where: $f'(\hat{x}) = \begin{cases} w_{12} & \text{forcase1} \\ \frac{a_1\bar{w}_{12}}{(\bar{w}_{12}+\hat{x})^2} & \text{forcase2}\end{cases}$

$$g'(\hat{x}) = \begin{cases} w_{13} & \text{forcase1} \\ \frac{d_1\bar{w}_{13}}{(\bar{w}_{13}+\hat{y})^2} & \text{forcase 2}\end{cases}$$

4. A pitch-fork bifurcation provided that:

$$\left.\begin{array}{r} \frac{\hat{b}}{\hat{x}}\left(w_9+w_{10}f(\hat{x})+w_{11}g(\hat{y})-\frac{w_8\hat{y}}{\hat{b}}\right)(w_1+w_2\hat{y}) \\ = w_8\hat{y}+w_{10}f'(\hat{x})\hat{b}^2 \\ w_8(w_1+\hat{x}) = w_2\left(w_9+w_{10}f(\hat{x})+w_{11}g(\hat{y})\right. \\ \left.-\frac{w_8\hat{y}}{\hat{b}}\right)\hat{b}+w_{11}g'(\hat{y})\hat{b}^2 \end{array}\right\} \quad \text{... (59a)}$$

$$N = \Psi_1\left(\frac{w_6-w_3w_4}{(w_3w_4-w_5)\hat{b}}\right)^2 - 2\Psi_2\left(\frac{(w_6-w_5)(w_6-w_3w_4)}{(w_3w_4-w_5)^2\hat{b}^2}\right) + \Psi_3\left(\frac{w_6-w_5}{(w_3w_4-w_5)\hat{b}}\right)^2 \neq 0 \quad (59b)$$

where:

$$\Psi_1 = \frac{2\hat{b}}{\hat{x}}\left[(w_1+w_2\hat{y})\left(w_9+w_{10}f(\hat{x})+w_{11}g(\hat{y})-\frac{w_8\hat{y}}{\hat{b}}\right)-w_8\hat{y}\right] + w_{10}f''(\hat{x})\hat{b}^3$$
$$\Psi_2 = \frac{w_2\hat{b}}{\hat{x}}\left(w_9+w_{10}f(\hat{x})+w_{11}g(\hat{y})-\frac{w_8\hat{y}}{\hat{b}}\right)(w_1-\hat{x}+w_2\hat{y}) + w_8(w_1+\hat{x}-w_2\hat{y})$$
$$\Psi_3 = 2w_2\left[(w_1+\hat{x})w_8 - w_2\hat{b}\left(w_9+w_{10}f(\hat{x})+w_{11}g(\hat{y})-\frac{w_8\hat{y}}{\hat{b}}\right)\right] + w_{11}g''(\hat{y})\hat{b}^3$$

$$f''(\hat{x}) = \begin{cases} 0 & \text{forcase1} \\ \frac{-2a_1\bar{w}_{12}}{(\bar{w}_{12}+\hat{x})^3} & \text{forcase2}\end{cases} \quad g''(\hat{x}) = \begin{cases} 0 & \text{forcase1} \\ \frac{-2d_1\bar{w}_{13}}{(\bar{w}_{13}+\hat{y})^3} & \text{forcase 2}\end{cases}$$

Proof. Follows directly by applying Sotomayor theorem as shown in proof of theorem (13). ■

Theorem 15. Assume that conditions (39a) and (39b) hold and the parameter w_3 passes through the value $\bar{w}_3 = w_5\bar{x} + \frac{w_6\bar{z}}{\bar{b}}$, where $\bar{b}$ is given in (38a), then system (6) near the equilibrium E_{xz} has:

1. No saddle-node bifurcation.
2. A transcritical bifurcation but no pitch-fork bifurcation can occur provided that one of the following condition holds:

$$M_1 = \frac{\Gamma_1}{\Gamma_2}(-w_5+w_6) + w_4\left(w_5\bar{x}+\frac{w_6\bar{z}}{\bar{b}}\right) - w_6 \neq 0 \quad (60)$$

A pitch-fork bifurcation provided that

$$M_1 = 0 \quad (61a)$$
$$M_2 = \left(\frac{\Gamma_1}{\Gamma_2}\right)\left[w_6\bar{z}\left(\frac{\Gamma_1}{\Gamma_2}\right) - 2w_2w_6\bar{z} - w_6\left(\frac{\Gamma_1\Gamma_3}{\Gamma_2}-\Gamma_4\right)\right] + w_2{}^2w_6\bar{z} + w_2w_6\left[\frac{\Gamma_1\Gamma_3}{\Gamma_2}-\Gamma_4\right] \neq 0 \quad (61b)$$

here

$$\Gamma_1 = w_1w_8 + (w_8 - w_2w_7)\bar{x} - w_{11}g'(0)\bar{b}^2$$
$$\Gamma_2 = w_1w_7 - w_{10}f'(\bar{x})\bar{b}^2$$
$$\Gamma_3 = -\bar{b}^2 + \bar{z},\ \Gamma_4 = -\bar{b}^2 + w_2\bar{z}$$

Where $f'(\bar{x})$ and $g'(0)$ are given in (38b)
Proof. Follows directly by applying Sotomayor theorem as shown in proof of theorem (13). ■

Theorem 16. Assume that conditions (41b) and (41c) hold and the parameter w_1 passes through the value $\widetilde{w}_1 = \frac{w_2\widetilde{y}(\widetilde{y}-1)+\widetilde{z}}{1-\widetilde{y}}$, then system (6) near the equilibrium E_{yz} has:

1. No saddle-node bifurcation.
2. A transcritical bifurcation but no pitch-fork bifurcation can occur provided that one of the following condition holds:

$$K_1 = \frac{\sigma_1 w_7 - \sigma_2(1-\widetilde{y})}{\sigma_1 w_8 - \sigma_4}\left[w_6 - w_3 w_4\right] + w_5 - w_6 \neq 0 \ldots (62)$$

3. A pitch-fork bifurcation provided that
4.

$$K_1 = 0 \qquad (63a)$$

$$\begin{aligned} K_2 = \widetilde{z} + w_6&\left[\frac{\sigma_3(\sigma_1 w_7 - \sigma_2(1-\widetilde{y}))}{w_6(\sigma_1 w_8 - \sigma_4)} - \frac{(-w_5\widetilde{b}^2 + w_6\widetilde{z})}{w_6}\right] \\ &+ \frac{\sigma_1 w_7 - \sigma_2(1-\widetilde{y})}{\sigma_1 w_8 - \sigma_4}\left[w_2{}^2\widetilde{z}\left(\frac{\sigma_1 w_7 - \sigma_2(1-\widetilde{y})}{\sigma_1 w_8 - \sigma_4}\right)\right. \\ &\left.- w_2\left(\frac{\sigma_3(\sigma_1 w_7 - \sigma_2(1-\widetilde{y}))}{w_6(\sigma_1 w_8 - \sigma_4)}\right) - 2w_2\widetilde{z}\right] \neq 0 \end{aligned} \qquad (63b)$$

here:

$$\sigma_1 = w_2\widetilde{y}(\widetilde{y}-1) + \widetilde{z}$$

$$\sigma_2 = (w_8 - w_2 w_7)\widetilde{y} + w_{10} f'(0)\widetilde{b}^2$$

$$\sigma_3 = -w_3 w_4 \widetilde{b}^2 + w_2 w_6 \widetilde{z}$$

$$\sigma_4 = (1-\widetilde{y}) w_{11} g'(\widetilde{y})\widetilde{b}^2$$

with $f'(0)$ and $g'(\widetilde{y})$ are obtain from Eq. (38b).
Proof. Follows directly by applying Sotomayor theorem as shown in proof of theorem (13). ■

7. Hopf bifurcation

Finally, in order to investigate the Hopf bifurcation of the model system (6), we will follow the Liu approach [18] as shown in the following theorem.

Theorem 17. Assume that the coexistence equilibrium point of system (6) exists and let in addition to conditions (45a)-(45c), the following conditions hold:

$$w_8 > \frac{w_2 w_7 x^*}{w_1 + x^*} \qquad (64a)$$

$$\begin{aligned} &\left(D_5 - w_{10} b^{*2} f'(x^*)\right)\left[(w_6 D_2 - D_4)\right. \\ &\cdot(-w_6 y^* D_4 - x^* D_3) - \frac{x^*}{y^*}(-x^* D_1 - w_6 y^* D_2) \\ &\left.\cdot(-w_6 D_1 + D_3)\right] > \frac{x^*}{b^* z^*}(-x^* D_1 - y^* D_4) \\ &\qquad\cdot(D_1 D_4 - D_2 D_3)(-w_6 D_1 + D_3) \end{aligned} \qquad \ldots\ldots\ldots\ldots (64b)$$

Then a simple Hopf bifurcation of the model system (6) occurs at

$$\begin{aligned} w_{11} \equiv w_{11}{}^* = &\frac{b^{*3}}{y^* z^*(-w_6 y^* D_4 - x^* D_3) g'(y^*)} \\ &\left[-\frac{x^* y^*}{b^{*6}}(x^* D_1 + y^* D_4)(D_1 D_4 - D_2 D_3)\right. \\ &+ \frac{x^* z^*}{b^{*5}}\left(D_5 - w_{10} b^{*2} f'(x^*)\right) \\ &\cdot(-x^* D_1 - w_6 y^* D_2) \\ &\left.+ \frac{y^* z^*}{b^{*5}} D_6(-x^* D_3 - w_6 y^* D_4)\right] \end{aligned}$$

where $D_i, i = 1,2,3,4,5,6$ and $f'(x^*)$ are given in Eqs. (42a) and (42b) respectively.
Proof. According to the Liu approach a simple Hopf bifurcation occurs if and only if $A_1(\mu_*) > 0,\ A_3(\mu_*) > 0,\ \Delta(\mu_*) = 0\ and\ \left.\frac{d\Delta}{d\mu}\right|_{\mu=\mu_*} \neq 0$ where μ_* is a critical value of the key parameter and $A_i\ for\ i = 1,3\ and\ \Delta$ are given in equations (44a), (44b) and (44c). Note that it is clear that $D_6 > 0$ under the condition (64a) and hence $w_{11}{}^*$ is positive under the conditions (45a)-(45c). Now, by substituting the value of $w_{11}{}^*$ in these equations we obtain:

$$A_1(w_{11}{}^*) = \frac{-1}{b^{*2}}\left(x^* D_1 + y^* D_4\right)$$

which is positive due to condition (45a);

$$\begin{aligned} A_3(w_{11}{}^*) = &\frac{x^* y^* z^*}{b^{*5}(-w_6 y^* D_4 - x^* D_3)} \\ &\cdot\left[\frac{x^*}{b^* z^*}(x^* D_1 + y^* D_4)(D_1 D_4 - D_2 D_3)\right. \\ &\cdot(-w_6 D_1 + D_3) + \left(D_5 - w_{10} b^{*2} f'(x^*)\right) \\ &\cdot\left[(w_6 D_2 - D_4)(-w_6 y^* D_4 - x^* D_3)\right. \\ &\left.\left.- \frac{x^*}{y^*}(-x^* D_1 - w_6 y^* D_2)(-w_6 D_1 + D_3)\right]\right] \end{aligned}$$

Clearly, $A_3(w_{11}{}^*) > 0$ under the conditions (45a), (45b), (45c) and (64b). Now rewrite equation (44c) gives that

$$\begin{aligned} \Delta = &-\frac{x^* y^*}{b^{*6}}\left(x^* D_1 + y^* D_4\right)\left(D_1 D_4 - D_2 D_3\right) \\ &+ \frac{x^* z^*}{b^{*5}}\left(D_5 - w_{10} b^{*2} f'(x^*)\right)\left(-x^* D_1 - w_6 y^* D_2\right) \\ &+ \frac{y^* z^*}{b^{*5}} D_6\left(-w_6 y^* D_4 - x^* D_3\right) \\ &- w_{11}\frac{y^* z^* g'(y^*)}{b^{*3}}\left(-w_6 y^* D_4 - x^* D_3\right) \end{aligned}$$

where $g'(y^*)$ is given in Eq.(42b). Hence it easy to verify that $\Delta(w_{11}{}^*) = 0$. Finally, since

$$\left.\frac{d\Delta}{dw_{11}}\right|_{w_{11}=w_{11}{}^*} = \frac{-y^*z^*g'(y^*)}{b^{*3}}\left(-w_6y^*D_4 - x^*D_3\right) \neq 0$$

Thus, a simple Hopf bifurcation occurs in system (6) at $w_{11} = w_{11}{}^*$. ■

8. Numerical Analysis

In this section the global dynamics of system (6), for case 1 and case 2, is studied numerically. In both the cases, system (6) is solved numerically for different sets of parameters and different sets of initial conditions, and then the attracting sets and their time series are drown.

For the following set of parameters

$$\begin{aligned}
&w_1 = 0.5, w_2 = 0.75, w_3 = 1, w_4 = 1.1,\\
&w_5 = 0.75, w_6 = 1, w_7 = 0.5, w_8 = 0.5,\\
&w_9 = 0.1, w_{10} = 0.2, w_{11} = 0.2, w_{12} = 0.75, \qquad (65)\\
&w_{13} = 0.75, \overline{w}_{12} = 0.7, \overline{w}_{13} = 0.7,\\
&a_1 = 1, d_1 = 1
\end{aligned}$$

The attracting sets along with their time series of system (6) are drown in figure 1 and figure 2, starting from different sets of initial conditions, for case 1 and case 2 respectively. Note that from now onward, in time series figures, we will use solid line type for x, dash line type for y and dot line type for z.

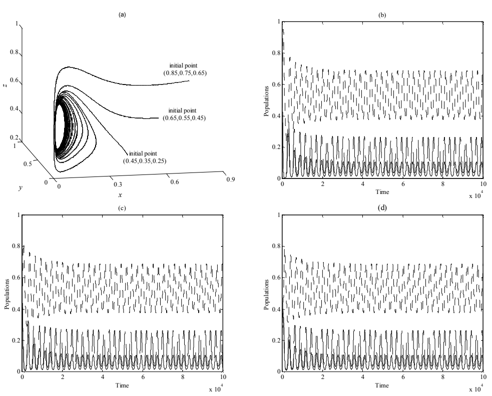

Figure 1- The phase plot of system (6) in case 1. (a) The solution of system (6) approaches asymptotically to stable limit cycle initiated at different initial points. (b) Time series of the attractor in (a) initiated at (0.85, 0.75, 0.65). (c) Time series of the attractor in (a) initiated at (0.65, 0.55, 0.45). (d) Time series of the attractor in (a) initiated at (0.45, 0.35, 0.25).

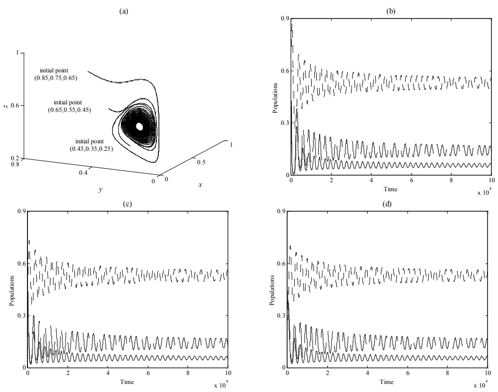

Figure 2- The phase plot of system (6) in case 2. (a) The solution of system (6) approaches asymptotically to stable limit cycle initiated at different initial points. (b) Time series of the attractor in (a) initiated at (0.85, 0.75, 0.65). (c) Time series of the attractor in (a) initiated at (0.65, 0.55, 0.45). (d) Time series of the attractor in (a) initiated at (0.45, 0.35, 0.25).

Obviously, these figures show that, system (6) approaches to the globally asymp-totically stable limit cycle in the $Int.R^3_+$ starting from different sets of initial conditions. However, for the set of parameters values (65) with $w_1 = 0.6$, system (6) approaches asymptotically to coexistence equilibrium point E_{xyz} in the $Int.R^3_+$ starting from different sets of initial conditions, see figure 3 and figure 4 for case 1 and case 2 respectively.

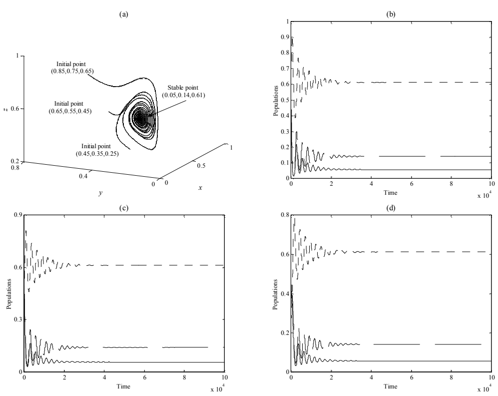

Figure 3- The phase plot of system (6) in case 1. (a) The solution of system (6) approaches asymptotically to stable positive point initiated at different initial points. (b) Time series of the attractor in (a) initiated at (0.85, 0.75, 0.65). (c) Time series of the attractor in (a) initiated at (0.65, 0.55, 0.45). (d) Time series of the attractor in (a) initiated at (0.45, 0.35, 0.25).

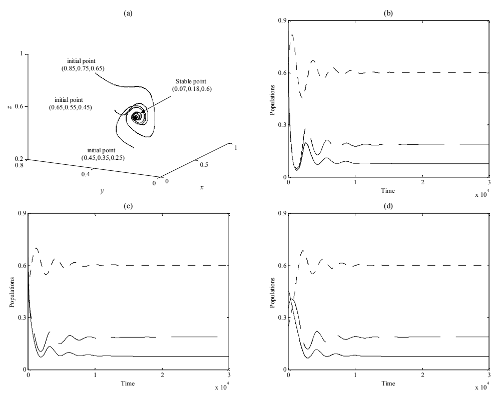

Figure 4- The phase plot of system (6) in case 2. (a) The solution of system (6) approaches asymptotically to stable positive point initiated at different initial points. (b) Time series of the attractor in (a) initiated at (0.85, 0.75, 0.65). (c) Time series of the attractor in (a) initiated at (0.65, 0.55, 0.45). (d) Time series of the attractor in (a) initiated at (0.45, 0.35, 0.25).

Further analysis for the role of changing the half saturation constant w_1 on the dynamics of system (6) is performed, and the following results are obtained: **In case 1**, system (6) has a periodic dynamic in the $Int.R_+^3$ for $w_1 \le 0.54$ with the rest of parameters as given in Eq. (65), while for $0.55 \le w_1 \le 1.18$ the system (6) has a globally asymptotically stable positive point in the $Int.R_+^3$. Finally, it approaches asymptotically to zooplankton free equilibrium point $E_{xy} = (0.28, 0.71, 0)$ in the $Int.R_+^2$ of $xy-$ plane when $w_1 \ge 1.19$ as shown in figure 5a. However, **in case 2**, the system (6) has a periodic dynamic in the $Int.R_+^3$ when $w_1 \le 0.5$, while the system (6) has a globally asymptotically stable positive point in the $Int.R_+^3$ for $0.51 \le w_1 \le 1.10$ and finally the system approaches asymptotically to zooplankton free equilibrium point $E_{xy} = (0.28, 0.71, 0)$ in the $Int.R_+^2$ of $xy-$ plane for $w_1 \ge 1.11$ as shown in figure 5b.

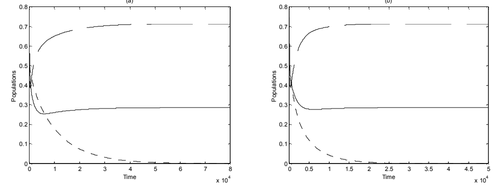

Figure 5- The trajectory of system (6) approaches asymptotically to zooplankton free equilibrium point $E_{xy} = (0.28, 0.71, 0)$ in the $Int.R_+^2$ of $xy-$ plane for $w_1 = 1.25$ with other parameters as in (65). (a) The time series of the system (6) in case 1. (b) The time series of the system (6) in case 2.

Obviously the parameter w_1 plays a vital role in the extinction of zooplankton species causes losing the persistence of system (6).

Now, the effect of varying the intrinsic growth rate of the second phytoplankton y, the parameter w_3, on the dynamics of system (6) is studied. It is observed that, **for case 1**, system (6) has periodic dynamics in the $Int.R_+^2$ of the $xz-$plane for parameters values given by (65) with $w_3 \le 0.94$, while it approaches to periodic attractor in the $Int.R_+^3$ for the range of values $0.95 \le w_3 \le 1.01$; finally the system (6) approaches asymptotically to the first phytoplankton free equilibrium point in the $Int.R_+^2$ of $yz-$plane when $w_3 \ge 1.02$, see figure 6. However, **for case 2,** the system (6) approaches to periodic dynamics in the $Int.R_+^2$ of $xz-$plane for $w_3 \le 0.94$, while it has a periodic attractor in the $Int.R_+^3$ where $0.95 \le w_3 \le 1$, further the solution of system (6) return to approach at the positive equilibrium point in the $Int.R_+^3$ at the values of $w_3 = 1.01$. Finally, it is observed that for $w_3 \ge 1.02$ system (6) approaches asymp-totically to the first phytoplankton free equilibrium point in the $Int.R_+^2$ of $yz-$plane as shown in figure 7.

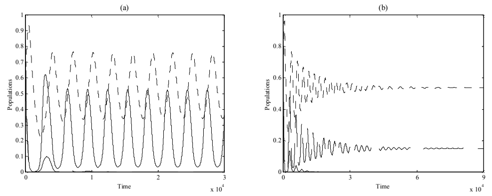

Figure 6- Time series of the solution of system (6), in case 1, for different values of w_3 with other parameters as given in (65). (a) Periodic attractor in the $Int.R_+^2$ of $xz-$plane at $w_3 = 0.9$. (b) Stable point $E_{yz} = (0, 0.15, 0.53)$ in the $Int.R_+^2$ of $yz-$plane at $w_3 = 1.05$.

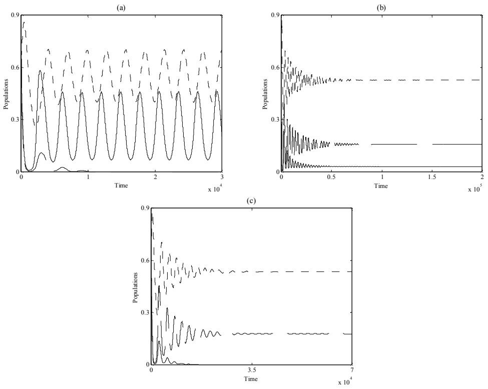

Figure 7- Time series of the solution of system (6), in case 2, for different values of w_3 with other parameters as given in (65). (a) Periodic attractor in the $Int.R^2_+$ of $xz-$plane at $w_3 = 0.9$. (b) stable point $E_{xyz} = (0.03,0.15,0.52)$ in the $Int.R^3_+$. (c) Stable point $E_{yz} = (0,0.17,0.53)$ in the $Int.R^2_+$ of $yz-$plane at $w_3 = 1.05$.

According to the figures 6 and 7, it is clear that the persistence of system (6) is very sensitive to the changing in the intrinsic growth rate of the second phytoplankton that represented by the parameter w_3 in both the cases and hence it represent a bifurcation parameters. Moreover, on contrast of case 1, in case 2 the system (6) still has two types of attractors in the $Int.R^3_+$ (stable point and periodic) as the parameter w_3 passes through some critical values.

Now before we summarize our obtained numerical results for other parameters on the dynamical behavior of system (6) in the form of tables, we will show the occurrence of Hopf bifurcation in both the cases as a function of the conversion rate of zooplankton from the first phytoplankton that represented by the parameter w_7.

It is observed that, **for case 1**, the system (6) approaches asymptotically to the positive equilibrium point E_{xyz} in the $Int.R^3_+$ when $w_7 < 0.27$ keeping other parameters fixed as in (65), while it is approaches to a periodic dynamic in the $Int.R^3_+$ as the parameter $w_7 \geq 0.27$, see Figure 8. However, **for case 2**, system (6) approaches to the positive equilibrium point E_{xyz} in the $Int.R^3_+$ for the values of $w_7 < 0.49$ keeping other parameters fixed as in (65), while it has a periodic attractor in $Int.R^3_+$ as the parameter $w_7 \geq 0.49$, see figure 9.

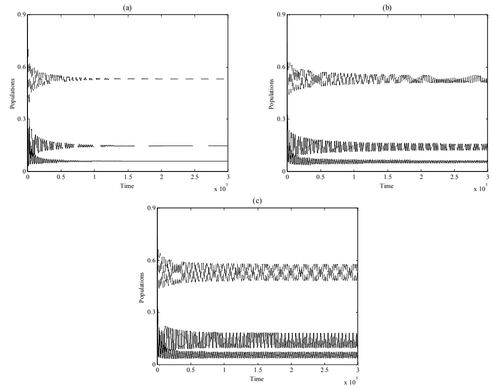

Figure 8- Hopf bifurcation in system (6) as a function of w_7 with other parameters fixed as in (65) in case 1. (a) Stable point for $w_7 = 0.24$. (b) Small periodic for $w_7 = 0.26$. (c) Periodic attractor for $w_7 = 0.28$.

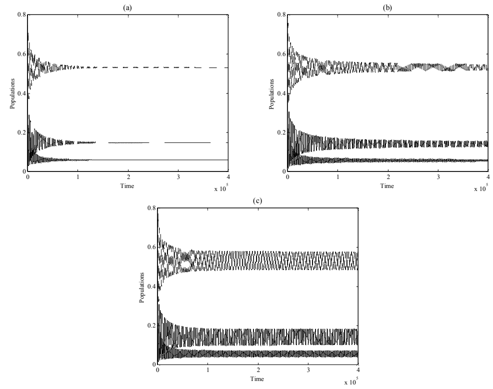

Figure 9- Hopf bifurcation in system (6) as a function of w_7 with other parameters fixed as in (65) in case 2. (a) Stable point for $w_7 = 0.46$. (b) Small periodic for $w_7 = 0.5$. (c) Periodic attractor for $w_7 = 0.52$.

According to the figures 8 and 9, it is clear that. Although increasing the parameter w_7 has destabilizing effects of the dynamics of system (6) due to transfer of the dynamical behavior from stability at positive equilibrium point in the $Int.R^3_+$ to periodic dynamic in the $Int.R^3_+$ too this means the occurrence of the Hopf bifurcation, the system (6) still persists and all the species coexist in the $Int.R^3_+$. In the following, we summarize our obtained numerical simulation results for other parameters in case 1 and case 2 as shown in table (1) and table (2) respectively.

Table 1- Numerical behavior and persistence of system (6) in case 1 as changing in a specific parameter keeping other parameters fixed as in Eq. (65)

Parameters varied in system (6)	**Numerical behavior of system (6)**	**Persistence of system (6)**
$w_2 \le 0.57$ $0.58 \le w_2 \le 1.96$ $w_2 \ge 1.97$	Approaches to stable point in $Int.R_+^3$ Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $xy-$plane	Persists Persists Not persists
$w_4 \le 1.02$ $1.03 \le w_4 \le 2.14$ $w_4 \ge 2.15$	Approaches to periodic dynamic in $yz-$plane Approaches to periodic dynamic in $Int.R_+^3$ Approaches to periodic dynamic in $xz-$plane	Not persists Persists Not Persists
$w_5 \le 0.97$ $w_5 \ge 0.98$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to periodic dynamic in $xz-$plane	Persists Not persists
$w_6 \le 0.98$ $0.99 \le w_6 \le 1.05$ $w_6 \ge 1.06$	Approaches to stable point in $yz-$plane Approaches to periodic dynamic in $Int.R_+^3$ Approaches to periodic dynamic in $xz-$plane	Not persists persists Not persists
$w_8 \le 0.23$ $0.24 \le w_8 \le 0.4$ $w_8 \ge 0.41$	Approaches to stable point in $xy-$plane Approaches to stable in $Int.R_+^3$ Approaches to periodic dynamic in $Int.R_+^3$	Not persists persists persists
$w_9 \le 0.12$ $0.13 \le w_9 \le 0.22$ $w_9 \ge 0.23$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point in $xy-$plane	Persists Persists Not persists
$w_{10} \le 0.68$ $0.69 \le w_{10} \le 0.83$ $w_{10} \ge 0.84$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point in $xy-$plane	Persists Persists Not persists
$w_{11} \le 0.39$ $0.40 \le w_{11} \le 0.58$ $w_{11} \ge 0.59$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point in $xy-$plane	Persists Persists Not persists
$w_{12} \le 2.53$ $2.54 \le w_{12} \le 3.14$ $w_{12} \ge 3.15$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point in $xy-$plane	Persists Persists Not persists
$w_{13} \le 1.47$ $1.48 \le w_{13} \le 2.17$ $w_{13} \ge 2.18$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point in $xy-$plane	Persists Persists Not persists

Table 2- Numerical behavior and persistence of system (6) in case 2 as changing in a specific parameter keeping other parameters fixed as in Eq. (65)

Parameters varied in system (6)	Numerical behavior of system (6)	Persistence of system (6)
$w_2 \le 0.65$ $0.66 \le w_2 \le 1.53$ $1.54 \le w_2 \le 1.70$ $w_2 \ge 1.71$	Approaches to stable point in $Int.R_+^3$ Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point in $xy-$plane	Persists Persists Persists Not persists
$w_4 \le 1.01$ $1.02 \le w_4 \le 2.56$ $w_4 \ge 2.57$	Approaches to periodic dynamic in $yz-$plane Approaches to periodic dynamic in $Int.R_+^3$ Approaches to periodic dynamic in $xz-$plane	Not persists Persists Not Persists
$w_5 \le 0.98$ $w_5 \ge 0.99$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to periodic dynamic in $xz-$plane	Persists Not persists
$w_6 \le 0.98$ $0.99 \le w_6 \le 1.06$ $w_6 \ge 1.07$	Approaches to stable point in $xy-$plane Approaches to periodic dynamic in $Int.R_+^3$ Approaches to periodic dynamic in $xz-$plane	Not persists persists Not persists
$w_8 \le 0.27$ $0.28 \le w_8 \le 0.47$ $w_8 \ge 0.48$	Approaches to stable point in $xy-$plane Approaches to stable in $Int.R_+^3$ Approaches to periodic dynamic in $Int.R_+^3$	Not persists persists persists
$w_9 \le 0.1$ $0.11 \le w_9 \le 0.22$ $w_9 \ge 0.23$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point in $xy-$plane	Persists Persists Not persists
$w_{10} \le 0.27$ $0.28 \le w_{10} \le 0.63$ $w_{10} \ge 0.64$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point in $xy-$plane	Persists Persists Not persists
$w_{11} \le 0.23$ $0.24 \le w_{11} \le 0.44$ $w_{11} \ge 0.45$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point in $xy-$plane	Persists Persists Not persists
$\overline{w}_{12} \le 0.03$ $0.04 \le \overline{w}_{12} \le 0.48$ $\overline{w}_{12} \ge 0.49$	Approaches to stable point in $xy-$plane Approaches to stable point in $Int.R_+^3$ Approaches to periodic dynamic in $Int.R_+^3$	Not persists Persists Persists
$\overline{w}_{13} \le 0.57$ $\overline{w}_{13} \ge 0.58$	Approaches to stable point in $Int.R_+^3$ Approaches to periodic dynamic in $Int.R_+^3$	Persists Persists
$a_1 \le 1.39$ $1.40 \le a_1 \le 3.16$ $a_1 \ge 3.17$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point in $xy-$plane	Persists Persists Not persists
$d_1 \le 1.17$ $1.18 \le d_1 \le 2.22$ $d_1 \ge 2.23$	Approaches to periodic dynamic in $Int.R_+^3$ Approaches to stable point in $Int.R_+^3$ Approaches to stable point in $xy-$plane	Persists Persists Not persists

9. Conclusions and Discussions

In this paper, a mathematical model consisting of two harmful phytoplankton interacting with a single zooplankton has been proposed and analyzed. It is assumed that the two phytoplankton producing a toxin substance as a defensive strategy against the predation by zooplankton. The effect of toxin producing plankton on the dynamical behavior of phytoplankton-zooplankton is considered. Two different scenarios of the distribution of the toxin substance, through Holling type-I and Holling type-II, are studied. In both the cases, the dynamical behavior of system (6) has been investigated locally as well as globally. The conditions for the system (6) to be persists have been derived. The occurrence of local bifurcation as well as Hopf bifurcation in system (6) is investigated. Finally the effects of changing the parameters on the dynamics of system (6) are studied numerically and the trajectories of the system are shown in the form of figures. According to these figures the following conclusions are obtained.

1. Gradually increasing the parameters w_1, w_9, w_{10}, w_{11}, w_{12} and w_{13} in both the cases, which stand for half saturation constant, natural death rate of zooplankton, the liberation rates of toxin substance from first and second phytoplankton, and the maximum ingestion rates of zooplankton to the toxin from the first and second phytoplankton respectively, have stabilizing effect on the dynamics of system (6) and keep the system persist. However, further increasing for these parameters causes extinction of zooplankton and hence the system (6) losing the persistence. Consequently, hopf bifurcation may occurs as the above parameters decrease passing through a critical value.

2. Similar results are obtained as those stated in the above point, if we increase the parameters a_1 and d_1, which stand for the maximum zooplankton ingestion rates to the toxin produced from the first and second phytoplankton, in case 2.

3. The intrinsic growth rate of the second phytoplankton represented by w_3 plays a vital role on the coexistence of all the species in system (6). However increasing this parameter above a critical value causes extinction of the first phytoplankton.

4. The parameters w_7 and $\overline{w}_{13}$ (in case 2), which represent the conversion rate of zooplankton from the first phytoplankton and the half saturation constant of zooplankton to the toxin produced from the second phytoplankton, have destabilizing effect on the dynamics of system (6). In fact as changing these parameters the system (6) still persists. They are Hopf bifurcation parameters of system (6) too.

5. Increasing the parameter w_2 that represents the preference rate of zooplankton causes destabilizing of system (6) in the $Int.R_+^3$ first. However further increasing to this parameter leads to extinction of zooplankton in case 1, while the system return to stability at the positive equilibrium point first and then goes to extinction of zooplankton in case 2.

6. The parameters w_4 and w_6, which represent the intra-specific competition and the maximum attack rate of the second phytoplankton, represent critical parameters for the coexistence of all the species. Indeed increasing those parameters slightly leads to persistence of system (6). However, further increasing for them cause extinction in the second phytoplankton.

7. Keeping the inter-specific competition parameter represented by w_5 sufficiently small make the system persists. However, increasing this parameter more than a critical value causes losing the persistence due to extinction of the second phytoplankton.

8. Finally, the parameters w_8 and $\overline{w}_{12}$ (in case 2), which represent the conversion rate of zooplankton from the second phytoplankton and the half saturation constant of zooplankton to the toxin produced from the first phytoplankton, play a vital role on the coexistence of all the species by keeping the system (6) persists as they increase. In fact they have destabilizing effect on the dynamics of the system.

References

1. Falkowski, P.G. **1984**. The role of phytoplankton photosynthesis in global biogeochemical cycles. Photosynth. Res., 39, pp:235–258.
2. Truscott, J.E., Brindley, J. **1994**. Ocean plankton populations as excitable media. Bull. Math. Biol., 56, pp:981–998.
3. Hallegraeff, G.M. **1993**. A review of harmful algal blooms and the apparent global increase. Phycologia, 32, pp:79–99
4. Graneli, E., et al. **1989**. From anoxia to fish poisoning: the last ten years of phytoplankton blooms in Swedish marine waters. In: Cosper, E.M., et al. (Eds.),

Novel Phytoplankton Blooms. Springer-Verlag, pp:407–427.

5. Shimizu, Y. **1989**. Toxicology and pharmacology of red tides: an overview. In: Red Tides: Biology, Environmental Science and Toxicology. Proceedings of the First International Symposium on Red Tides. Elsevier, pp:17–22.

6. Yasumoto, T., Murata, M., **1993**. Marine toxins. *Chem. Rev.,* 93, pp:897–909.

7. Okaichi, T., Nishio, S. **1976**. Identification of ammonmia as the toxic principle of red tide of Noctiluca miliaris. Bull. Plankton Soc. Jpn., 23, pp:25–30.

8. Chattopadhyay, J., Sarkar, R.R., Mandal, S., **2002a.** Toxin-producing plankton may act as a biological control for planktonic bloomsfield study and mathematical modeling. *Journal of Theor. Biol*., 215, pp:333–344.

9. Sarkar, R.R., Pal, S., Chattopadhyay, J. **2005**. Role of two toxin-producing plankton and their effect on phytoplankton–zooplankton system – a mathematical study supported by experimental findings. BioSystems, 80: pp:11–23.

10. Smayda, T.J. **1992**. Global epidemic of noxious phytoplanktonb blooms and food chain consequences in large ecosystems.

11. In: Sherman, K.L., et al. (Eds.), Food Chains, Models and Management of Large *Marine Ecosystem. Westview,* pp:257–307.

12. Burkholder, J.M., Glasgow Jr., H.B. **1995**. Interactions of a toxic estuarine dinoflagellate with microbial predators and prey. Arch. Protistenkd., 145, pp:177–188.

13. Lewitus, A.J., et al. **1995**. Discovery of the “phantom” dinoflagellate in Chesapeake Bay. Estuaries, 18, pp:373–378.

14. Chattopadhyay, J., Sarkar, R.R., El abdllaoui, A. **2002b.** A delay differential equation model on harmful algal blooms in the presence of toxic substances. *IMA Journal of Math. Appl. Med. Biol*., 19, pp:137–161.

15. Chattopadhyay, J., Sarkar, R.R., Pal, S., **2004**. Mathematical modeling of harmful algal blooms supported by experimental findings. Ecol. Complexity 1, pp:225–235.

16. Sarkar, R.R., Chattopadhyay, J. **2003.** Occurrence of planktonic blooms under environmental fluctuations and its possible control mechanism – mathematical models and experimental observations. *Journal of Theor. Biol*., 224, pp:501–516.

17. Gard, T.C. and T, Hallam G. **1979.** Persistence in food web. Lotka-Volttera food chains*, Bull. Math. Biol*., 41, pp:877-891.

18. Perko, L. 2000. Differential Equations and Dynamical Systems. Third Edition, Springer Verlag, Berlin., pp:337- 338.

19. Liu W.M. **1994.** Criterion of Hopf Bifurcations without using Eigenvalues. *Journal of of Mathematical Analysis and Application*, 182, pp:250-256.