On Purely Y–Extending Modules

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Abstract
In this note we consider a generalization of the notion of a purely extending modules, defined using y– closed submodules.
We show that a ring R is purely y – extending if and only if every cyclic nonsingular R – module is flat. In particular every nonsingular purely y extending ring is principal flat.
Key words : Pure submodule ; y – closed submodule ; Extending modules

مقاسات التوسع النقي من النمط *y*

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الخلاصة
تم في هذا البحث اعطاء تعميم لمفهوم مقاسات التوسع النقية باستخدام تعريف المقاسات الجزئية المغلقة من النمط y برهنا ان الحلقة R تكون حلقة توسع نقي من النمط y اذا وفقط اذا كل مقاس دائري غير شاذ على الحلقة R يكون مقاساً مسطحاً. بشكل خاص كل حلقة توسع نقي من النمط y غير شاذة تكون حلقة مسطحة رئيسياً.

Introduction:
Throughout R will be an associative ring with identity and all modules will be unital left R – modules. A sub module N of an R- module M is said to be closed if it has no proper essential extension in M.
A module M is said to be extending if provided every closed submodule of M is a direct summand, see for example [1-3].
A submodule N of an R – module M is said to be an y – closed submodule of M provided $\frac{M}{N}$ is nonsingular, see [4]. Clearly that every y-closed is closed. The converse is true if M is nonsingular.
A submodule N of an R – module M is said to be pure in M if IN = IM $\cap$ N, for every finitely generated ideal I in R, see [5].
Clearly that every direct summand is pure, but the converse is not true, see [5].
A module M is said to be purely extending if every closed submodule of M is a pure submodule, see [3].
In this paper we define purely y-extending modules as a generalization of purely extending modules.
The notation K $\subseteq_e$ M indicates that K is an essential submodule of M.

1- Purely y-Extending Modules.

Definition 1.1: an R-module M is called purely y-extending if every y-closed submodule of M is a pure submodule of M.
Trivially, every purely extending is purely y-extending,
The converse is not true as the following example shows :

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Example 1.2: Consider the module M = $Z_8 \oplus Z_2$ as a Z-module. Since M is singular, then M is the only y-closed submodule of M and hence M is purely y-extending module. Now let A = ((2,1)) be the submodule generated by (2,1).It is easily checked that A is closed in M. But A is not pure in M, where (4,0) = 4(1,0) ∈ 4($Z_8 \oplus Z_2$) ∩ A, but (4,0) ∉ 4A = (0,0).
Thus M is not purely extending.
In the following two Lemmas We recall some basic properties of the pure submodules that are relevant to our work, for more details, see [6], [5].
Lemma1.3: Let M be an R-module and let A and B be submodules of M such that A ⊆ B.
1- if A is pure in B and B is pure in M, then A is pure in M.
2- if A is pure in M, then A is pure in B.
3- if B is pure in M, then $\frac{B}{A}$is pure in $\frac{M}{A}$.
4- if A is pure in M and $\frac{B}{A}$ is pure in $\frac{M}{A}$, then B is pure in M.
Lemma 1.4: let M = $\oplus_{i\in I} M_i$, where M_i is a submodule of M, ∀i∈I and let W_i be a sub module of M_i, ∀i∈I. Then $\oplus_{i\in I} W_i$ is pure in M if and only if W_i is pure in M_i, ∀i∈I.
The following result shows that purely y – extending modules behave like purely extending in terms of direct summands
Lemma 1.5: Any direct summand of purely y-extending module is purely y-extending module.
Proof : let M = A ⊕ B, for some submodules A and B of M
Let K be a y-closed submodule of A. Since
$\frac{M}{K\oplus B} = \frac{A\oplus B}{K\oplus B} \simeq \frac{A}{K}$,
Then K ⊕ B is a y-closed submodule of M.So that K ⊕ B is a pure submodule of M. Thus, K is a pure submodule of A.
Proposition 1.6: Any y-closed submodule of a purely y-extending module is a purely y-extending.
Proof: Let A be a y-closed submodule of M and K be a y-closed submodule of A. Now consider the following exact sequence

$$0 \rightarrow \frac{A}{K} \xrightarrow{i} \frac{M}{K} \xrightarrow{\pi} \frac{\frac{M}{K}}{\frac{A}{K}} \rightarrow 0,$$ where i is the inclusion map and π is the natural epimorphism. Since $\frac{\frac{M}{K}}{\frac{A}{K}} \simeq \frac{M}{A}$, is nonsingular and $\frac{A}{K}$ is non singular then K is y-closed in M by [4]. So K is a pure in M and hence K is pure in A.
Let M be an R-module. Recall that M is said to have the purely intersection property (briefly PIP) if the intersection of any two pure submodules of M is pure in M, see[٧].
Proposition 1.7: Let R be a nonsingular ring and A be a pure submodule of a purely y-extending module M. If M has the PIP, then A is a purely y-extending module.
Proof : Let K be a y-closed submodule of A. Then there exists a y-closed submodule B in M such that K ⊆ B and $\frac{B}{K}$ is singular, by (prop 2.3 of [4]) So B is a pure in M. Since M has the PIP, then A ∩ B is pure in M. Clearly that $\frac{A\cap B}{K}$ = $\frac{A}{K}\cap\frac{B}{K}$. But $\frac{A}{K}$ is non singular and $\frac{B}{K}$ is singular, therefore A ∩ B = K. Thus K is pure in M and hence K is pure in A.

Proposition 1.8: If an R-module M is purely y-extending and A is a y-closed submodule of M, then $\frac{M}{A}$ is purely y-extending.
Proof : Let $\frac{B}{A}$ be a y-closed submodule of $\frac{M}{A}$. Since $\frac{\frac{M}{A}}{\frac{B}{A}} \simeq \frac{M}{B}$, then B is pure in M. Thus $\frac{B}{A}$ is pure in $\frac{M}{A}$.
Before we give our next result, we need the following :

Remark 1.9[4,p.49] Let A be a submodule of an R-module M. By Zorns Lemma, there is a smallest y-closed submodule H of M containing A called the y-closure of A in M {we denote it by A^{-y}}.

Proposition 1.10: An R-module M is purely y-extending if and only if A^{-y} is pure in M, for every submodule A of M.
Proof : Let M be purely y-extending and Let A be a sub module of M.Since A^{-y} is y-closed in M, then A^{-y} is pure in M.
The converse, Let A be a y-closed sub module of M, Then A^{-y} = A. Thus A is pure in M.

Theorem 1.11: An R-module M is purely y-extending if and if A ∩ M is pure in M, for every direct summand A of E(M) the injective hull of M with A ∩ M is y-closed in M.
Proof : Let A be a y-closed submodule of M and B be a relative complement of A in M. Thus A ⊕ B $\subseteq_e$ M. By [4],A ⊕ B $\subseteq_e$E(M) and hence E(A) ⊕ E(B) = E(A⊕B) = E(M). Since A=A∩M $\subseteq_e$ E(A) ∩ M and hence $\frac{E(A)\cap M}{A}$ is singular by [4]. But $\frac{E(A)\cap M}{A} \subseteq \frac{M}{A}$ and $\frac{M}{A}$ is non singular,

therefore A = E(A) ∩ M. By our assumption, A is pure in M.
The converse is clear.
Recall that an R-module M is a flat R-module if IM ≃ I ⊗M, for every finitely generated ideal I of R, see [8], [9].
Before we give our next result, we need the following theorem.

Theorem 1.12[9] Let M be any R-module and P a submodule of M :
1) if $\frac{M}{P}$ is a flat R-module, then P is a pure submodule of M.
2) if M is a flat R- module, then $\frac{M}{P}$ is a flat R-module if and only if P is a pure sub module of M.
3) if P is a pure submodule of a flat R-module M, then P is a flat R-module.

Proposition 1.13 : Let M be an R – module such that for any direct summand A of the injective hull E(M) of M with A ∩ M is y-closed in M, A + M is flat. Then M is purely y-extending module.
Proof : Let A be a direct summand of E(M) with A ∩M is y-closed in M. Consider the following short exact sequences

$$0 \to A \cap M \xrightarrow{i_1} M \xrightarrow{f_1} \frac{M}{A \cap M} \to 0$$

$$0 \to A \xrightarrow{i_2} A+M \xrightarrow{f_2} \frac{A+M}{A} \to 0$$

Where i_1, i_2 are the inclusion maps and f_1, f_2 are the natural epimorphisms. Since A is a direct summand of E(M), then A is a direct summand of A + M and hence the second sequence is splits.But A + M is flat, so $\frac{A}{A \cap M} \simeq \frac{A+M}{A}$ is flat.
Thus A ∩ M is pure in M.
Recall that an R-module M is called a multiplication R-module if N = (N:M)M, for every sub module N of M, see [10].

Proposition 1.14: Let M be a faithful multiplication R-module.
If R is purely y-extending module, then M is purely y-extending module.
Proof : Let A be a y-closed submodule of M. Since M is multiplication, then A = [A : M]M. claim that (A:M) is y-closed in R, assume not, so there exits r ∈ R such that r + [A:M] ≠ [A:M] and ann (r + [A:M]) $\subseteq_e$R.
Then there exists m_0∈ M such that rm_0∉A. One can easily show that ann (r + [A:M]) ⊆ann (rm_0+ A). Thus ann (rm_0+ **A)**$\subseteq_e$**R.** But $\frac{M}{A}$ is non singular, so rm_0+ A = A which is a contradiction since R is purely y-extending, then [A:M] is pure in R.
Now let I be a finitely generated ideal of R, then IA = I (A:M)M = (I ∩ (A:M))M = IM ∩ (A:M)M = IM ∩ A, by [10]. Thus A is a pure submodule of M.

2-The direct sum of purely y-extending modules

A ring R is called PF if each of its principal ideals is flat, see [11].

Theorem 2.1: A ring R is purely y-extending if and only if every cyclic nonsingular R-module is flat. In particular every nonsingular purely extending ring is principal flat (PF).
Proof : assume R is purely y-extending and let M = Ra be a cyclic nonsingular R-module generated by a. Define f : R → Ra by f(n) = ra. It is easily seen that f is an epimorphism.
Thus $\frac{R}{Kerf} = \frac{R}{ann(a)} \simeq$ Ra and hence Kerf is pure in R.
But R is flat, therefore Ra is flat, by Th. 1.12–2.
The converse let C be a y-closed ideal of R. Hence $\frac{R}{C}$is cyclic and nonsingular. By our assumption $\frac{R}{C}$ is flat.
Thus C is pure in R, by Th 1.12 – 2.
It is known that there exists a non – singular R-module M such that M is not flat, see Prop 5.16 of [8].

Theorem 2.2: Let R be a ring, then R ⊕ R is purely y-extending if and only if every nonsingular two generated R-module is flat.
Proof : Let M = $Rm_1 + Rm_2$ be a nonsingular R-module and Let f = R ⊕ R → M be a map defined by $f(r_1, r_2) = r_1m_1 + r_2m_2$.
Clearly that f is an epimorphism and hence $\frac{R \oplus R}{kerf} \simeq$ M. Thus kerf is y-closed in R ⊕ R. By our assumpitation kerf is pure in R ⊕ R. But R ⊕ R is flat, therefore M is flat, by Th 1.12 – 2.
The converse, Let C be a y-closed submodule of R ⊕ R. Hence $\frac{R \oplus R}{C}$ is a nonsingular two generated R-module. Thus $\frac{R \oplus R}{C}$ is flat and hence C is pure in R.
By the same argument, we can prove.

Theorem 2.3: Let R be a ring and I be a finite index set, then $\oplus_I R$is purely y-extending if and only if every non singular I – generated R-module is flat.

Recall that a ring R is called a flat ring if every ideal of R is flat, see [8].

Proposition 2.4: Let R be a commutative integral domain.
Then the following statements are equivalent
1) R is a flat ring
2) R ⊕ R is extending module.
3) R ⊕ R is a purely extending.
4) R ⊕ R is a purely y-extending.
5) For each n ∈ N, $\oplus_n R$ is an extending.
6) For each n ∈ N, $\oplus_n$ R is a purely extending.
7) For each n ∈ N, $\oplus_n R$ is a purely y-extending.

Proof : since R is nonsingular, then clearly that (3) ↔ (4) and (6) ↔ (7)
(1) ↔ (2) ↔ (3) ↔ 5 ↔ 6, see prop 1.6, [3].

Proposition 2.5: Let R be a ring. The following are equivalent
1) $\oplus_{\alpha\in\Lambda} R$ is purely y-extending, for every index set Λ ;
2) every projective *R*-module is purely y-extending ;
3) every nonsingular R-module is flat.

Proof : (1) → (2) Let M be a projective R-module. Then there exists an apimorphism f : $\oplus_I R \to$ M, for some index set, by [12]. But M is projective, then by [12] the following short exact sequence is splits. $0 \to \ker f \xrightarrow{i} \oplus_I R \xrightarrow{f} M \to 0$, where i is the inclusion map. So $\oplus_I R \simeq \ker f \oplus M$ by [12]. Since $\oplus_I R$ is purely y-extending, then M is purely y-extending.
(2) → (1) clear.
(1) → (3) Let M be a nonsingular R-module. By [12], there is a free R-module $\oplus_I R$ and an epimorphism f : $\oplus_I R \to$ M.
Thus $\frac{\oplus_I R}{\ker f} \simeq$ M and hence kerf is a y-closed ideal of $\oplus_I R$. Thus M is flat, by Th 1.12-2.
(3) → (1) Let C be a y-closed submodule of $\oplus_\Lambda R$ and hence $\frac{\oplus_\Lambda R}{C}$ is non singular. By our assumption $\frac{\oplus_\Lambda R}{C}$ is flat. Thus C is pure in $\oplus_\Lambda R$, by Th 1.12 -1.

Proposition 2.6 : Let M = $\oplus_{\alpha\in\Lambda} M_\alpha$ be an R-module such that every y-closed submodule of M is fully invariant, then M is purely y-extending if and only if M_α is purely y-extending, ∀α∈Λ.

Proof : →) clear by prop 1.5
←) Let A be a y-closed submodule of M. For each i ∈I, Let $\pi_\alpha : M \to M_\alpha$ be the projection map. Now Let x ∈ A, then x = $\sum_{i\in I} m_\alpha$, $m_\alpha \in M_\alpha$ and m_α = 0, for all except a finite number of α∈Λ. Since A is fully invariant, then $\pi_\alpha(x) = m_\alpha \in A\cap M_\alpha$. Thus A = $\oplus_{\alpha\in\Lambda}(A\cap M_\alpha)$. Since $\frac{M}{A} = \frac{\oplus_\Lambda M_\alpha}{\oplus_\Lambda (A\cap M_\alpha)} \simeq \oplus_\Lambda \frac{M_\alpha}{A\cap M_\alpha}$. So $\frac{M_\alpha}{A\cap M_\alpha}$ is nonsingular, ∀α∈Λ and hence $A\cap M_\alpha$ is pure in M_α, for each α∈Λ. Now Let I be a finitely generated ideal of R.
IA = I $(\oplus_\Lambda (A\cap M_\alpha)$ = $\oplus_\Lambda I (A\cap M_\alpha)$ = $\oplus_\Lambda$ $((IM_\alpha)\cap(A\cap M_\alpha)) = (\oplus_\Lambda IM_\alpha)\cap \oplus_\Lambda (A\cap M_\alpha)$
= I $(\oplus_\Lambda M_\alpha)\cap \oplus_\Lambda (A\cap M_\alpha)$
= IM ∩ A
Thus A is pure in M and M is purely y-extending.

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