On The Static Reliability Of Binary-State And Multi-State K-Out-Of-N:G Systems

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Abstract

The k-out-of-n:G (or k/n:G) system structure is a very popular of redundancy in fault-tolerant systems, with wide applications in so many fields. This paper presents two states of multi-state k/n:G systems. The first part, we present the definition that introduced by Al-Neweihi et al [1], where the k^{th} values are the same with respect to all system states and we show that there exists an alternative equivalent definition to Al-Neweihi's definition. In the second part of this paper we give more general definition proposed by Huang et al [2], where it allows different k^{th} values with respect to different system states and we provide there exists an equivalent definition to Huang's definition when the k^{th} values are increasing. We show it is simply to generalize the mathematical theory of static reliability of binary k/n:G systems to the multistate k/n:G systems. Several new results are established concerning the evaluation of the stochastic of the system static performance measures, together with their computer algorithm (Belfour algorithm).

حول المعولية الساكنة لنظم k-out-of-n:G ثنائية و متعددة الحالة

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الخلاصة

تعد المنظومة التي يتبع هيكلها الهندسي k-out-of-n:G (او k/n:G) واحدة من اهم النظم الشائعة الاستخدام وذلك لتمتعها بخاصية الفائض، اي السماح لبعض مركباتها بالفشل دون احداث اي تغير نوعي في عمل المنظومة نفسها. يقدم هذا البحث حالتان من الانظمة k/n:G متعددة الحالات. في الجزء الاول من هذا البحث نقدم التعريف الذي عرف من قبل Al-Neweihi [1] حيث قيم k متساوية عند كل حالة. ثم برهنت وجود تعريف مكافئ لتعريف Al–Neweihi . في الجزء الثاني من هذا البحث نقدم تعريف Huang [2] الاكثر شمولية حيث قيم k تختلف باختلاف الحالات ثم برهنا ان هنال تعريف مكافئ لتعريف Huang عندما تكون قيم k متزايدة. وبناءا على هذين التعريفين الجديدين تم التوسع رياضياتيا لنظرية المعولية الساكنة لنظم k/n:G ثنائية الحالات الى نظم k/n:G متعددة الحالات. عدة نتائج برهنت فيما يخص ايجاد المقاييس التصادفية كذلك اعطاء خوارزمية احد هذه الاجزاء الحسابية مع البرانامج بلغة بيسك.

1. Introduction

One inherent weakness of traditional binary reliability theory is that both the system and its components are considered to take only two possible states "performance levels": working or failed. This approach represents an over simplification in many real- life situation [3] where the system and their components are capable of assuming a whole range (more than two) of levels of performance, varying from

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perfect functioning to complete failure. Thus, reliability theory must not only take into account a yes or no (functioning or failure) for each the system and its components, but the possibility of working with a slightly degraded system. This motivates so called multistate system with multistate components.

consider a system with n components, let C={1,2,…,n} denotes the set of components indices, where the system and each of its components have finite number (M+1) of distinct states, d={0,1,2,…,M} representing various level of performances such that they ranging from perfect functioning denoted by (state M) to complete failure (state 0). Let the random variable $X_i \in d$; i=1,2,…,n, be the state or performance level of component i, and $\underline{X}=(X_1,X_2,\ldots,X_n) \in d^n$ be the components state random vector. Assume that the performance of the system depends deterministically on the performance of each of its components viewed at a fixed moment of time. Hence, we can assume that a system state or performance level is a random variable determined by the function Ψ.

Definition (1.2) A multistate system with multistate components is said to be multistate monotone system (MMS) iff its structure function Ψ satisfies:

1) Ψ is a non-decreasing function in each argument,
2) $\Psi(\underline{j}) = j$; j = 1, 2, …, M, where $\underline{j}$ =(j, j,…, j).

we assume throughout this paper that $X_1,X_2,\ldots,X_n$ are stochastically independent, where the following notations are adopted, let $P_r(X_i = j) = P_{ij}$; i =1,2,…,n; j = 0, 1, 2, …, M ; $\sum_{j=0}^{M} P_{ij} = 1$, be the probability of component i is in state j. Also, let $P_r(\Psi(\underline{X}) = j) = P_j$; j = 0,1,2,…, M; $\sum_{j=0}^{M} P_j = 1$, be the probability of the system is in state j. Thus, both P_{ij} and P_j; j= 0,1,2,…, M, represents the state or performance distribution of the multistate component i; i=1,2,…,n, and the multistate system, respectively. Another important measure given by ESP [1] is called the performance function defined by: $h = E\,\Psi(\underline{X})$.

Other related vital notations are given by: $P_r(X_i \geq j) = \sum_{r=j}^{M} P_{ir} = P_i(j)$; i =1,2, …,n ; j = 0, 1,2,…,M, the probability of component i in state j or above. Also, $P_r(\Psi(\underline{X}) \geq j) = \sum_{r=j}^{M} P_r = P(j)$; j = 0,1,2,…, M, the probability of the multistate system in state j or above.

2. Simple multistate k-out-of-n:g system

Definition (2.1) A multistate system with multistate components is called a simple multistate k/n:G system, denoted by (SM) k/n:G system, iff its structure function Ψ is given by:

$$\forall\, \underline{X};\ \underline{X} \in d^n \rightarrow \Psi(\underline{X}) = X(n-k+1),$$

where $X(1) \leq X(2) \leq \ldots \leq X(n)$ is a non-decreasing arrangement of $X_1,X_2,\ldots,X_n$ and k =1,2,…,n is independent of the system performance level j ; j = 0,1,2,…,M.

Another vital property that can be derived is given by the following lemma.

Lemma(2.1) A(SM) k/n:G system with structure function Ψ is an (MMS).

Proof. By using definition (2.1), we want to show that:

1) Ψ is a non-decreasing function in each argument,
2) $\Psi(\underline{j}) = j$; j = 1, 2, …, M, where $\underline{j}$ = (j, j,…, j).

Now, for $\underline{X}, \underline{Y} \in d^n$, assume that $\underline{X} \geq \underline{Y}$ then for every k; k=1,2,…,n, we have X(n−k+1) is equal or larger to (n-k+1) arguments X_i of the vector $\underline{X}$. Since $\underline{X} \geq \underline{Y}$, so X(n−k+1) is equal or larger to (n−k+1) arguments Y_i or the vector $\underline{Y}$, equivalently to Y(n−k+1).
Hence, $\Psi(\underline{X}) = X(n-k+1) \geq Y(n-k+1) = \Psi(\underline{Y})$. The proof of (1) is completed. Also, let $\underline{j}$ =(j, j,…, j) ; j = 1,2, …, M. From definition (3.2.1), we have:
$X(1) = X(2) = \ldots = X(n) = j$; ∀ j=1,2,…,M.
So, $\Psi(\underline{j}) = j$; ∀ j = 1, 2, …, M

This lemma expresses that improving component performance of a (SM) k/n:G system cannot harm the system, and if all components are in a certain state the system itself will also be in this state. In particular, $\Psi(\underline{0})=0$ and $\Psi(\underline{M})=M$ merely states if all components are in the complete failure then the system is in the complete failure state, and if all components are functioning perfectly, the system functions perfectly.

Until (2003) a little have been said about the definition of a (SM)k/n:G system. At that time, Huang et al [4] investigate extensively this definition and they established the following important property that we represented in the following lemma.

Lemma(2.2) A(SM) k/n:G system with structure function Ψ is in state j or above iff at least k components are in state j or above, for each *j* ; *j* = 1,2,…,M. Equivalently, for each *j*;*j*= 1,2,…,M, we have: $\Psi(\underline{X}) \geq j$ iff ∃ at least k components are in state ≥ j.

Proof. By the definition of a (SM) k/n:G system, we have the state of the system is determined by the state of the best k components. The proof is completed.

With the aid of the above lemma we suggest another alternative equivalent definition to a (SM) k/n:G system. To do this, we need the following notations, let for all *j*; *j* = 1, 2, …, M, and for all $\underline{X} \in d^n$; $\underline{X}_j=(X_{1j}, X_{2j}, \ldots, X_{nj})$ be a random state vector of a binary indicator functions X_{ij} ; *i* = 1, 2,…, n, such that:

$$X_{ij} = \begin{cases} 1 \text{ iff } X_i \geq j \\ 0 \text{ iff } X_i < j, \end{cases}$$

This shows, with respect to any given component of level j; j =1, 2,…, M, the states of each component *i*; *i* =1, 2,…,*n*, are divided in two separate groups: the functioning states {j, j+1, …,M } and the failure states {0,1, …, j−1 }, i.e. component i working if $X_i \geq j$ (X_{ij}=1) and failed if $X_i < j$ (X_{ij}= 0). Next, let Φ be a binary structure function defined on $\underline{X}_j$ such that for each j; j = 1, 2,…, M,

$$\Phi(\underline{X}_j) = \begin{cases} 1 \text{ iff } \sum_{i=1}^{n} X_{ij} \geq k, k = 1,2,\ldots,n, \\ 0 \text{ iff } \sum_{i=1}^{n} X_{ij} < k. \end{cases}$$

In other words, $\Phi(\underline{X}_j)$=1 iff at least k components in state ≥ j .

Thus, we conclude that Φ constitutes a binary structure function of a binary k/n:G system, and it is the same structure function for each level of performance *j*; *j* = 1, 2, …, M.

From this construction, we suggest the following definition.

Definition(2.2) A multistate system with multistate components and a structure function Ψ is said to be a (SM) k/n:G system iff for all *j*; *j* = 1, 2, …, M, and all $\underline{X} \in d^n$ and $\underline{X}_j \in S^n$ there exists a binary k/n:G system with structure function Φ such that the following relation is satisfied, $\Psi(\underline{X}) \geq j \Leftrightarrow \Phi(\underline{X}_j) = 1$.

Based on this definition it follows that the structure function Ψ of a (SM) k/n:G system specified by ESP is very closely related to the structure function of a binary k/n:G system, and exploiting this relationship makes it easy to generalize results from binary k/n:G system to a (SM) k/n:G system.

The following gives a unique correspondence between the two structure functions Ψ and Φ.

Lemma (2.3) Given a (SM) k/n:G system, then for every $\underline{X} \in d^n$ we have:

1) $\Psi(\underline{X}) < j \Leftrightarrow \Phi(\underline{X}_j) = 0$; j = 1,2,…, M,
2) $\Psi(\underline{X}) = j \Leftrightarrow \Phi(\underline{X}_1) = 0$,
3) $\Psi(\underline{X}) = j \Leftrightarrow \Phi(\underline{X}_j) - \Phi(\underline{X}_{j+1}) = 1$; j = 1, 2,…, M−1,
4) $\Psi(\underline{X}) = M \Leftrightarrow \Phi(\underline{X}_M) = 1$,
5) $\Psi(\underline{X}) = \sum_{j=1}^{M} \Phi(\underline{X}_j)$.

Proof. We give the proof of (3) and (5) only, others are follow immediately from the above definition.

3) Since by lemma (2.1) Φ is a non-decreasing function, then

$\Phi(\underline{X}_j) - \Phi(\underline{X}_{j+1}) = 1 \Leftrightarrow \Phi(\underline{X}_j) = 1 \wedge \Phi(\underline{X}_{j+1}) = 0 \Leftrightarrow \Psi(\underline{X}) \geq j \quad \Psi(\underline{X}) < j+1 \Leftrightarrow \Psi(\underline{X}) \geq j \wedge \Psi(\underline{X}) \leq j \Leftrightarrow j \leq \Psi(\underline{X}) \leq j \Leftrightarrow \Psi(\underline{X}) = j$.

5) For any $\underline{X}$; $\underline{X} \in d^n$, we have:

$\underline{X}_1 \geq \underline{X}_2 \geq \ldots \geq \underline{X}_M$

and since Φ is a non-decreasing function (see lemma (2.1)) we have:

$\Phi(\underline{X}_1) \geq \Phi(\underline{X}_2) \geq \ldots \geq \Phi(\underline{X}_M)$.

Thus, if $\Phi(\underline{X}_j)$=1 then $\Phi(\underline{X}_r)$=1;for r=1,2,…, j−1, and if $\Phi(\underline{X}_j)$=0 then $\Phi(\underline{X}_s)$=0 ; for s= j+1, j+2, …,M. The proof follows from (3).

Next, in terms of the static performance distributions of multistate components P_{ij} or P_i(j); *i* =1,2,…, *n*; j =1, 2,…,M, we consider the problem of evaluation the static stochastic performance measures of a (SM) k/n:G system, namely, the exact and bounds of :

1) The system performance distribution P_j or *P*(j) ; j= 0,1,2,…, M,
2) The system performance function *h*= E $\Psi(\underline{X})$.

To do this, we begin first with the two easy cases (k = *n*) and (k =1) given in the following lemma, where, R(k, n, j) = $P_r(\Psi(\underline{X}) \geq j) = P_j$; j = 0, 1, 2,…, M.

The next lemma evaluates the static performance measures for a general (SM) k/n:G system, where: R_j(k, n, 1) = $P_r(\Phi(\underline{X}_j) = 1)$; j = 1,2,…,M, is the reliability function of a binary k/n:G system with the structure function Φ defined on $\underline{X}_j$, $\underline{X}_j \in S^n$.

Lemma (2.4) Given a (SM) k/n:G system with structure function Ψ having a binary structure function Φ of a binary k/n:G system, then:

1) For each j, j =1,2,…,M, we have: R(k, n, j) = $P_r(\Psi(\underline{X}) \geq j) = P_r(\Phi(\underline{X}_j) = 1) = E\,\Phi(\underline{X}_j)$

= $R_j(k,n,1)$ =$R_j(\underline{P}(j))$. Where, $\underline{P}(j) = (P_1(j), P_2(j),\ldots, P_n(j))$.

2) $P_0 = P_r(\Psi(\underline{X}) = 0) = 1- R_1(k,n,1)$,

3) $P_j=P_r(\Psi(\underline{X})=j)=R_j(k,n,1)-R_{j+1}(k,n,1)$; $j=1,2,\ldots, M-1$,

4) $P_M = P_r(\Psi(\underline{X}) = M) = R_M(k,n,1)$,

5) $h= \sum_{j=1}^{M} R_j(k,n,1) = \sum_{j=1}^{M} R\ (k,n,j)$.

proof. 1) Follows definition (2.2).

2) $P_0 = P_r(\Psi(\underline{X}) = 0) = P_r(\Phi(\underline{X}_1) = 0) = 1- P_r(\Phi(\underline{X}_1) = 1) = 1-R_1(k, n, 1)$.

3) From lemma (2.3), we have:

$P_j = P_r\ (\Psi(\underline{X}) = j) = P_r(\ [\Phi(\underline{X}_j) - \Phi(\underline{X}_{j+1})] = 1) = E\ [\Phi(\underline{X}_j) - \Phi(\underline{X}_{j+1})] = E\Phi(\underline{X}_j) - E\Phi(\underline{X}_{j+1})$
$= R_j(k, n, 1) - R_{j+1}(k, n, 1)$.

4) $P_M=P_r(\Psi(\underline{X})=M)= P_r\ (\Phi(\underline{X}_M)= 1)=R_M(k, n,1)$.

5) $h=E\ \Psi(\underline{X})=E \sum_{j=1}^{M} \Phi(\underline{X}_j) = \sum_{j=1}^{M} E\ \Phi(\underline{X}_j)$.

This lemma shows clearly that the stochastic performance measures of both systems, the (SM) k/n:G and the binary k/n:G, are closely related. This relation makes it easy to evaluate the performance distribution and the performance function of the (SM) k/n:G system from the binary k/n:G system, simply by employing the formula: $R(k, n, j) = P_r(\Psi(\underline{X}) \geq j) = R_j(k, n,1) = P_r(\Phi(\underline{X}_j) = 1)$; j =1,2,…, M.

3. Generalized multistate k-out-of-n:g systems

Huang et al [2] proposed the following definition of generalization

Definition (3.1) A multistate system with multistate components is called a generalized multistate k/n:G system iff for each j; j = 1, 2,…, M, $\Psi(\underline{X}) \geq j$ if there exists an integer value l; l = j, j+1, …, M, such that at least k_l components are in state $\geq l$.

Note that, in this definition, the k_l do not have to the same for different system states j;j =1,2,…,M. This means that the structure of this system can be different for different system state levels. Generally speaking k_l values are not necessarily in a monotone ordering. As a special the constant case, when k_l is a constant, that is; $k_1 = k_2 =\ldots= k_M = k$, say, the structure of the system is the same for all state levels. This reduce the definition of the generalized multistate k/n:G system to the definition of the (SM) k/n:G system.

We shall be particularly interested in the following case given by Huang, where all the concepts and results of a binary k/n:G system, again, can be easily extended.

Definition (3.2) A generalized multistate k/n:G system is called an increasing generalized multistate k/n:G system, denoted by (IGM) k/n:G system, iff: $k_1 \leq k_2 \leq \ldots \leq k_M$.

In this case, for the system to be in a higher state level ≥ j, a large number of components must be in state ≥ j. That is, an increasing requirement on the number of components that must be in a certain state or above for the system to be in a higher state or above. That is why it is called an (IGM) k/n:G system.

For an (IGM) k/n:G system we have the following lemma.

Lemma (3.1) When $k_1 \leq k_2 \leq \ldots \leq k_M$, the definition (3.2) of a generalized multistate k/n:G system is equivalent to: $\forall\ \underline{X}$; $\underline{X} \in d^n$ and $\forall$ j; j=1, 2,…, M, we have : $\Psi(\underline{X}) \geq j$ iff $\exists$ at least k_j components are in state ≥ j.

Proof. For a given state vector $\underline{X}$, let N_j be the number of components in $\underline{X}$ that are in state ≥ j, so we have: $N_j \geq N_{j+1} \geq \ldots \geq N_M$.

The definition (3.1) can be rephrased as $\Psi(\underline{X}) \geq$ j iff at least one of the following inequalities is satisfied: $N_j \geq k_j$, $N_{j+1} \geq k_{j+1}$, …, $N_M \geq k_M$.

Assume that for some p; $j \leq p \leq M$, that $N_p \geq k_p$.

Then we have: $N_j \geq N_p \geq k_p \geq k_j$.

Hence, $\Psi(\underline{X}) \geq j$ iff $N_j \geq k_j$.

Thus, based on the above lemma and as far as state level j; j=1, 2, …, M, is concerned: if at least k_j components are in state ≥ j then these components can be considered "functioning", while the system be in state ≥ j the system is considered to be "functioning". We will suggest an alternative equivalent definition to the (IGM) k/n:G system. To do this, again, let for all j; j=1, 2,…,M and for all $\underline{X}$, $\underline{X} \in d^n$:

$\underline{X}_j = (X_{1j}, X_{2j}, \ldots, X_{nj})$be a random state vector of a binary indicator functions X_{ij} ; i = 1, 2,…, n, such that:

$$X_{ij} = \begin{cases} 1 iff\ X_i \geq j \\ 0 iff\ X_i < j, \end{cases}$$

That is, the state levels of each component i; i=1,2,…,n, are divided in two separate groups: The functioning states {j, j+1,…,M} and the failure states {0,1,…, j−1}, i.e. component i working if $X_i \geq j$ (X_{ij}=1) and failed if $X_i < j$ (X_{ij}= 0). Next, let Φ_j ; j = 1, 2,…, M,be a binary structure function defined on $\underline{X}_j$ such that ;

$$\Phi(\underline{X}_j) = \begin{cases} 1 iff \sum_{i=1}^{n} X_{ij} \geq k_j, k_j = 1,2,\ldots,n, \\ 0 iff \sum_{i=1}^{n} X_{ij} < k_j. \end{cases}$$

That is, for each j; j=1, 2, …, M, Φ_j ($\underline{X}_j$) = 1 iff at least k_j components are in state ≥ j.

Notice that Φ_j constitute a binary structure function of a binary k_j/n:G system, and they have different system structure for different system state levels j; j=1, 2,…, M.

From this construction we suggest the following equivalent definition to the given (IGM) k/n:G system.

Definition (3.3) A multistate system with multistate components and a structure function Ψ is said to be an (IGM) k/n:G system iff for j; j = 1, 2, …, M, and all $\underline{X}$; $\underline{X} \in d^n$ and $\underline{X}_j \in S^n$, there exists a binary k_j/n:G system with structure function Φ_j such that the following relation is satisfied; $\Psi(\underline{X}) \geq j \iff \Phi_j(\underline{X}_j) = 1$

Before we proceed further, it is worth to mention that the above definition is consistent with the general definition of a multistate coherent system suggested by Natvig [5] were most of the theory for the traditional binary coherent system can be extended to this system. Thus, most of the following can be considered as consequences of the Natvig proposal, where it is extensively studied by Raheem [6].

The binary structure functions Φ_j; j=1, 2,…, M, are with the following property.

Lemma (3.2) Given an (IGM) k/n:G system. Then for all $\underline{X}$; $\underline{X} \in d^n$ we have:

$\Phi_j(\underline{X}_j) \geq \Phi_{j+1}(\underline{X}_{j+1})$; j=1, 2, …, M−1.

Proof. For a given $\underline{X} \in d^n$, let $\Phi_{j+1}(\underline{X}_{j+1}) = 1$, to show that $\Phi_j(\underline{X}_j) = 1$.

Since, $\Phi_{j+1}(\underline{X}_{j+1}) = 1$ iff $\sum_{i=1}^{n} X_{ij+1} \geq k_{j+1}$, and $\underline{X}_j \geq \underline{X}_{j+1}$, we have:

$$\sum_{i=1}^{n} X_{ij} \geq \sum_{i=1}^{n} X_{ij+1} \geq k_{j+1} \geq k_j .$$

Hence, $\sum_{i=1}^{n} X_{ij} \geq k_j$ or $\Phi_j(\underline{X}_j) = 1$.

The next lemma gives unique correspondence between Ψ and Φ_j; j=1,2,…, M.

Lemma (3.3) Given an (IGM) k/n:G system, then $\forall \underline{X}$, $\underline{X} \in d^n$ we have:

1) $\Psi(\underline{X}) < j \iff \Phi_j(\underline{X}_j)=0$; j =1,2,…, M,
2) $\Psi(\underline{X}) = 0 \iff \Phi_1(\underline{X}_1) = 0$,
3) $\Psi(\underline{X}) = j \iff [\Phi_j(\underline{X}_j) - \Phi_{j+1}(\underline{X}_{j+1})] = 1$; j = 1, 2,…, M−1,
4) $\Psi(\underline{X}) = M \iff \Phi_j(\underline{X}_M) = 1$,
5) $\Psi(\underline{X}) = \sum_{j=1}^{M} \Phi_j(\underline{X}_j)$.

Proof. 1) Follows from the definition (3.3).

2) $\Phi_1(\underline{X}_1) = 0 \iff \Psi(\underline{X}) < 1 \iff \Psi(\underline{X}) = 0$.

3) From lemma (3.3.2), we have:

$[\Phi_j(\underline{X}_j) - \Phi_{j+1}(\underline{X}_{j+1})] = 1 \iff \Phi_j(\underline{X}_j) = 1 \wedge \Phi_{j+1}(\underline{X}_{j+1}) = 0$.

But, $\Phi_j(\underline{X}_j) = 1 \iff \Psi(\underline{X}) \geq j$, and $\Phi_{j+1}(\underline{X}_{j+1}) = 0 \iff \Psi(\underline{X}) < j+1 \iff \Psi(\underline{X}) \leq j$.

Hence, $j \leq \Psi(\underline{X}) \leq j$ or $\Psi(\underline{X}) = j$.

4) Follows from the definition (3.3.3).

5) Since, $\Psi(\underline{X}) = j \iff \Phi_j(\underline{X}_j)= 1 \wedge \Phi_{j+1}(\underline{X}_{j+1}) = 0$, it follows from lemma (3.2) that:

$\Phi_p(\underline{X}_p) = 1$ for p= 1, 2, …, j−1, and $\Phi_s(\underline{X}_s) = 1$ for s= j+2, j+3, …, M.

Hence, we have: $\Psi(\underline{X}) = \sum_{j=1}^{M} \Phi_j(\underline{X}_j)$

From this lemma we have: starting out with Φ_j; j = 1, 2,…, M, then Ψ is uniquely determined and vice-versa.

Next, in terms of the static performance distribution of multistate components P_{ij} or P_i(j); i=1,2,…,n; j=1,2,…,M, we consider the problem of evaluation the static stochastic performance measures of an (IGM) k/n:G system, namely, the exact and bounds of :

1) The system performance distribution P_j or P(j) ; j = 0, 1, 2, …, M,
2) The system performance function h=E $\Psi(\underline{X})$.

To do this, define: $R(k_j,n,j) = P_r(\Psi(\underline{X}) \geq j) = P(j)$;j=1,2,…,M, and, $R_j(k_j, n, 1) = P_r(\Phi_j(\underline{X}_j) = 1) = P_j$; j = 1, 2, …, M.

The following lemma evaluates the static performance measures for an (IGM) k/n:G system.

Lemma (3.4) Given an (IGM) k/n:G system with structure function Ψ having a binary structure function Φ_j of a binary k_j /n:G system; j = 1, 2,…,M, then:

1) For each j; j = 1, 2, …, M, we have: $R(k_j, n, j)= P_r(\Psi(\underline{X}) \geq j) = P_r(\Phi_j(\underline{X}_j) = 1) = E\,\Phi_j(\underline{X}_j) = R_j(k_j, n, 1) = R_j(\underline{P}(j))$.
Where, $\underline{P}(j) = (P_1(j), P_2(j),…, P_n(j))$.
2) $P_0 = P_r(\Psi(\underline{X}) = 0) = 1 - R_1(k_1, n, 1)$,
3) $P_j = P_r(\Psi(\underline{X}) = j) = R_j(k_j,n,1) - R_{j+1}(k_{j+1}, n, 1)$; j = 1, 2, …, M−1,
4) $P_M = P_r(\Psi(\underline{X}) = M) = R_M(k_M\, n, 1)$,
5) $h= \sum_{j=1}^{M} R_j(k_j,n,1) = \sum_{j=1}^{M} R_j(k_j,n,j)$

proof. Similar to proof of lemma (2.4).

The above relations between the (IGM) k/n:G system and the binary k_j/n:G system makes it easy to evaluate the performance distribution and the performance function of the (IGM) k/n:G system, simply by employing the formula: $P(j) = R(k_j, n, j) = R_j(k_j, n, 1)$; j = 1, 2, …, M.

This suggests the following evaluation algorithm that can be summarized by the steps:

Step-1:Given n, M, P_{ij};i=1,2,…,n; j=0,1,2,…, M.

Step-2: Calculate P_i(j) ; i=1,2,…, n; j=1,2,…, M.

Step-3: Given k_j; j = 1, 2,…, M; treat for an state level j the P_i(j) as P_{ij} in the algorithm given in Belfore approach [7] to calculate $R_j(k_j,n, 1)$.

Step-4: The results are R(k_j, n, j); j = 1,2, …, M.

Step-5: Both the performance distribution $P_j= P_r(\Psi(\underline{X})=j)$; j=0,1,2,…,M, and the performance function h are easy to compute through:

1) $P_0 = 1-R(k_1, n, 1)$.

2) $P_j = R(k_j , n, j) - R(k_{j+1} , n, j+1)$; j = 1, 2, …, M−1.

3) $P_M = R(k_M , n, M)$.

4) $h = \sum_{j=1}^{M} R_j(k_j,n,1)$

Based on the above algorithm we give in the next computer program in “BASIC” language,

```
10  INPUT "N= "; N
20  INPUT " the maximum state level="; M
30  MM=M+1
40  DIM  D(N), PROD(N),  A(N),  S(N), G(N,M), H(N,M), B(M), C(MM)
50  FOR I = 1 TO N
60  FOR J = 1 TO M
70  PRINT "P(";I;",";J;")=";: INPUT G(I,J)
80  NEXT J
90  NEXT I
100 FOR I = 1 TO N
110 FOR J = 1 TO M
120 FOR T = J TO M
130 H(I,J) = H(I,J) + G(I,T)
140 NEXT T
150 NEXT J
160 NEXT I
170 DIM A(N+1)
180 FOR R=1 TO M
190 PRINT "K(";R;")="; INPUT K
200 ALPHA = 1
210 FOR I = 1 TO N
220 P = H(I,R)
230 D(I) = (1-P)/P
240 ALPHA = ALPHA*P
250 PROD(I) = 1
260 NEXT I
270 SUMS = 1
280 IEND = N+1
290 FOR I = 1 TO N-K
300 PRDTMP = 0
310 IEND = IEND – 1
320 FOR J = 1 TO IEND
330 JTEMP = J
340 ILOC = IEND +1-J
350 PRDTMP = PRDTMP + D(ILOC)*PROD(JTEMP)
360 PROD(JTEMP) = PRDTMP
370 NEXT J
380 S(I) = PROD(JTEMP)
390 SUMS = SUMS + S(I)
400 NEXT I
410 S = ALPHA * SUMS
420 PRINT"RELIABILITY FUNCTION = R(";K;",";N;",";R;")= ";S
430 B(R)= S
440 h = h + S
450 NEXT R
460 C(1)= 1−B (1)
470 FOR I=1 TO M−1
480 C(I+1)= B(I)−B(I+1)
490 C(MM) =B (M)
500 FOR I=1 TO MM
510 J = I−1
520 PRINT “At level “; J
530 PRINT “The performance distribution is”; C(I)
540 NEXT I
550 PRINT “The performance function is”; h
560 END
```

References

1. AL-Neweihi, E., Proschan, F., and Sethuraman, J.**1978**. Multistate coherent system. *Journal of applied probability,* 15, pp:675-688.
2. Huang, J. et al. **2000**. Generalized multi-state k-out-of-n:G systems, *IEEE Transactions on Reliability*, 4(1), pp:105-111.
3. Huang, J. et al. **1986**.. Near-Catastrophe at Le-Bugey *Journal of Nature*, No.(321), pp:462.
4. Huang, J. et al. **2003**. Multi-state consecutive k-out-of-n systems, *IEE Transactions*, 35, pp:527-534.
5. Natvig, B.**1982**. Two suggestions of how to define a multi-state coherent system, *Applied probability*, 14, pp:391-402.
6. Raheem M. A. F. **2011**. On the reliability of multistate systems with multistate components, *M.Sc. thesis, College of science University of Baghdad,*.
7. Belfore, L. A. **1995**. An $O(n(\log_2(n))^2$ algorithm for computing the reliability of k-out-of-n:G and k-to-*l*-out-of-n:G systems, *IEEE Transactions on Reliability*, 44(1), pp:132-136.