Orthogonal Derivations and Orthogonal Generalized Derivations on ΓM-Modules

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Abstract

Let M be Γ-ring and X be ΓM-module, Bresar and Vukman studied orthogonal derivations on semiprime rings. Ashraf and Jamal defined the orthogonal derivations on □-rings M. This research defines and studies the concepts of orthogonal derivation and orthogonal generalized derivations on ΓM -Module X and introduces the relation between the products of generalized derivations and orthogonality on ΓM -module.

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المشتقات المتعامدة واعمام المشتقات المتعامدة على المقاسات من النمط − ΓM

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الخلاصة

لتكن M حلقه من النمط −Γ و X مقاس من النمط −ΓM درس كل من Bresar و Vukman المشتقات المتعامدة على الحلقات شبه الاولية وبينما درس كل من Ashraf و Jamal المشتقات المتعامدة على الحلقات من النمط −Γ . في هذا البحث قدمنا ودرسنا المفاهيم التالية الاشتقاق المتعامد واعمام الاشتقاق المتعامد على المقاس X من النمط −□□ وقدمنا العلاقة بين اعمام جداء المشتقات والتعامد على المقاسات من النمط −ΓM .

1. Introduction

Nobusawa [1] presented the definition of Γ-ring and generalized by Barnes [2] as following:

Let M and Γ be additive abelian groups. Suppose that there is a mapping from $M\times\Gamma\times M\rightarrow M$ (the image of (a,α,b) being denoted by $a\alpha b$, $a,b\in M$ and $\alpha\in\Gamma$) satisfying for all $a,b,c\in M$, $\alpha\in\Gamma$:

i) $(a+b)\alpha c= a\alpha c+ b\alpha c$

$a(\alpha+\beta)c = a\alpha c + a\beta c$

$a\alpha(b + c) = a\alpha b + a\alpha c$

ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$

Then M is Γ-ring. M is said to be 2-torsion free if 2a = 0 implies a=0 for all $a\in M$. Besides, M is called a prime Γ-ring if for all $a,b\in M$, $a\Gamma M\Gamma b = (0)$ implies either a=0 or b=0, M is called semiprime if $a\Gamma M\Gamma a = (0)$ with $a\in M$ implies a=0. Note that every prime Γ-ring is obviously semiprime [3].

Let M be a Γ-ring and X be an additive abelian group, X is a left ΓM- module if there exists a mapping $M\times\Gamma\times X\rightarrow X$ (sending (m,α,x) into $m\alpha x$ where $m\in M$, $\alpha,\beta\in\Gamma$ and

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$x \in X$) satisfying for all $m, m_1, m_2 \in M$, $\alpha, \beta \in \Gamma$ and $x, x_1, x_2 \in X$ [4]:
i) $(m_1+m_2)\alpha x= m_1\alpha x+m_2\alpha x$
ii) $m(\alpha+\beta)x=m\alpha x+m\beta x$
iii) $m\alpha(x_1+x_2)=m\alpha x_1+m\alpha x_2$
iv) $(m_1\alpha m_2)\beta x= m_1\alpha(m_2\beta x)$

X is called a right ΓM- module if there exists a mapping $X\times\Gamma\times M \rightarrow X$, X is called ΓM-module if X is both left and right ΓM- module, X is called a left prime (right prime) if $a\Gamma M\Gamma b=(0)$ then a=0 or b=0, $a\in M$, $b\in X$ ($a\in X, b\in M$) respectively and X is prime if its both left and right prime. X is called semipeime if $a\Gamma M\Gamma a=(0)$ where $a\in X$ implies a=0, X is called 2-torsion free if 2x=0 implies x=0 for all $x\in X$ [4]. Jing [5] defined a derivation on Γ-ring as following, an additive map $d:M\rightarrow M$ is said to be a derivation of M if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$, for all $a,b\in M$ and $\alpha\in\Gamma$.Ceven and Ozturk [6] defined a generalized derivation on Γ-ring M as follows, an additive map $D:M\rightarrow M$ is said to be generalized derivation on M if there exists a derivation d: $M\rightarrow M$ such that $D(a\alpha b) = D(a)\alpha b + a\alpha d(b)$, for all $a,b\in M$ and $\alpha\in\Gamma$.

Paul and Halder [4] defined a left derivation and Jordan left derivation of Γ-ring M onto ΓM-module X as follows $d:M\rightarrow X$ is a left derivation if $d(a\alpha b)=a\alpha d(b)+b\alpha d(a)$, a Jordan left derivation $d(a\alpha a)=2a\alpha d(a)$. Also Paul and Halder proved that every Jordan left derivation of Γ-ring M into ΓM-module is a left derivation. Salih [7] defined derivation and Jordan derivation on ΓM-module as follows $d:M\rightarrow X$ is a derivation if $d(a\alpha b)=d(a)\alpha b+a\alpha d(b)$, a Jordan derivation $d(a\alpha a)=d(a)\alpha a + a\alpha d(a)$ and proved that every Jordan derivation of Γ-ring M into ΓM-module is a derivation as well as in [8] defined generalized derivation and Jordan generalized derivation as follow f: $M\rightarrow X$ is generalized derivation if there exist a derivation $d:M\rightarrow X$ such that $f(a\alpha b) = f(a)\alpha b+a\alpha d(b)$ and prove that every Jordan generalized derivation of Γ-ring M into ΓM-module is a generalized derivation.

The study of orthogonal derivation in rings was initiated by Bresar and Vukman [9]. In fact, they obtained some results on orthogonal derivations in semiprime rings related to product of derivations. Ashraf and Jamal [10] defined and studied the orthogonal derivation on Γ-rings and generalized the result of Bresar and Vukman into Γ-ring. Argac , Nakajima and Albas [11] presented the definition of orthogonal generalized derivations on rings.

In the present paper, we define and study the concepts of orthogonal derivations and orthogonal generalized derivations on ΓM-module and obtained some results parallel to those earlier obtained by Bresar, Vukman in [9], Argac, Nakajima and Albas in [11].

2. Orthogonal Derivations on ΓM-Module:

In this section, this research presents and studies the definition of orthogonal derivations on ΓM-module, we prove that if M is a Γ-ring and X a 2-torsion free semiprime ΓM-module. Suppose that d and g are derivations of M into X. If
$d^2=g^2$ or $d(x)\alpha d(x) = g(x)\alpha g(x)$ for all $x\in M$ and $\alpha\in\Gamma$,Then d+g and d-g are orthogonal.

Definition 2.1:

Let M be a Γ-ring and X a ΓM-module, the derivations d and g of M into X are said to be orthogonal if
$d(x)\Gamma M\Gamma g(y) = g(y)\Gamma M\Gamma d(x)$, for all $x,y\in M$.

Now, we give an example of orthogonal derivation:

Example 2.2:

Let d and g be derivations of a ring R, M = $\mathbf{Z}\oplus\mathbf{Z}$ and $\Gamma=\mathbf{Z}\oplus\mathbf{Z}$ where **Z** is the set of integer numbers, then M is Γ-ring, $X=R\oplus R$, then X is ΓM-module we define d_1 and g_1 on X by
$d_1((x,y)) = (d(x) , 0)$ and $g_1((x,y)) = (0 , g(y))$ for all $x,y\in R$.
then d_1 and g_1 are orthogonal.

Now we give the following lemma we need later:

Lemma 1:

Let M be Γ-ring and X a 2-tortion free semiprime ΓM-module and a,b the elements of X. Then the following conditions are equivalent:
i) $a\Gamma M\Gamma b=(0)$
ii) $b\Gamma M\Gamma a=(0)$
iii) $a\Gamma M\Gamma b+b\Gamma M\Gamma a=(0)$
If one of these conditions are fulfilled then $a\Gamma b=b\Gamma a=(0)$.

Proof:

(i)$\rightarrow$(ii) Suppose that $a\Gamma M\Gamma b=(0)$.
Then $(b\Gamma M\Gamma a)\Gamma M\Gamma(b\Gamma M\Gamma a)=(0)$, by semiprimeness of X we get $b\Gamma M\Gamma a=(0)$.

(ii)$\rightarrow$(iii) Suppose that bΓMΓa=(0), that is aΓMΓb=(0), this implies aΓMΓb+bΓMΓa=(0).
(iii)$\rightarrow$(i) Suppose that aΓMΓb+bΓMΓa=(0) that is aΓMΓb=-bΓMΓa
Let m and m' be two arbitrary elements of M. Then by hypothesis, we have:
(aΓmΓb)Γm'Γ(aΓmΓb)
=- ΓmΓa) Γm'Γ(aΓmΓb)
=-(bΓ(mΓaΓm') Γa) ΓmΓb
=(aΓ(mΓaΓm') Γb) ΓmΓb
=aΓmΓ(aΓm'Γb) ΓmΓb
=-aΓmΓ(bΓm'Γa) ΓmΓb
= - (aΓmΓb)Γm'Γ(aΓmΓb)
Thus 2((aΓmΓb) Γm'Γ(aΓmΓb))=0, since X is 2-torsion free, therefore, (aΓmΓb)Γm'Γ(aΓmΓb) = 0. By the semiprimeness of X, then aΓmΓb=0 for all m∈M. Hence we get, aΓmΓb = bΓmΓa =0 for all m∈M. ♦

Lemma 2:
Let X be a semiprime ΓM-module. Suppose that additive mappings f and h of Γ-ring M into X satisfy f(x)ΓMΓh(x)=(0), for all x∈M then f(x)ΓMΓh(y)=(0), for all x,y∈M.
Proof:
Suppose f(x)α zβh(x)=0 for all x,z∈M and $\alpha,\beta\in\Gamma$. On linearizing we get :
0=f(x+y)α zβh(x+y)
=(f(x)+f(y)) α zβ(h(x)+h(y))
=f(x)α zβh(x) +f(x)α zβh(y) + f(y)α zβh(x) + f(y)α zβh(y)
= f(x)α zβh(y) + f(y)α zβh(x)
Therefore by our assumption we get
f(x)α zβh(y)γ tδ f(x)α zβh(y)
=- f(x)α zβh(y)γ tδ f(y)α zβh(x)
=0
Since X is semiprime, this implies
f(x)α zβh(y)=0 for all x,y,z∈M and $\alpha,\beta\in\Gamma$. ♦

In the following lemmas we give the properties of orthogonal derivation on ΓM-module:

Lemma 3:
Let X be a 2-tortion free semiprime ΓM-module and let d and g be derivations of Γ-ring M into X. Derivations d and g are orthogonal if and only if d(x)α g(y)+g(y)α d(x) =0, for all x,y∈M, $\alpha\in\Gamma$.
Proof:
Suppose d(x)α g(y)+g(x)α d(y) =0, for all x,y∈M, $\alpha\in\Gamma$.Replace y by yβx, to get
0= d(x)α g(yβx)+ g(x)α d(yβx)
= d(x)α g(y)βx + d(x)α yβg(x) + g(x)α d(y)βx + g(x)α yβd(x)
= d(x)α yβg(x) + g(x)α yβd(x)
Hence by Lemma 1
d(x)α yβg(z) = 0, for all x,y,z∈M and $\alpha,\beta\in\Gamma$.
Therefore, d and g are orthogonal.
Conversely, if d and g are orthogonal, then
d(x)α zβg(y) = 0 =g(y)α zβd(x).
Therefore by Lemma 1,
d(x)α g(y)= 0 = g(y)α d(x).
This implies that
d(x)α g(y) + g(y)α d(x) =0, for all x,y ∈M and $\alpha\in\Gamma$. ♦

Lemma 4:
Let M be Γ-ring and X a 2-torsion free semiprime ΓM-module. Suppose d and g are derivations of M into X. Then d and g are orthogonal if and only if dg=0.
Proof:
Suppose that dg=0, and x,y∈M, $\alpha\in\Gamma$.
0=dg(xα y) = d(g(x)α y+ xα g(y))
=dg(x)α y + g(x)α d(y) + d(x)α g(y) + xα dg(x)
= g(x)α d(y) + d(x)α g(y)
Therefore by Lemma 3, d and g are orthogonal.
Conversely, suppose that d and g are orthogonal. Then d(x)α yβg(z)= 0 for all x,y,z∈M , $\alpha,\beta\in\Gamma$.
Hence
0 = d(d(x)αyβg(z))
= d^2(x)α yβg(z) + d(x)α d(y)βg(z) + d(x)α yβdg(x)
Then d(x)α yβdg(x) =0
Replacing x by g(x) and using semiprimeness of X, we find that dg(z)=0
for all z∈M, hence dg=0. ♦

Lemma 5:
Let X be 2- torsion free semiprime ΓM-module. Suppose d and g are derivations of Γ-ring M into X, then d and g are orthogonal if and only if dg + gd =0.
Proof:
Suppose that dg + gd =0. Then we have for all x,y ∈M, $\alpha,\beta\in\Gamma$

0= (dg+gd)(xαy)
=g(d(x)αy + xαd(y)) + d(g(x)αy + xαg(y))
=gd(x)αy + d(x)αg(y) + g(x)αd(y) + xαgd(y) + dg(x)αy + g(x)αd(y) + d(x)αg(y) +xαdg(y)
=(gd + dg)(x)αy + 2(d(x)αg(y) + g(x)αd(y)) + xα(gd + dg)(y)
= 2(d(x)αg(y) + g(x)αd(y))
Since X is 2-torsion free
d(x)αg(y) + g(x)αd(y)= 0
hence by Lemma 3, d and g are orthogonal.
Conversely, since d and g are orthogonal, then by Lemma 4, dg = 0 = gd, therefore dg + gd = 0. ♦

Lemma 6:
Let M be Γ-ring and X a 2-tortion free semiprime ΓM-module. Suppose d and g are derivations of M into X. Then d and g are orthogonal if and if dg is a derivation.
Proof:
Since d and g are derivations, we have for all x,y∈M, α∈Γ
dg(xαy)= dg(x)αy + xαdg(y) …(1)
on the other hand
dg(xαy)= d(g(x)αy + xαg(y))
=dg(x)αy+g(x)αd(y)+d(x)αg(y)+xαdg(y)… (2)
Comparing (1) and (2) we have
g(x)αd(y) + d(x)αg(y)=0
Hence, by Lemma 3, d and g are orthogonal.
Conversely, suppose d and g are orthogonal, by Lemma 4 we have dg=0 therefore dg is a derivation. ♦

Corollary 7:
Let M be Γ-ring and X a 2-tortion free semiprime ΓM-module. If d a derivation of M into X such that d^2 is also derivation, then d=0.

Lemma 8:
Let X be a semiprime ΓM-module. Suppose that d and g are derivations of Γ-ring M into X. Then d and g are orthogonal if and only if there exists a,b∈M and
α, β ∈Γ such that dg(x) = aαx + xβb for all x∈M.
Proof:
Suppose that dg(x) = aαx + xβb for all x∈M. Replacing x by xδy we have
dg(xδy)= aαxδy + xδyβb
d(g(x)δy + xδg(y))= aαxδy + xδyβb
dg(x)δy + g(x)δd(y) + d(x)δg(y) + xδdg(y)= aαxδy + xδyβb
xδbβy + xγaαy + d(x)δg(y) + g(x)δd(y) = 0, for all x∈M and δ ∈Γ. ....(1)
Replacing y by yδx in (1), we have
0 = xβbγyδx + xγaαyδx + d(x)γg(yδx) + g(x)γ d(yδx)
= xβbγyδx + xγaαyδx + d(x)γg(y)δx +d(x)γyδg(x)+g(x)γd(y)δx+ g(x)γyδd(x).
=(xβbγy+ xγaαy+ d(x)γg(y) + g(x)γd(y))δx+d(x)γyδg(x)+g(x)γyδd(x).
Hence using (1), we find that
d(x)γyδg(x) + g(x)γyδd(x) = 0
By Lemma 1, we have
d(x)γyδg(z) = 0.
Thus, d and g are orthogonal.
Conversely, since d and g are orthogonal, dg = 0, so we can choose a=b=0 and $\alpha, \beta \in \Gamma$ so that dg(x) = aαx + xβb. ♦

Corollary 9:
Let X be a semiprime ΓM-module. Suppose that d and g are derivations of Γ-ring M into X. Then d and g are orthogonal if and only if there exists a,b∈M and α, β ∈Γ such that d^2(x) = aαx + xβb for all x∈M then d=0.

Theorem 10:
Let M be a Γ-ring and X be a 2-torsion free semiprime ΓM-module. Suppose that d and g are derivations of M into X. Suppose d^2=g^2. Then d+g and d-g are orthogonal.
Proof:
Suppose d^2=g^2, for all x∈M
((d-g)(d+g) + (d+g)(d-g))(x) =(d-g)(d+g)(x) + (d+g)(d-g)(x)
=d^2(x) + dg(x) − gd(x) −g^2(x) + d^2(x) −dg(x)+ gd(x) −g^2(x)
= 0
Therefore (d-g)(d+g) + (d+g)(d-g)= 0, hence by Lemma 5, d+g and d-g are orthogonal. ♦

Theorem 11:
Let M be a Γ-ring and X be a 2-torsion free semiprime ΓM-module. Suppose that d and g are derivations of M into X. If d(x)αd(x) = g(x)αg(x) for all x∈M and α ∈Γ, then d+g and d-g are orthogonal.
Proof:
For all x∈M and α ∈Γ

(d-g) (x) α (d+g)(x) + (d+g)(x) α (d-g)(x)
= (d(x)– g(x))α(d(x)+ g(x))+
(d(x)+ g(x))α(d(x)– g(x))
= d(x) α d(x) + d(x) α g(x) – g(x) α d(x) – g(x) α g(x) +d(x) α d(x) – d(x) α g(x) + g(x) α d(x) – g(x) α g(x)g(x) α g(x)= 0
Hence, by Lemma 5, then d+g and d-g are orthogonal. ♦

3. Orthogonal Generalized Derivations on ΓM-Module:

In this section we generalize the results of section two by present the definition of orthogonal generalized derivations on ΓM-module, also we introduce the conditions which mark derivation and generalized derivation on ΓM-module are orthogonal we start by the following definition:

Definition 3.1:
Two generalized derivations D and G with derivations d and g respectively of Γ-ring M into ΓM-module X are said to be orthogonal if D(x)ΓMΓG(y) = 0 = G(y)ΓMΓD(x), for all x,y $\in$ M.

Now, we give the example of orthogonal generalized derivation

Example 3.2:
Let X, M and Γ as in Example 2.2 and D, G are generalized derivations on R we define D_1 and G_1 by
D_1 ((x,y)) = (D(x) , 0) and G_1((x,y)) = (0, G(y)) , for all x,y $\in$ R
Then D_1 and G_1 are orthogonal generalized derivations.
In the following theorem we give the relations which mark the derivation and generalized derivation are orthogonal on ΓM-module:

Theorem 12:
If D and G are orthogonal generalized derivations and d, g are derivations associative with D and G respectively of Γ-ring M into ΓM-module X then the following relations hold:
i) D(x) α G(y) = G(x) α D(y) = 0, hence D(x) α G(y) + G(x) α D(y) = 0, for all x,y $\in$ M, $\alpha \in \Gamma$.
ii) d and G are orthogonal, and d(x) α G(y) = G(y) α d(x) = 0, for all x,y $\in$ M, $\alpha \in \Gamma$.
iii) g and D are orthogonal, and g(x) α D(y) = D(y) α g(x) = 0, for all x,y $\in$ M, $\alpha \in \Gamma$.
iv) d and g are orthogonal derivations.

Proof:
i)By the hypothesis we have
D(x) α z β G(y) = 0, for all x,y,z $\in$ M, $\alpha, \beta \in \Gamma$.
Hence by Lemma 1 we get
D(x) α G(y) = G(x) α D(x) = 0, for all x,y $\in$ M, $\alpha \in \Gamma$.

ii) By D(x) α G(y) = 0 and D(x) α z β G(y) = 0, for all x,y,z $\in$ M, $\alpha, \beta \in \Gamma$, we get
0 = D(r β x) α G(y) = (D(r) β x + r β d(x)) α G(y)
= D(r) β x α G(y) + r β d(x) α G(y)
= r β d(x) α G(y) , for all x,y,r $\in$ M, $\alpha \in \Gamma$.
Then d(x) α G(y) = 0 , for all x,y $\in$ M, $\alpha \in \Gamma$.
Then we have
0 = G(y) α D(r β x) = G(y) α (D(r) β x + r β d(x))
= G(y) α D(r) β x + G(y) α r β d(x)
= G(y) α r β d(x) for all x,y,r $\in$ M, $\alpha, \beta \in \Gamma$.
Then by Lemma 1 we obtain
G(y) α d(x) = 0 for all x,y $\in$ M, $\alpha \in \Gamma$.
iii) 0 = D(x) α G(m β y)
= D(x) α (G(m) β y + m β g(y))
= D(x) α G(m) β y + D(x) α m β g(y)
= D(x) α m β g(y) for all x,y,m $\in$ M, $\alpha, \beta \in \Gamma$.
Then d(x) α G(y) = 0 for all x,y $\in$ M, $\alpha \in \Gamma$.
Hence we have
0 = g(x β m) α D(y)
= (g(x) β m + x β g(m)) α D(y)
= g(x) β m α D(y) + x β g(m) α D(y)
= g(x) β m α D(y) for all x,y,m $\in$ M, $\alpha, \beta \in \Gamma$.
Since by Lemma 1 we get
g(x) α D(y) = 0 for all x,y $\in$ M, $\alpha \in \Gamma$.
iv) We have
0 = D(x α z) β G(y α w) for all x,y,z,w $\in$ M, $\alpha, \beta \in \Gamma$.
= (D(x) α z + x α d(z)) β (G(y) α w + y α g(w))
= D(x) α z β G(y) α w + D(x) α z β y α g(w) + x α d(z) β G(y) α w + x α d(z) β y α g(w)
By hypothesis and (ii) , (iii) we get
x α d(z) β g(y) α w = 0 for all x,y,z,w $\in$ M, $\alpha, \beta \in \Gamma$. Hence d and g are orthogonal. ♦

Lemma 13:
Let X be a semiprime ΓM-module and D be a generalized derivation of Γ-ring M into X with derivation d of M into X. If D(x) α D(y) = 0, for all x,y $\in$ M, $\alpha \in \Gamma$, then D = d = 0.

Proof:

By the hypothesis we have

$0 = D(x)\alpha D(y\beta z)$

$= D(x)\alpha (D(y)\beta z + y\beta d(z))$

$= D(x)\alpha D(y)\beta z + D(x)\alpha y\beta d(z)$

$= D(x)\alpha y\beta d(z)$, for all $x,y,z\in M, \alpha,\beta\in\Gamma$.

Hence by Lemma 1 we have

$d(z)\alpha D(x) = 0$ all $x,z\in M, \alpha\in\Gamma$

Replacing x by $x\beta z$ in last relation, we get

$0 = d(z)\alpha D(x\beta z)$

$= d(z)\alpha (D(x)\beta z + x\beta d(z))$

$= d(z)\alpha D(x)\beta z + d(z)\alpha x\beta d(z)$ for all $x,z\in M, \alpha,\beta\in\Gamma$

By the semiprimeness of X, we obtain d = 0.

$0 = D(x\beta y)\alpha D(y)$

$= (D(x)\beta y + x\beta d(y))\alpha D(y)$

$= D(x)\beta y\alpha D(y) + x\beta d(y)\alpha D(y)$

Hence $D(x)\beta y\alpha D(y) = 0$ for all $x,y\in M, \alpha,\beta\in\Gamma$. By the semiprimeness of X, we obtain D = 0. ♦

Lemma 14:

Let D and G be generalized derivations of Γ-ring M into ΓM-module X, with derivations d and g respectively. If the following relation hold for all $x,y\in M$ and $\alpha\in\Gamma$.

i) $D(x)\alpha G(y) + G(x)\alpha D(y) = 0$

ii) $d(x)\alpha G(y) + g(x)\alpha D(y) = 0$

then D and G are orthogonal.

Proof:

If we take $x\beta z$ instead of x in (i) we get

$D(x\beta z)\alpha G(y) + G(x\beta z)\alpha D(y) = 0$

$(D(x)\beta z + x\beta d(z))\alpha G(y) + (G(x)\beta z + x\beta g(z))\alpha D(y) = 0$

$D(x)\beta z\alpha G(y) + x\beta d(z)\alpha G(y) + G(x)\beta z\alpha D(y) + x\beta g(z)\alpha D(y) = 0$

$D(x)\beta z\alpha G(y) + G(x)\beta z\alpha D(y) = 0$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. Thus by Lemma 1 we have $D(x)\Gamma M\Gamma G(y) = G(x)\Gamma M\Gamma D(y) = 0$, for all $x,y\in M$. ♦

Lemma 15:

Let X be 2-torsion free semiprime ΓM-module D and G are generalized derivations of Γ-ring M into X, with derivations d and g respectively, such that

$D(x)\alpha G(y) = d(x)\alpha G(y) = 0$, for all $x,y\in M$ and $\alpha\in\Gamma$ then D and G are orthogonal.

Proof:

If we take $x\beta z$ instead of x in $D(x)\alpha G(y) = 0$ we get

$D(x\beta z)\alpha G(y) = 0$

$(D(x)\beta z + x\beta d(z))\alpha G(y) = 0$

$D(x)\beta z\alpha G(y) + x\beta d(z)\alpha G(y) = 0$

$D(x)\beta z\alpha G(y) = 0$

Thus by Lemma 1 we get D and G are orthogonal. ♦

Lemma 16:

Let X be 2-torsion free semiprime ΓM-module, D and G are generalized derivations of Γ-ring M into X, with derivations d and g respectively, such that

$D(x)\alpha G(y) = 0$, for all $x,y\in M$ and $\alpha\in\Gamma$ and dG=dg=0. Then D and G are orthogonal.

Proof:

Since dg=0, we have

$dG(x\alpha y) = 0$

$d(G(x)\alpha y + x\alpha g(y)) = 0$

$dG(x)\alpha y + G(x)\alpha d(y) + d(x)\alpha g(y) + x\alpha dg(y) = 0$

$G(x)\alpha d(y) = 0$, for all $x,y\in M$ and $\alpha\in\Gamma$. ...(1)

Replacing x by $x\beta z$ in (1) we have

$G(x\beta z)\alpha d(y) = 0$

$(G(x)\beta z + x\beta g(z))\alpha d(y) = 0$

$G(x)\beta z\alpha d(y) + x\beta g(z)\alpha d(y) = 0$

$G(x)\beta z\alpha d(y) = 0$, for all $x,y,z\in M, \alpha,\beta\in\Gamma$

Now, $d(y\gamma z)\alpha G(x) = 0$

$(d(y)\gamma z + y\gamma d(z))\alpha G(x) = 0$

$d(y)\gamma z\alpha G(x) + y\gamma d(z)\alpha G(x) = 0$

$d(y)\gamma z\alpha G(x) = 0$

Hence by Lemma 1 we get $d(y)\alpha G(x) = 0$, for all $x,y\in M$ and $\alpha\in\Gamma$. Thus by Lemma 15 we get D and G are orthogonal. ♦

Lemma 17:

Let X be 2-torsion free semiprime ΓM-module, D and G are generalized derivations of Γ-ring M into X, with derivations d and g respectively, such that

DG is generalized derivation of Γ-ring M into X with derivation dg and $D(x)\alpha G(y)= 0$, for all $x,y\in M$ and $\alpha\in\Gamma$. then D and G are orthogonal.

Proof:

Since DG is generalized derivation with derivation dg

$DG(x\alpha y) = DG(x)\alpha y + x\alpha dg(y)$ for all $x,y\in M$ and $\alpha\in\Gamma$ ...(1)

on the other hand

$DG(x\alpha y) = D(G(x)\alpha y + x\alpha g(y))$

$=DG(x)\alpha y+G(x)\alpha d(y)+D(x)\alpha g(y)+x\alpha Dg(y)$ for all $x,y\in M, \alpha \in\Gamma$...(2)

Compare (1) and (2) we get

$G(x)\alpha d(y) + D(x)\alpha g(y) = 0$, for all $x,y\in M$ and $\alpha \in\Gamma$.

Since $D(x)\alpha G(y)= 0$, for all $x,y\in M$ and $\alpha \in\Gamma$. ...(3)

Replacing y by $y\beta z$ in (3) we get

$D(x)\alpha G(y\beta z)= 0$

$D(x)\alpha (G(y)\beta z + y\beta g(z)) = 0$

$D(x)\alpha G(y)\beta z + D(x)\alpha y\beta g(z) = 0$

$D(x)\alpha y\beta g(z) = 0$, for all $x,y\in M$ and $\alpha \in\Gamma$.

Now, replacing x by $x\beta z$ in (3) we get

$D(x\beta z)\alpha G(y)= 0$

$(D(x)\beta z + x\beta d(z))\ \alpha G(y)= 0$

$D(x)\beta zD(x)\beta z + x\beta d(z)\alpha G(y) = 0$

$x\beta d(z)\alpha G(y) = 0$

Hence $d(z)\alpha G(y) = 0$, for all $x,y\in M$ and $\alpha \in\Gamma$.

Thus by Lemma 15 we get D and G are orthogonal. ♦

4. Products of Generalized Derivations on ΓM-Module:

In this section, we introduce and study the relation between the products of generalized derivations and orthogonality on ΓM-module. We state by the following theorem:

Theorem 18:

Let D and G are generalized derivations of Γ-ring M into semiprime ΓM-module X, with derivations d and g respectively. Then DG is generalized derivation with derivation dg, if and only if D and g are orthogonal, also G and d are orthogonal.

Proof:

Assume that DG is a generalized derivation with derivation dg, we obtain

$DG(x\alpha y) = DG(x)\alpha y + x\alpha dg(y)$, for all $x,y\in M$ and $\alpha \in\Gamma$. ...(1)

On the other hand

$DG(x\alpha y) = D(G(x)\alpha y + x\alpha g(y))$

$= DG(x)\alpha y + G(x)\alpha d(y) + D(x)\alpha g(y) + x\alpha dg(y)$ for all $x,y\in M$ and $\alpha \in\Gamma$. ...(2)

Compare (1) and (2) we get

$G(x)\alpha d(y) + D(x)\alpha g(y) = 0$, for all $x,y\in M$ and $\alpha \in\Gamma$,for all $x,y\in M$ and $\alpha \in\Gamma$ ......(3)

Replacing y by $y\beta z$ in relation (3) where $z\in M$ and $\beta \in\Gamma$ we get

$G(x)\alpha d(y\beta z) + D(x)\alpha g(y\beta z) = 0$

$G(x)\alpha (d(y)\beta z + y\beta d(z)) + D(x)\alpha (g(y)\beta z + y\beta g(z)) = 0$

$G(x)\alpha d(y)\beta z + G(x)\alpha y\beta d(z) + D(x)\alpha g(y)\beta z + D(x)\alpha y\beta g(z) = 0$

By using (3) we get

$G(x)\alpha y\beta d(z) + D(x)\alpha y\beta g(z) = 0$, for all $x,y,z\in M$ and $\alpha,\beta \in\Gamma$. ...(4)

Since DG is generalized derivation with derivation dg. Therefore d and g are orthogonal by Lemma 16 and Theorem 12(iv). Thus we have:

$0=G(x)\gamma g(z)\alpha y\beta d(z) + D(x)\gamma g(z)\alpha y\beta g(z)$ $= D(x)\gamma g(z)\alpha y\beta g(z)$, for all $x,y,z\in M$ and $\alpha,\beta,\gamma \in\Gamma$

Hence we get $D(x)\gamma g(z)\Gamma M\Gamma D(x)\gamma g(z) = 0$, for all $x,z\in M$ and $\gamma \in\Gamma$, by the semiprimeness of X, we obtain

$D(x)\gamma g(z) = 0$, for all $x,z\in M$ and $\gamma \in\Gamma$.

Thus $D(x)\alpha y\beta g(z) = 0$, for all $x,y,z\in M$ and $\alpha,\beta \in\Gamma$, and by (4) we have

$G(x)\alpha y\beta d(z) = 0$, for all $x,y,z\in M$ and $\alpha,\beta \in\Gamma$.

Assume that D and g are orthogonal also G and d are orthogonal.

Since D and g are orthogonal, we get

$D(x)\alpha y\beta g(z) = 0$, for all $x,y,z\in M$ and $\alpha,\beta \in\Gamma$. ...(5)

Substituting $r\gamma x$ for x in relation (5) we get

$D(r\gamma x)\alpha y\beta g(z) = 0$

$(D(r)\gamma x + r\gamma d(x))\alpha y\beta g(z) = 0$

$D(r)\gamma x\alpha y\beta g(z) + r\gamma d(x)\alpha y\beta g(z) = 0$

By (5) we get $r\gamma d(x)\alpha y\beta g(z) = 0$ for all $r,x,y,z\in M$ and $\alpha,\beta,\gamma \in\Gamma$.

Hence $d(x)\alpha y\beta g(z) = 0$, for all $x,y,z\in M$ and $\alpha,\beta \in\Gamma$. Thus by Lemma 16 and Theorem 12(iv) we conclude that dg is a derivation .

Moreover since $D(x)\alpha y\beta g(z) = 0$, for all $x,y,z\in M$ and $\alpha,\beta \in\Gamma$. we also get $D(x)\alpha g(z)\ \Gamma M\ \Gamma D(x)\beta g(z) = 0$ and so by semiprimeness of X we get $D(x)\alpha g(z) = 0$ for all $x,z\in M$ and $\alpha \in\Gamma$, similarly, since G and d are orthogonal, we have $G(x)\alpha d(y) = 0$ for all $x,y\in M$ and $\alpha \in\Gamma$. Thus we obtain

DG(x α y) = DG(x) α y + x α dg(y), for all x,y $\in$ M and $\alpha \in \Gamma$, which means that DG is a generalized derivation with derivation dg. ♦

Corollary 19:

Let D and G be generalized derivations with derivations d and g respectively of Γ-ring into semiprime ΓM-module X. Then GD is a generalized derivation with derivation gd of M into X if and only if D and g are orthogonal, also G and d are orthogonal.

Corollary 20:

Let D be a generalized derivation with derivation d of Γ-ring M into semiprimeness ΓM-module X. If D^2 is a generalized derivation with derivation d^2, then d=0.

Proof:

Since d^2 is a derivation, d and d are orthogonal by Lemma 17 and Theorem18. Hence we have d(x) α y β d(x) = 0 for all x, y $\in$ M, $\alpha, \beta \in \Gamma$. Therefore by the semiprimeness of X, we get d(M) = 0 ♦

Lemma 21:

Let D be a generalized derivation with derivation d of Γ-ring M into semiprimeness ΓM-module X. If D(x) α D(y) = 0, for all x,y $\in$ M, $\alpha \in \Gamma$, then D=d=0.

Proof:

Since D(x) α D(y) = 0, for all x,y $\in$ M, $\alpha \in \Gamma$, ...(1)

Replacing y β z by y in (1) where z $\in$ M, $\beta \in \Gamma$ we get

0= D(x) α D(y β z)

= D(x) α (D(y) β z + y β d(z))

= D(x) α D(y) β z + D(x) α y β d(z) for all x,y,z $\in$ M, $\alpha, \beta \in \Gamma$.

Hence by Lemma 1 we get

d(z) β D(x)=0 for all x,z $\in$ M, $\beta \in \Gamma$. …(2)

Replacing x by x γ z in (2) we get

0 = d(z) β D(x γ z)

= d(z) β (D(x) γ z + x γ d(z))

= d(z) β D(x) γ z + d(z) β x γ d(z) for all x,z $\in$ M, $\beta, \gamma \in \Gamma$.

By the semiprimeness of X, we obtain d= 0.

Then we have:

0 = D(x α y) β D(y)

= (D (x) α y + x α d(y)) β D(y)

= D(x) α y β D(y) + x α d(y) β D(y), for all x,y $\in$ M, $\alpha, \beta \in \Gamma$.

By the hypothesis and semiprimeness of X we get D = 0. ♦

By using the similar argument we can prove the following lemma:

Lemma 22:

Let X be a 2-torsion free prime ΓM-module. If D and G are generalized derivations with derivations d and g respectively of M into X satisfy one of the following conditions:

i) DG is a generalized derivation with derivation dg on X.

ii) GD is a generalized derivation with derivation gd on X.

iii) D and g are orthogonal, and G and d are orthogonal.

Then D=d =0 or G= g=0.

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