Iteration $x_{n+1} = \propto_{n+1} f(x_n) + (1 - \propto_{n+1})Tx_n$ for a Non-Self $\propto$-Strongly Pseudocontractive Map

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Abstract

Let X be a Banach space, C be a nonempty closed convex subset of X, and Tbe self nonexpansive map. The sequence $\{x_n\}$ generated by the iterative method $x_{n+1} = \propto_{n+1} f(x_n) + (1 - \propto_{n+1})Tx_n$, where $f: C \rightarrow C$ be a contractive mapping and $\{\propto_n\}$ is a sequence in $[0,1]$. We generalize the mapping T to non-sel$\propto$-Strongly Pseudocontractive .

التكرار $x_{n+1} = \propto_{n+1} f(x_n) + (1 - \propto_{n+1})Tx_n$ للدالة T من النمط $\propto$-شبه الانكماشي بشدة

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الخلاصة

ليكن X فضاء بناخ، C مجموعة غير خالية مغلقة محدبة جزئية من X و T دالة غير قابلة للانكماش. المتتابعة $\{x_n\}$ المتولدة بأستخدام التكرار $x_{n+1} = \propto_{n+1} f(x_n) + (1 - \propto_{n+1})Tx_n$ حيث انه $f: C \rightarrow C$ هي دالة قابلة للانكماش و $\{\propto_n\}$ هي متتابعة في $[0,1]$. سوف نعمم النتيجة وذلك باستخدام الدالة T الى دالة غير ذاتية من النمط $\propto$-شبه الانكماشي بشدة.

1-Introduction

Let X be a Banach space, C be a nonempty closed convex subset of X, and $T: C \rightarrow X$ be a finite family of $\propto$-stronglypseudocontractivemapping, $f: C \rightarrow C$ be a contraction mapping with contractive constant;$a \in (0,1)$. In this paper we use the following iteration to study the strong convergence to a common fixed point q.

$$x_{n+1} = \propto_{n+1} f(x_n) + (1 - \propto_{n+1})Tx_n \quad \forall n \geq 0 \quad \ldots\ldots (1.1)$$

There are several types of iterations used to compute the fixed point of the systems of mappings. In 1992, Wittman [1] had obtained the following iteration

$$x_{n+1} = \propto_{n+1} u + (1 - \propto_{n+1})Tx_n \quad \ldots\ldots (1.2)$$

where$u \equiv f(x) \in C$ in (1.1). In 1996, Bauschke[2] considered the iteration (1.1),where $u \equiv f(x) \in C$and$\{T_i: C \rightarrow C; i = 1,2, \ldots, N\}$ are nonexpansive mappings and X be a Hilbert space then (1.1) is equal to the iteration as bellow

$$x_{n+1} = \propto_{n+1} u + (1 - \propto_{n+1})T_{n+1}x_n$$
$$\forall n \geq 0, \; T_n = T_{n(mod N)} \quad \ldots\ldots (1.3)$$

In 2005, Jung [3] introduced iteration (1.3) when X be a uniformly smooth Banach spaces with the weakly sequentially continuous duality mapping. When$f: C \rightarrow C$ is a contractive mapping, and $T: C \rightarrow C$ is a

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nonexpansivemapping , therefore (1.1) is equivalent to the iteration as bellow:

$x_{n+1} = \propto_{n+1} f(x_n) + (1 - \propto_{n+1}) T x_n \; \forall n \geq 0.$ (1.4)

It was obtain and studied by Moudafi[3] in Hilbert spaces and Xu[4] in uniformly smooth Banach spaces. Chang[5] in 2006, proved the iteration as bellow in uniformly smooth Banach spaces$X$,$\{T_i : C \to C : i = 1,2,\dots,N\}$ are nonexpansive mappings,

$x_{n+1} = \propto_{n+1} f(x_n) + (1 - \propto_{n+1}) T_{n+1} x_n$

$\forall n \geq 0, \quad T_n = T_{n(mod\, N)}$ …………… (1.5)

The main result of this paper is to extend the mappings$T$ to $\propto$-stronglypseudocontractive mapping.

2- Preliminaries

Suppose$X$ is a realBanach space, and $X^*$is the dual space of X and Suppose that C is a nonempty closed convex subset of X, $D(T)$ the domain of $T$and that $F(T)$ is the set of all fixed points of mapping T. Denote the generalized duality paring between X and X^* by $\langle .,. \rangle$(where $\langle .,. \rangle$is afunctional from X into X^*) and the identity mapping by I. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$J(x) = \{j \in X^* : \langle x,j \rangle = \|x\| \cdot \|j\| = \|j\|^2 = \|x\|^2\}$

, $x \in X$ (2.1)

If $\{x_n\}$ is a sequence in X, then $x_n \to x$(resp., $x_n \rightharpoonup x, x_n \rightharpoonup^* x$) denotes strong(resp.,weak and weak*) convergence of the sequence$\{x_n\}$ to x.

Definition 2.1([7]):

A mapping $T : D(T) = X \to X$ is said to be accretive if for all $x, y \in X$, there exists $j(x-y) \in J(x-y)$ such that

$\langle Tx - Ty, j(x-y) \rangle \geq 0.$

The mapping T is said to be strongly accretive if there exists a constant $k \in (0,1)$ such that for all $x, y \in X$, there exists $j(x-y) \in J(x-y)$such that

$\langle Tx - Ty, j(x-y) \rangle \geq k\|x-y\|^2$

and is said to be $\propto$-strongly accretive if there is a strictly increasing function $\propto : [0,\infty) \to [0,\infty)$ with $\propto(0) = 0$ such that for any $x, y \in X$ there exists$j(x-y) \in J(x-y)$such that

$\langle Tx - Ty, j(x-y) \rangle \geq \propto(\|x-y\|)\|x-y\|.$

Definition2.2 [6]:

The mapping $T : X \to X$ is calledstrongly pseudocontractiveif there exists $k \in (0,1)$ and for all $x, y \in X$, there exists $j(x-y) \in J(x-y)$such that

$\langle Tx - Ty, j(x-y) \rangle \leq (1-k)\|x-y\|^2.$

Definition 2.3[8]:

The mapping $T : X \to X$ is called $\propto$-strongly pseudocontractive and there exists a strictly increasing function $\propto : [0,\infty] \to [0,\infty]$ with $\propto(0) = 0$ such that for all$x, y \in X$ there exists$j(x-q) \in J(x-q)$such that

$\langle Tx - Ty, j(x-y) \rangle$

$\leq \|x-y\|^2 - \propto(\|x-y\|)\|x-y\|.$

Definition 2.4[8]:

The mapping T is pseudocontractive if and only if $I - T$ is accretive and is strongly pseudocontractive if and only if $I - T$is stronglyaccretive (respectively, $\propto$-strongly pseudocontractive).

Definition 2.5 [10]:

A Banach space is said to admit a weakly sequentially continuous normalized duality mapping J, if $J : X \to X^*$ is single-valued and weak-weak* sequentially continuous. i.e., if $x_n \to x \text{ in } X$then$J(x_n) \rightharpoonup^* J(x) \text{ in } X^*$.

We will give the definition of the concepts of non-self contraction mappings

Definition 2.6[11]:

Let C be a nonempty subset of a Banachspace X. For $x \in C$, the inward set of x relative to C is the set

$I_C(x) := \{x + t(y-x) : y \in C \text{ and } t \geq 0\}$

And the outward set of x relative to C is the set

$O_C(x) := \{x - t(y-x) : y \in C \text{ and } t \geq 0\}$

Now, we will give the definition of weakly inward and weakly outward mappings:

Definition 2.7 [12]:

Let C be a nonempty subset of a Banach space X and $T : C \to X$ mapping. Then T is said to be a

(i) Inward mapping if $Tx \in I_C(x)$ for all $x \in C$.

(ii) Weakly inward mapping if $T_x \in \overline{I_C(x)}$for all $x \in C$.

(iii) Weakly outward mapping if $T_x \in \overline{O_C(x)}$for all $x \in C$.

In the following we will obtain the definitions of smoothness and convexity of a Banach space.

Definition 2.8 [12]:

Let $U = \{x \in X: \|x\| = 1\}$. $X$is said to be a smooth Banach space, if the limit $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$.

Definition 2.9[11]:

The modulus of convexity of a Banach space X is the function $\delta_X: [0,2] \to [0,1]$ defined by

$$\delta_x(\epsilon) = inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

Note That [11]:

1- For any$\epsilon > 0$ the number of $\delta_x(\epsilon)$is the largest number for which the following implication always holds for $x, y \in X$,

$$\left.\begin{array}{r}\|x\| \le 1\\ \|y\| \le 1\\ \|x-y\| \ge \epsilon\end{array}\right\} \Rightarrow \left\|\frac{x+y}{2}\right\| \le 1 - \delta_x(\epsilon).$$

(2.1)

2- For later reference we note that (2.1) has the following equivalent formulation. For $x, y, p \in X, R > 0$, and $r \in [0,2R]$,

$$\left.\begin{array}{r}\|x\| \le R\\ \|y\| \le R\\ \|x-y\| \ge r\end{array}\right\} \Rightarrow \left\|p - \frac{x+y}{2}\right\| \le \left(1 - \delta_x\left(\frac{r}{R}\right)\right)R.$$

(2.2)

Lemma 2.10[8]

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three non-three nonnegative real sequences satisfying the following conditions:

$a_{n+1} \le (1-t_n)a_n + b_n + c_n$, for all $n \ge n_0$,

wheren_0 is some nonnegative integer, with $t_n \subset [0,1)$, with$\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and$\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0$as $n \to \infty$.

Lemma 2.11[6]

Let X be a real Banach space, and $J: X \to 2^{X^*}$ be the normalized duality mapping. Then for any $x, y \in X$, the following conclusion holds:

$\|x\|^2 + 2\langle y, j(x)\rangle \le \|x+y\|^2$

$\le \|x\|^2 + 2\langle y, j(x+y)\rangle$,

for all $j(x) \in J(x)$, $j(x+y) \in J(x+y)$.

Corollary 2.12[12]:

Let C be a nonempty closed convex subset of a Banach space X and $T: C \to X$ a weakly inward continuous α-strongly pseudocontractive mapping. Then T has a unique fixed point in C.

3. Main results

The following theorem gives a generalize of the most essential mapping need to prove the main theorem in this paper.

Theorem3.1:

Let X be uniformly convex Banach space, C is a closed convex subset of X, and $T: C \to X$ be α-strongly pseudocontractive mapping. Then the mapping $f = I - T$ is demiclosed on C. That is, for any sequence $\{x_n\}$in X, if $x_n \to x$ and $(x_n - Tx_n) \to y$ then $(I-T)x = y$.

To proof the above theorem we need the following two propositions.

Proposition 3.2:

Suppose C is a bounded, convex subset of a uniformly convex space X and suppose $T: C \to X$ be α-strongly pseudocontractive mapping. Then for $\{u_n\}$, $\{v_n\}$ in C and $z_n = \frac{1}{2}(u_n + v_n)if\lim_{n\to\infty}\|u_n - Tu_n\| = 0$ and $\lim_{n\to\infty}\|v_n - Tv_n\| = 0$then $\lim_{n\to\infty}\|z_n - Tz_n\| = 0$.

Proof:

Let $\epsilon > 0$ and there are two sequences $\{u_n\}$, $\{v_n\}$ belongs to C such that $\lim_{n\to\infty}\|u_n - Tu_n\| = 0$ and $\lim_{n\to\infty}\|v_n - Tv_n\| = 0$, suppose $z_n = \frac{1}{2}(u_n + v_n)$ such that $\lim_{n\to\infty}\|z_n - Tz_n\| \ge \epsilon$, $\epsilon > 0$ Suppose $d = diam\, C$ and for some $r > 0$, we have $\lim_{n\to\infty}\|u_n - z_n\| = \lim_{n\to\infty}\|v_n - z_n\| = r$.

Choose

$t < \epsilon/d$where$t > 0$.Therefore

$t < \epsilon/[2\|u_n - z_n\| - \alpha(\|u_n - z_n\|)]$

Also if$n \to \infty$,

$t < \epsilon/[\|u_n - Tu_n\| + 2\|u_n - z_n\| - \alpha(\|u_n - z_n\|)]$

And,

$\|u_n - Tz_n\| \le \|u_n - Tu_n\| + \|Tu_n - Tz_n\|$

$\le \|u_n - Tu_n\| + \|u_n - z_n\| - \alpha(\|u_n - z_n\|)$

(3.1)

If u_n is replaced by v_n in (3.1) we have the same result.

Now, by using the inequality (2.2) and when $n \to \infty$ then

$\|u_n - v_n\| \le \left\|u_n - \frac{1}{2}(z_n + Tz_n)\right\| + \left\|v_n - \frac{1}{2}(z_n + Tz_n)\right\|$

$\le [1 - \delta(t)(\|u_n - Tu_n\| + \|u_n - z_n\| - \alpha(\|u_n - z_n\|))]$

$$+ \quad [1-\delta(t)(\|v_n - Tv_n\| + \|v_n - z_n\| - \alpha(\|v_n - z_n\|))]$$

$$\leq ([\|u_n - Tu_n\| + \|u_n - z_n\| - \alpha(\|u_n - z_n\|)] + [\|v_n - Tv_n\| + \|v_n - z_n\| - \alpha(\|v_n - z_n\|)])(1-\delta(t))$$

If $n \to \infty$ we obtain the contradiction

$$2r \leq [r(-\alpha(\|u_n - z_n\|) - \alpha(\|v_n - z_n\|))](1-\delta(t))$$

The next proposition shows that T has a fixed point in C if T is α-strongly pseudocontractive which satisfies $inf\{\|x - Tx\|: x \in C\} = 0$.

Proposition 3.3:

Suppose C is a bounded, closed and convex subset of uniformly convex space X and suppose $T: C \to X$ is α-strongly pseudocontractive mapping which satisfies $inf\{\|x - Tx\|: x \in C\} = 0$. Then T has a fixed point in C.

Proof:

For any numbers $r > 0$, we can define R as $R := \{r: B(0,r) \cap C \neq \emptyset\}$, suppose $c_0 = \inf R$. Therefore $c_0 < \infty$, clearly $c_0 > 0$ or $c_0 = 0$. So we done $c_0 = 0$ implies that $0 \in C$ and $T0 = 0$. Now, if $c_0 > 0$. Choose for each n, $x_n \in B(c_0, \frac{1}{n}) \cap C$ satisfying $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Suppose $\{x_{n_k}\}$ be strongly convergent subsequence of $\{x_n\}$ such that $\|x_{n_k} - x_{n_{k+1}}\| \geq \epsilon$, $k = 1,2,\ldots$, and it have a limit point as it's a fixed point of T. Let $m_k = 1/2(x_{n_k} + x_{n_{k+1}})$ for all k. By using inequality (2.2) and take $t > 0$ is any number smaller than ϵ/c_0, $t \leq \epsilon/\|x_k|$ as $k \to \infty$ we have

$$\|m_k\| \leq (c_0 + (1/n_k))(1-\delta(t)). \quad (3.2)$$

From inequality (3.2) we get $\lim_{k\to\infty} sup\|m_k\| \leq c_0(1-\delta(t)) < c_0$.

Therefore Proposition (3.2) yields $\lim_{k\to\infty}\|m_k - Tm_k\| = 0$, but this contradiction with the definition of c_0.

Now, we will give the proof of theorem (3.1)

Proof of theorem(3.1):

Let $K_n = \overline{conv}\{u_n, u_{n+1}, \ldots\}$, where $\{u_n\}$ be a weakly convergent sequence to u in C such that $\lim_{n\to\infty}\|u_n - Tu_n\| = 0$. Note that these limits are preserved under the weakly translation. Now, we use propositions (3.3) to K_n and there are $y_n \in K_n$ satisfy $y_n = Ty_n$ and it have weak subsequential limit belong to $\bigcap_{n=1}^{\infty} K_n = \{u\}$. Therefore $\lim_{n\to\infty} y_n = u$ .Then u lies in the weak closure of the fixed point set, $F(T)$ of T. Then $F(T)$ is closed and convex, hence weakly closed since X is uniformly convex which implies X is strictly convex and reflexive. So $u \in F(T)$, completing the proof.

Before starting the main theorems of this paper, we will give the definition of the mapping S_t^f as follows:

Definition 3.4[10]

Let T be α-strongly pseudocontractive mappings from a nonempty closed convex subset C of a real Banach space X to X. Define $f: C \to C$ is any given Banach space contraction mapping with a contractive constant $a \in (0,1)$ and a mapping $S_t^f: C \to X$ by

$$S_t^f(x) = tf(x) + (1-t)T(x) \quad x \in C . \quad (3.3)$$

Clearly from (3.3) that $S_t^f: C \to X$ is a Banach contraction mapping. z_t is the unique solution in C for the following

$$z_t = tf(z_t) + (1-t)T(z_t) \qquad (3.4)$$

for any given $t \in (0,1)$.

Theorem 3.5:

Let X be a reflexive Banach space which admits a weakly sequentially continuous normalize duality mapping J from X to X^*, and C be a nonempty closed convex subset of X. Assume that $f: C \to C$ is a given Banach contraction with a contractive constant $a \in (0,1)$, and $\{z_t: t \in (0,1)\}$ is the net defined by (3.4). Suppose $T: C \to X$ is a α-stronglypseudocontractive mapping, then as $t \to 0$, $\{z_t\}$ converges strongly to a common fixed point $q \in F(T)$ such that q is the unique solution in $F(T)$ for the following variational inequality

$$\langle (I-f)q, j(q-u) \rangle \leq 0, for\ all\ u \in F(T) \quad (3.5)$$

Proof:

Now, we will prove that $\{z_t: t \in (0,1)\}$ is bounded.By use (3.4) and let $u \in F(T)$ we have

$\|z_t - [tf(z_t) + (1-t)u]\| = (1-t)\|T(z_t) - u\| \leq (1-t)\|z_t - u\|$ (3.6)

By applying lemma (2.11) we have

$\|z_t - [tf(z_t) + (1-t)u]\|^2 = \|(1-t)(z_t - u) + t(z_t - f(z_t))\|^2 \geq (1-t)^2\|z_t - u\|^2 + 2t(1-t)\langle z_t - f(z_t), j(z_t - u)\rangle$ (3.7)

We can deduce by (3.6) and (3.7) that

$2t(1-t)\langle z_t - f(z_t), j(z_t - u)\rangle \leq \|z_t - [tf(z_t) + (1-t)u\|^2 - (1-t)^2\|z_t - u\|^2] \leq 0$ .. (3.8)

From (3.8) we get for all $u \in F(T)$ there is $j(z_t - u) \in J(z_t - u)$ such that

$\langle z_t - f(z_t), j(z_t - u)\rangle \leq 0.$ .... (3.9)

It follows by a Banach's contraction principle, and for each $u \in F(T)$

$\langle f(z_t) - f(u), j(z_t - u)\rangle \leq a\|z_t - u\|^2$ (3.10)

Also

$\langle z_t - f(z_t), j(z_t - u)\rangle =$

$\langle z_t - u + u - f(u) + f(u) - f(z_t), j(z_t - u)\rangle$

$= \|z_t - u\|^2 + \langle u - f(u), j(z_t - u)\rangle$

$+\langle f(u) - f(z_t), j(z_t - u)\rangle$

$\geq \|z_t - u\|^2 + \langle u - f(u), j(z_t - u)\rangle - \|f(u) - f(z_t)\| \cdot \|z_t - u\|$

$\geq (1-a)\|z_t - u\|^2 + \langle u - f(u), j(z_t - u)\rangle$......(3.11)

By (3.9) and (3.10) we get that

$(1-a)\|z_t - u\|^2 + \langle u - f(u), j(z_t - u)\rangle \leq 0$ (3.12)

It follows from (3.12)

$(1-a)\|z_t - u\|^2 \leq \langle u - f(u), j(z_t - u)\rangle \leq \|u - f(u)\| \cdot \|u - z_t\|.$

(3.13)

Then

$\|u - z_t\| \leq \frac{\|u - f(u)\|}{1-a}$.................. (3.14)

This prove that $\{z_t : t \in (0,1)\}$ is bounded. Also $\{T(z_t) : t \in (0,1)\}$ and $\{f(z_t) : t \in (0,1)\}$ are bounded.

Now, we get by (3.4) that

$\|z_t - T(z_t)\| \leq t\|f(z_t) - T(z_t)\| \to 0$ as $t \to 0$.

Therefore

$\lim_{t\to 0}\|z_t - T(z_t)\| = 0.$ .... (3.15)

To see that $\{z_t : t \in (0,1)\}$ is relatively compact, note that X is reflexive and $\{z_t : t \in (0,1)\}$ is bounded.

Let $\{z_{t_n}\}$ be a subsequence of $\{z_t\}$ with $\{t_n\} \in (0,1)$, also there is a subsequence of $\{z_{t_n}\}$ (we denote it by $\{z_{t_n}\}$) satisfy

$z_{t_n} \rightharpoonup q$ when $t_n \to 0$ (3.16)

We can deduce by (3.15)

$\|z_{t_n} - T(z_{t_n})\| \to 0$ when $t_n \to 0$

It follows by (3.16) and theorem (3.1) yields $I - T$ satisfy the demiclosed principle. Hence

$q \in F(T)$ (3.17)

Replace u by q and t by t_n, then

$\|z_{t_n} - q\|^2 \leq \frac{\langle q - f(q), j(q - z_{t_n})\rangle}{1-a}$ (3.18)

Since J is weakly sequentially continuous, we have

$\lim_{t_n\to 0}\|z_{t_n} - q\|^2 \leq \lim_{t_n\to 0}\frac{\langle q - f(q), j(q - z_{t_n})\rangle}{1-a} = 0$ (3.19)

The end step of the prove in this theorem that the entire net $\{z_t : t \in (0,1)\}$ converges strongly to q.

Suppose there is another subsequence $\{z_{t_i}\}$ of $\{z_t\}$ such that $z_{t_i} \to \acute{q}$ as $t_i \to 0$. And use the same argument as given above, then $\acute{q} \in F(T)$ Now, we will prove that

$\langle (I - f)\acute{q}, j(\acute{q} - u)\rangle \leq 0$, for all $u \in F(T)$

It follows that $\{z_t - u\}$ and $\{z_t - f(z_t)\}$ are bounded for all $u \in F(T)$.

Since J be normalized duality mapping and $\lim_{t_i\to 0} z_{t_i} = \acute{q}$ that

$$|\langle (I-f)z_{t_i}, j(z_{t_i} - u)\rangle - \langle (I-f)\acute{q}, j(\acute{q} - u)\rangle|$$
$$= |\langle (I-f)z_{t_i} - (I-f)\acute{q}, j(z_{t_i} - u)\rangle + \langle (I-f)\acute{q}, j(z_{t_i} - u) - j(\acute{q} - u)\rangle|$$
$$\leq \|(I-f)z_{t_i} - (I-f)\acute{q}\| \cdot \|z_{t_i} - u\| + |\langle (I-f)\acute{q}, j(z_{t_i} - u) - j(\acute{q} - u)\rangle|$$
$$\to 0 \text{ as } t_i \to 0.$$

Thus, by (3.9) we have

$\langle (I - f)\acute{q}, j(\acute{q} - u)\rangle = \lim_{t_i\to 0}\langle (I - f)z_{t_i}, j(z_{t_i} - u)\rangle \leq 0$ ....(3.20)

And we may also prove that

$\langle (I - f)q, j(q - u)\rangle \leq 0$ .... (3.21)

It is clear that

$\langle (I - f)q - (I - f)\acute{q}, j(q - \acute{q})\rangle \leq 0.$

Then

$\|q - \acute{q}\|^2 \leq \langle f(q) - f(\acute{q}), j(q - \acute{q})\rangle \leq a\|q - \acute{q}\|^2.$

From the above inequality we have $q = \acute{q}$, hence the theorem has been proved.

Theorem 3.6 :

Let X be a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping J from X to X^*, and

C be a nonempty closed subset of X. Assume $f: C \to C$ is a given Banach contraction with a contractive constant $a \in (0,1)$. Let $\{z_t : t \in (0,1)\}$ be the net defined by (3.4) and $T: C \to X$ be non-self of $\propto$-strongly pseudocontractive mapping.

Let $x_0 \in C$ be any given point, and $\{x_n\}$ be generated by the iteration (1.1) and $x_0 \in C$. If, in addition, the following conditions hold:

(a) $\lim_{n\to\infty} \propto_n = 0$

(b) $\sum_{n=0}^{\infty} \propto_n = \infty$

(c) $\|x_n - Tx_n\| \to 0$, as $n \to \infty$

then$\{x_n\}$ generated by $x_0 \in C$ and iteration (1.1) converges strongly to $q = \lim_{t\to 0} z_t$ such that q is the unique solution in$F(T)$ for the following variational inequality

$\langle (I-f)q, j(q-u)\rangle \leq 0,$ for all $u \in F(T)$

Proof:

On the one hand, $\lim_{t\to 0} z_t = q \in F(T)$ from theorem (3.5). We want to prove $\{x_n\}$ generated by iteration (1.1) is bounded sequence.

So

$$\|x_{n+1} - q\| \leq \propto_{n+1} \|f(x_n) - q\| + (1-\propto_{n+1})\|T_{n+1}x_n - q\| \leq \propto_{n+1} (\|f(x_n) - f(q)\| + \|f(q) - q\|) + (1-\propto_{n+1})[\|x_n - q\| - \propto(\|x_n - q\|)]$$

$$\leq \propto_{n+1} \|f(q) - q\| + \propto_{n+1} a\|x_n - q\| + (1-\propto_{n+1})\|x_n - q\| - (1-\propto_{n+1}) \propto (\|x_n - q\|)$$

$$= \propto_{n+1} \|f(q) - q\| - (1-\propto_{n+1}) \propto (\|x_n - q\|) + (1-(1-a)\propto_{n+1})\|x_n - q\|$$

$$\leq \max\left\{\|x_n - q\|, \frac{\propto(\|x_n - q\|)}{1-a}, \frac{\|f(q)-q\|}{1-a}\right\}.$$

By using induction method, we have

$$\|x_n - q\| \leq max\left\{\|x_0 - q\|, \frac{\propto(\|x_0 - q\|)}{1-a}, \frac{\|x_0 - q\|}{1-a}\right\} \text{for each } n \geq 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.22)$$

Then, we get $\{x_n\}$ bounded. By theorem (3.5), we have $\{z_t\}$ be bounded, suppose there are a constant $M > 0$ satisfying

$\|z_t\|^2 + \|z_t\| + \|q\|^2 + \|q\| + \|x_n\|^2 + \|x_n\| < M$for all $n \geq 0$ and $t \in (0,1)$.

It follows by lemma (2.11) and (1.1),we have

$$\|x_{n+1} - q\|^2 \leq (1-\propto_{n+1})^2\|Tx_{n+1} - q\|^2 + 2\propto_{n+1} \langle f(x_n) - q, j(x_{n+1} - q)\rangle$$

$$\leq (1-\propto_{n+1})^2[\|x_n - q\| - \propto(\|x_n - q\|)]^2$$

$$+2\propto_{n+1} \langle f(x_n) - q, j(x_{n+1} - q)\rangle \quad (3.23)$$

By applying f is contraction mapping, we get

$$2\propto_{n+1} \langle f(x_n) - q, j(x_{n+1} - q)\rangle =$$

$$2\propto_{n+1} \langle f(x_n) - f(q) + f(q) - q, j(x_{n+1} - q)\rangle$$

$$\leq 2\propto_{n+1} a\|x_n - q\| \cdot \|x_{n+1} - q\| + 2$$

$$\propto_{n+1} \langle f(q)q, j(x_{n+1} - q)\rangle$$

$$\leq \propto_{n+1} a(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + 2\propto_{n+1} \langle f(q) - q, j(x_{n+1} - q)\rangle$$

$$\leq \propto_{n+1} a(\|x_n - q\|^2 + \|x_{n+1} - q\|^2)$$

$$+2\propto_{n+1} \gamma_{n+1} \quad (3.24)$$

Such that

$\gamma_n = max\{0, \langle f(q) - q, j(x_n - q)\rangle\}$for each $n \geq 0$(3.25)

By (3.23) and (3.24) we have

$$\|x_{n+1} - q\|^2$$

$$\leq (1-\propto_{n+1})^2[\|x_n - q\| - \propto(\|x_n - q\|)]^2$$

$$+\propto_{n+1} a(\|x_n - q\|^2 + \|x_{n+1} - q\|^2)$$

$$+2\propto_{n+1} \gamma_{n+1}.$$

Hence,

$$\|x_{n+1} - q\|^2 \leq 2(1-\propto_{n+1})^2\|x_n - q\|^2 - 2(1-\propto_{n+1})^2 (\propto(\|x_n - q\|))^2$$

$$+\propto_{n+1} a(\|x_n - q\|^2 + \propto_{n+1} a\|x_{n+1} - q\|^2)$$

$$+2\propto_{n+1} \gamma_{n+1}.$$

Therefore,

$$(1-\propto_{n+1} a)\|x_{n+1} - q\|^2$$

$$\leq ((1-\propto_{n+1})^2 + \propto_{n+1} a) + \|x_n - q\|^2$$

$$+2\propto_{n+1} \gamma_{n+1}.$$

Now, by use the condition (a) that there is a nonnegative n_1 such that

$4(1-a)\propto_{n+1} \in [0,1)and1-\propto_{n+1} a \geq \frac{1}{4}$

for all $n \geq n_1$.

Then, we get

$$\|x_{n+1} - q\|^2$$

$$\leq \frac{((1-\propto_{n+1})^2 + \propto_{n+1} a)}{1-\propto_{n+1} a}\|x_n - q\|^2 + \frac{2\propto_{n+1} \gamma_{n+1}}{1-\propto_{n+1} a}$$

$$\leq \left(1 - \frac{2\propto_{n+1}(1-a)}{1-\propto_{n+1} a}\right)\|x_n - q\|^2 + \frac{\propto_{n+1}{}^2}{1-\propto_{n+1} a}(\|x_n\| + \|q\|)^2$$

$$+\frac{2\propto_{n+1} \gamma_{n+1}}{1-\propto_{n+1} a}$$

$$\leq (1 - 2\propto_{n+1}(1-a))\|x_n - q\|^2 + \frac{\propto_{n+1}{}^2}{1-\propto_{n+1} a}$$

$$2(\|x_n\| + \|q\|)^2 + \frac{2\propto_{n+1}\gamma_{n+1}}{1-\propto_{n+1} a}$$

$$\leq (1 - 2(1-a)\propto_{n+1})\|x_n - q\|^2 + 4M(\propto_{n+1})^2 + 4\propto_{n+1}\gamma_{n+1}$$

$$\leq (1 - 2(1-a)\propto_{n+1})\|x_n - q\|^2 + 4\propto_{n+1}(M\propto_{n+1} + \gamma_{n+1}) \quad (3.26)$$

Finally, we will prove $\lim_{n\to\infty} \Upsilon_n = 0$.

Since X is reflexive and $\{x_n\}$ is bounded, we can get a subsequence

$\{x_{n_k}\} \subset \{x_n\}$satisfy

$$\limsup_{n\to\infty} \langle q - f(q), j(q - x_n)\rangle$$

$$= \lim_{k\to\infty} \langle q - f(q), j(q - x_{n_k})\rangle. \quad (3.27)$$

Also, there is a subsequence $\{x_{n_i}\} \subset \{x_{n_k}\}$ such that $x_{n_i} \rightharpoonup x_0$ as $i \to \infty$. Hence we conclude that by using (3.27)

$$\limsup_{n\to\infty} \langle q - f(q), j(q - x_n)\rangle$$
$$= \lim_{n\to\infty} \langle q - f(q), j(q - x_{n_i})\rangle.$$

By condition (c), we obtain

$$\|x_{n_i} - T(x_{n_i})\| \to 0 \text{ as } i \to \infty.$$

Now, we show by the theorem (3.1) that $x_0 \in F(T)$. Then, we have by (3.5)

$$\limsup_{n\to\infty} \langle q - f(q), j(q - x_n)\rangle$$
$$= \lim_{i\to\infty} \langle q - f(q), j(q - x_{n_i})\rangle$$
$$= \langle q - f(q), j(q - x_0)\rangle \leq 0.$$

It follows that for all given $\epsilon > 0$ there correspondingly exists a positive integer $n_2 > n_1$ satisfy

$\langle q - f(q), j(q - x_n)\rangle < \epsilon$, for all $n > n_2$.

This is yields $0 \leq \Upsilon_n < \epsilon$, and then $\Upsilon_n \to 0$. We set $\lambda_n = 2(1-a) \propto_{n+1}$,

$a_n = \|x_n - q\|^2$,

$b_n = 4 \propto_{n+1} (M \propto_{n+1} + \gamma_{n+1})$ and $c_n = 0$ for all $n > n_2$. It follows from (3.26) and lemma (2.10) that

$\lim_{n\to\infty} \|x_n - q\| = 0$,

i.e., $x_n \to q = \lim_{t\to 0} z_t$ and $q \in F(T)$.

This completes the proof of theorem (3.6).

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