Injectivity and chain conditions on y-closed submodules

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Abstract
Let R be a commutative ring with identity and let M be a unital left R-module.Goodearl introduced the following concept :A submodule A of an $R-$ module M is an $y-$ closed submodule of M if $\frac{M}{A}$ is nonsingular.In this paper we introduced an $y-$ closed injective modules andchain condition on $y-$ closed submodules.

Key words: y-closed submodules, y-closed injective, chain conditions on y-closed submodules

الاغماريه وشروط السلسلة على المقاسات الجزئية المغلقة من النمط –y

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الخلاصة
لتكن R حلقة ابدالية ذات عنصرمحايد و M مقاس احادي ايسر معرف عليها.
كوديرل قدم المفهوم التالي: يدعى المقاس الجزئي A من المقاس M بانه مقاس جزئي من النمط –yاذاكان $\frac{M}{A}$
غير شاذ. في هذاالبحث قدمنا مفهوم الاغمارية بالنسبة الى صنف المقاسات الجزئية المغلقة من النمط –y
وكذلك شرط السلسلة على المقاسات الجزئيةالمغلقة من النمط –y.

1.Introduction.

Following C.Gomes[1] .Let M_1 and M_2 be an $R-$ modules M_1 is called M_2-C- injective if every homomorphism $\alpha : K \rightarrow M_1$, where K is a closed submodule of M_2, there exists a homomorphism $\beta : M_2 \rightarrow M_1$ such that $\beta \circ i = \alpha$.Where i is the inclusion map .

This concept lead us to introduced the following: Let M and N be $R-$ modules . The module M is called $N-y-$ *closed injective* if every homomorphism $f : K \rightarrow M$, where K is an $y-$ closed submodule of N , there exists a homomorphism $g : N \rightarrow M$ such that $g \circ i = f$. Where i is the inclusion map.

Following Goodearl [2] . An $R-$ module M is called a module with ascending (respectively, descending) chain condition (briefly ACC, respectively, DCC) on closed submodules, if every ascending (respectively , descending) chain of closed submodules of M is finite.

This concept lead us to introduced the following : an $R-$ module M is called a module with ascending (respectively, descending) chain condition (briefly ACC, respectively, DCC) on $y-$ closed submodules, if every ascending (respectively , descending) chain of $y-$ closed submodules of M is finite.

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In this paper, we give properties of $y-$closed injective module and chain condition on $y-$closed submodules.

In section one, we introduced the concept of $y-$ closed injective modules with some examples and basic properties,We prove that for an $R-$module M and an $y-$closed submodule$A$ of M. If A is $M-y-$*closed injective* module , then A is a direct summand of M .

In section two, we introduced the concept of chain condition on $y-$closed submodules with some examples and basic properties,We prove that a ring R satisfies ascending chain condition on $y-$closed ideals if and only if $\frac{R}{A}$ satisfies ascending chain condition on $y-$closed ideals for every $y-$closed ideal A of R .

1-Injectivity on *y*-closed submodules.

Definitions (1.1) :

1- Let M and N be $R-$modules . the module M is called $N-y-$*closed injective* if for every homomorphism $f : K \to M$, where K is an $y-$closed submodule of N , there exists a homomorphism $g : N \to M$ such that $g \circ i = f$. where$i$ is the inclusion map .

2- M_1 and M_2 are said to be relatively $y-$*closed injective* modules if M_i is M_j – injective, for every $i, j = \{1,2\}, i \neq j$

3- An $R-$module M is called *self* $-y-$*closed injective* module if for every homomorphism $f : K \to M$,where K is an $y-$closed submodule of M, there exists a homomorphism $g : M \to M$ such that $f = g \circ i$. Where i is the inclusion map .

4- An $R-$module M is called $y-$*closed injective* module if M is $N-y-$closed injective , for every Rmodule N .

Proposition (1.2) :

Let M be an $R-$module , then M is $N-y-$*closed injective* , for every singular $R-$module N .

Proof :

Let K be an $y-$closed submodule of N and let $f : K \to M$ be any R-homomorphism.Since N is singular , then N is the only $y-$closed submodule of N, by (2.1.3) .One can easily show that M is $N-y-$*closed injective* .

It is clear that every injective $R-$module is $y-$ *closed injective* . The converse is not true . For example,

Consider the module Z_n as $Z-$module . Since Z_n as $Z-$module is singular, then Z_n is the only $y-$closed submodule of Z_n and hence Z_n is $y-$*closed injective*. But Z_n as $Z-$module is not divisible where $0 = n Z_n \neq Z_n$. Hence Z_n is not injective.

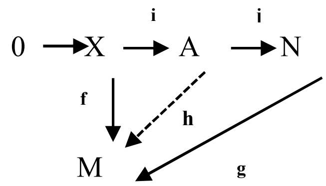
Now we give some basic properties of $y-$*closed injective* modules.

Proposition (1.3):

Let M be an $N-y-$*closed injective* module. If A is an $y-$closed submodule of N, then M is $A-y-$*closed injective* module .

Proof :

Let X be an $y-$closed submodule of A and let $f : X \to M$ be any $R-$homomorphism. Now consider the following diagram .



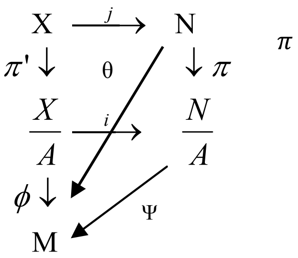
Where i,j are the inclusion maps. Since X is an $y-$closed submodule of A and A is an $y-$closed submodule of N, then X is an $y-$closed submodule of N, by (2.1.10) . But M is $N-y-$*closed injective*, therefore there exists a homomorphismg $: N \to M$ such that $f = g \circ j \circ i$. Take $h = g|_A : A \to M$. Clearly that $h \circ i = f$. Thus M is $A-yc-$injective .

Proposition (1.4):

Let M and N be $R-$modules and A is a submodule of N . If M is $N-y-$*closed injective* module then M is $\frac{N}{A}-y-$*closed injective* module .

Proof :

Let $\frac{X}{A}$ be an $y-$closed submodule of $\frac{N}{A}$ and let$\phi : \frac{X}{A} \to M$ be an $R-$homomorphism. We want to show that there exists a homomorphism $\Psi : \frac{N}{A} \to M$ such that $\Psi \circ i = \Phi$,where i is the inclusion map.Now consider the following digram.



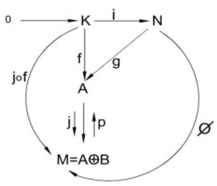
Where j is the inclusion map and π, π' are the natural epimorphism . Since $\frac{X}{A}$ is an $y-$closed submodule of $\frac{N}{A}$, then X is an $y-$closed submodule of N, by (2.1.5-2) . Since M be an $N-y-closed\ injective$, then there exists a homomorphism $\theta : N \to M$ such that $\Phi \circ \pi' = \theta \circ j$. $\theta(A) = \Phi \circ \pi'(A) = \Phi\ (0) = 0$ .So $ker\pi \subseteq ker\theta$. Now let $\Psi : \frac{N}{A} \to M$ be a map define as follows $\Psi(n + A) = \theta(n) \forall n \in N$. One can easily show that Ψ is well define. Now $\Psi \circ i\ (x+A) = \Psi(x+A) = \Psi \circ \pi(x) = \theta(x) = \Phi \circ \pi'(x) = \Phi(x+A)$. Thus M is $\frac{N}{A} - y - closed$ *injective* module.

Proposition (1.5) :

Let $M = A \oplus B$ be an $R-$module and N be an $R-$module .If M is $N-y-closed\ injective$, then A is $N-y-closed\ injective$.

Proof :

Let $M = A \oplus B$ be $N-y-closed\ injective$. To show that A is $N-y-closed\ injective$, let K be an $y-$closed submodule of N and let $f : K \to A$ be an $R-$homomorphism . Now consider the following diagram .



Where i , j are the inclusion maps and P is the projection map . Since M is $N-y-closed\ injective$, then there exists an $R-$homomorphism $\Phi : N \to M$ such that $j \circ f = \Phi \circ i$. Let $g = P \circ \Phi$, $g \circ i = P \circ \Phi \circ i = P \circ j \circ f = f$. Thus A is $N-y-closed\ injective$ module .

Recall that a submodule N of R-module M is called a fully invarientsubmodule of M ,if for evey endomorphis f:M→M, f(N)⊆N, see[1].

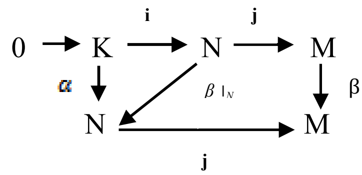
The following proposition gives a condition under which an $y-$closed submodule of an $y-$closed injective module is $y-$closed injective .

Proposition(1.6) :

Let M be a $self-y-closed\ injective$ module, then every fully invariant $y-$closed submodule N of M is $self-y-closed\ injective$.

Proof :

Suppose that M is $self-y-closed\ injective$ Let K be an $y-$closed submodule of N and let $\alpha : K \to N$ be a homomorphism . Since K is an $y-$closed submodule of N and N is an $y-$closed submodule of M, then K is an $y-$closedsubmodule of M , by (2.1.10).



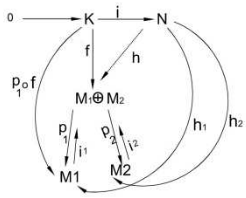
Since M is $self-y-closed\ injective$ module , then there exists a homomorphism $\beta : M \to M$ such that $j \circ \alpha = \beta \circ j \circ i$. Since N is fully invariant, then $\beta(N) \subseteq N$. And $\beta\ |_N : N \to N$ is a homomorphism and α is the restriction of this homomorphism .

Proposition (1.7):

Let M_1 ,M_2 and N be $R-$modules . If M_1 and M_2 are $N-y-closed\ injective$modules , then $M_1 \oplus M_2$is $N-y-closed\ injective$.

Proof :

Suppose that M_1 and M_2 are $N-y-closed\ injective$ modules . Let K be an $y-$closed submodule of N and let $f : K \to M_1 \oplus M_2$be an $R-$homomorphism. Now consider the following diagram .



Where i_1 ,i_2 are the inclusion maps , P_1 ,P_2 are the projection map . Since M_1 and M_2 are $N-y-closed\ injective$ modules , then there exists homomorphisms$h_1 : N \to M_1$ and $h_2 : N \to M_2$

such that $P_1 \circ f = h_1 \circ i$ and $P_2 \circ f = h_2 \circ i$. Define $h : N \to M_1 \oplus M_2$ as follows $h(n) = (h_1(n), h_2(n))$, $n \in N$. To show that $f = h \circ i$. Let $k \in K$,then $f(k) = (m_1, m_2)$, where $m_1 \in M_1$ and $m_2 \in M_2$. $h \circ i(k) = h(i(k)) = (h_1(i(k)), h_2(i(k))) = (P_1 \circ f(k), P_2 \circ f(k)) = (m_1, m_2)$ .Thus $M_1 \oplus M_2$ is $N - y -$ *closed injective* module .

Proposition (1.8) :

Let M be an R – module, and let A be an y – closed submodule of M. If A is $M - y -$ *closed injective* module , then A is a direct summand of M .

Proof :

Assume that A is $M - y -$ *closed injective* and let $I : A \to A$ be the identity map Since A is $M - y -$ *closed injective* module , then there exists a homomorphism $f : M \to A$ such that $I = f \circ i$. Claim that $M = \ker f \oplus A$. To show that , let $x \in M$, then one can easily show that $x - f(x) \in \ker f$. Thus $x = f(x) + (x - f(x)) \in A + \ker f$.

To show that $A \cap \ker f = 0$, let $x \in A \cap \ker f$, so $f(x) = 0$. But $f(x) = x$, therefore $x = 0$ .Thus A is a direct summand of M .

Recall that a module M is called a CLS-module if every y-closed submodule is a direct summand.

The following proposition gives a characterization of CLS – module in terms of y – *closed injective* modules .

Proposition (1.9) :

Let M be an R – module, then the following statements are equivalent .

1- M is a CLS – module .

2- Every module is $M - y -$ *closed injective* module .

3- Every y – closed submodule of M is $M - y -$ *closed injective* module .

Proof :

(1) $\Rightarrow$ (2)

Let M_1 be an R – module , let A be an y – closed submodule of M and let $\alpha : A \to M_1$ be an R – homomorphism.Since A is an y – closed submodule of M and M is CLS, then A is a direct summand of M . So $M = A \oplus A'$, for some submodule A' of M . Define $\beta : M \to M_1$ as follows .

$$\beta(x+y) = \begin{cases} \alpha(x) & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

Where $x \in A$ and $y \in A'$ .Hence β extends to α .

(2) $\Rightarrow$ (3) It is clear .

(3) $\Rightarrow$ (1) It is follow by proposition(1.8) .

Before we give the next proposition we need the following definitions.

Recall that an R – module M is called directly finite if $f \circ g = I_M$, implies that $g \circ f = I_M$ for all $f, g \in End(M)$, see [2] .

Recall that an R – module M is said to be co – hopfian if every monomorphism $f : M \to M$ is an isomorphism , see [6] .

Proposition (1.10) :

An y – *closed injective* R – module M is directly finite if and only if it is co – hopfian .

Proof :

$\Rightarrow$) Let $f : M \to M$ be a monomorphism and let $I : M \to M$ be the identity map . Since M is an y – closed submoduleof M and M is y – *closed injective*, then there exists a homomorphism $g : M \to M$ such that $g \circ f = I_M$. But M is directly finite, therefore $f \circ g = I_M$ and hence f is an isomorphism .Thus M is co – hopfian .

$\Leftarrow$) Let $f, g : M \to M$ be an R – homomorphism such that $g \circ f = I_M$, we want to show that $f \circ g = I_M$. Since $g \circ f = I_M$, then f is one to one. But M is co – hopfian, therefore f is an isomorphism . Claim that g is one to one, to show that . Let $g(m_1) = g(m_2)$, $m_1, m_2 \in M$. Since f is on – to, then $m_1 = f(x_1)$, $m_2 = f(x_2)$,$x_1, x_2 \in M$. So $g(f(x_1)) = g(f(x_2))$ and hence $m_1 = m_2$ One can easily show that $f \circ g = I_M$ .Thus M is directly finite.

Theorem (1.11) :

Let $M = M_1 \oplus M_2$ be an R – module such that M_1 is nonsingular and every submodule of M isomorphic to an y – closed submodule of M is itself y – closed in M . If for every y – closed submodule N of M such that $N \cap M_1 = 0$, there exists a submodule A of M such that $M = M_1 \oplus A$ and $N \subseteq A$, then M_1 is $M_2 - y -$ closed injective .

Proof :

Let K be an y – closed submodule of M_2 and let $f : K \to M_1$ be an R – homomrphism. Let $H = \{ -f(k) + k, k \in K \}$ and let $g : K \to H$ be a map defined by $g(k) = -f(k) + k$,$\forall k \in K$. One can easily show that g is an isomorphism . By the first isomorphism theorem $\frac{M}{M_2} \cong M_1$. Since M_1 is nonsingular , then M_2 is any – closed submodule of M. But K is any y – closed submoduleof M_2, therefore K is an y – closed submodule of M. But $K \cong H$, therefore H is an y – closed submoduleof M. Claim that $H \cap M_1 = 0$. To show this, let $x \in H \cap M_1$, so $x = -f(k) + k$, where $k \in M_2$. So $x + f(k) = k \in M_1 \cap M_2$ and hence $x = 0$. Now by hypothesis there exists a submodule A of M such that $H \subseteq A$ and $M = M_1 \oplus A$. Let $\pi : M_1 \oplus AM_1$. It is clear that $ker\ \pi = A$. Let $h = \pi \mid_{M2} : M_2 \to M_1$. Now $h \circ i(k) = h(k) = \pi(k) = \pi(f(k) - f(k) + k)$, $= \pi(f(k)) = f(k)$. Thus $h \circ i = f$.

We end this section by the following proposition.

Proposition(1.12) :
Let M be a *self – y – closed injective* module and let K be an y – closed submodule of M. If K is isomorphic to a direct summand of M, then K is a direct summand of M.

Proof :
Suppose that$M$ is a *self – y – closed injective* module and K be an y – closed submodule of Msuch that K is isomorphic to a direct summand B of M. Then there exists an isomorphism $\alpha : K \to B$be.Since M is *self – y – closed injective* module and B is a direct summand of M, then B is *M – y – closed injective* .So there exists a homomorphism $\beta : M \to B$ such that $\alpha = \beta \circ i$. Claim that $M = K \oplus kerf$ .To verify that. For every $x \in M$, there exists $y \in K$ such that $\beta(x) = \alpha(y) = \beta(y)$. Since $x - y \in ker\ \beta$, then $x = y + (x - y) \in K + ker\ \beta$ .Thus $M = K + kerf$ .To show $K \cap ker\ \beta = 0$, let $x \in K \cap ker\ \beta$, so $\beta(x) = 0$. But $\beta(x) = \alpha(x)$. Hence $x = 0$. Thus $M = K \oplus kerf$.

2. Chain condition on *y*-closed submodules.

Definition (2.1)
An R – module M is said to have the ascending chain condition (briefly *ACC*) on y – closed submodules if every ascending chain $A_1 \subseteq A_2 \subseteq \ldots$ of y – closed submodules of M is finite. That is, there exists $k \in Z_+$ such that $A_n = A_k$, for all $n \geq k$.

Definition (2.2)
An R – module M is said to have the descending chain condition (briefly *DCC*) on y – closed submodules if every descending chain $A_1 \supseteq A_2 \supseteq \ldots$ of y – closed submodule of M is finite. That is, there exists $k \in Z_+$ such that $A_n = A_k$, for all $n \geq k$.

Remarks and examples (2.3)
1- Every noetherain (respectively ,artinian) module satisfies (respectively DCC) on y – closed submoduls . For example , consider the module Z_6 as Z – module has *ACC* (respectively *DCC*) on y – closed submodules .
2- Every uniform module satisfies *ACC* (respectively DCC) on y – closed submoduls. For example ,the module Z as Z – module has *ACC* (respectively *DCC*) on y – closed submodules .
3- Every singular module satisfies*ACC* (respectively DCC) on y – closed submoduls.For example ,Z_4as a Z – module .
4- Consider the module $\oplus_{i \in N} Z_2$as Z_2 – module . One can easily show that $\oplus_{i \in N}$ Z_2 as Z_2 – module does not satisfies *ACC* and *DCC* on y – closed submodule .
5- If M satisfies *ACC* (respectively DCC) on closed submoduls, then M satisfies *ACC* (respectively DCC) on y – closed submoduls.The converse is true M is nonsingular .

The following example show that the converse is not true in general.
Consider the module $\oplus_{i \in N} Z_2$ as Z – module . Clearly that $\oplus_{i \in N} Z_2$ is singular and hence satisfies *ACC* and *DCC* on$y$ – closed submodule by (3) . But one can easily show that $\oplus_{i \in N} Z_2$ not satisfies *ACC* (*DCC*) on closed submodule**.**

Proposition (2.4)
Let $M = M_1 \oplus M_2$be an R – module . If M satisfies *ACC* on y – closed submodules, then M_1 satisfies *ACC* on y – closed submodules .

Proof :
Let $A_1 \subseteq A_2 \subseteq ..$, be ascending chain of y–closed submodules of M_1. Since $\frac{M_1 \oplus M_2}{A_i \oplus M_2} \cong \frac{M_1}{A_i} \oplus \frac{M_2}{M_2}$, by [7], then $A_i \oplus M_2$ is an y–closed submodule of $M_1 \oplus M_2 = M$, by (2.1.20) for each $i \in Z_+$. So we have an ascending chain $A_i \oplus M_2 \subseteq A_i \oplus M_2 \subseteq \ldots$, of y – closed submodules of M and hence there exists $k \in Z_+$ such that $A_n \oplus M_2 = A_k \oplus M_2 \forall n \geq k$ .Thus $A_n = A_k$.

Proposition (2.5)

Let M be an R- module and let A be an y – closed submodule of M. If M satisfies ACC (respectively DCC) on y – closed submodule, then A satisfies ACC (respectively DCC) on y – closed submodule .

Proof :

Let M satisfies ACC on y – closed submodule and let $A_1 \subseteq A_2 \subseteq \dots$, be ascending chain of y – closed submodule of A . Since A is an y – closed submodule of M, then A_i is an y – closed submodule of M and hence $A_1 \subseteq A_2 \subseteq \dots$, be ascending chain of y – closed submodule of M .So there exists $k \in Z_+$ such that $A_n = A_k$, $\forall n \geq k$.

By the same way we can prove the theorem for DCC on y – closed submodule .

Proposition (2.6)

Let M be an R- module and. If M satisfies ACC (respectively DCC) on y – closed submodules, then $\frac{M}{A}$ satisfies ACC (respectively DCC) on y – closed submodules.

Proof :

Suppose that M satisfies ACC on y – closed submodule and let $\frac{B_1}{A} \subseteq \frac{B_2}{A} \subseteq \dots$, be ascending chain of y – closed submodules of $\frac{M}{A}$, then B_i is an y – closed submodule of M for each $i \in Z_+$, by (2.1.5-2) . Hence $B_1 \subseteq B_2 \subseteq \dots$, be ascending chain of y – closed submodule of M. So there exists $k \in Z_+$ such that $B_n = B_k$,$\forall n \geq k$. Thus $\frac{B_n}{A} = \frac{B_k}{A}$ $\forall n \geq k$.

Using the same argument one can prove the theorem for DCC on y – closed submodules .

Proposition (2.7)

Let M be an R – module, then M satisfies ACCon y – closed submodules if and only if $\frac{M}{A}$ satisfies ACC on y – closed submodules, for every y – closed submodule$A$ of M .

Proof :

$\Rightarrow$) It is clear by proposition (2.6).

$\Leftarrow$)suppose that $\frac{M}{A}$ satisfies ACC of y – closed submodules, for every y – closed submodule$A$ of M and let $A_1 \subseteq A_2 \subseteq \dots$, be ascending chain of y – closed submodules of M . Since $A_1 \subseteq A_i$ and A_i is an y – closed submodule in M for each $i \in Z_+$, then $\frac{A_i}{A_1}$ is an y – closed submodule of $\frac{M}{A_1}$,for each $i \in Z_+$.

Thus we have the following ascending chain $\frac{A_1}{A_1} \subseteq \frac{A_2}{A_1} \subseteq \dots$ of y – closed submodules of $\frac{M}{A_1}$.

Since $\frac{M}{A_1}$ satisfies ACC on y – closed submodules, then there exists $k \in Z_+$ such that $\frac{A_n}{A_1} = \frac{A_k}{A_1}$, for each $n \geq k$. It is follow that $A_n = A_k$ for each $n \geq k$.

Proposition (2.8)

Let M be an R – module such that the sum of any two y – closed submodules of M is again an y – closed submodule . If A is an y – closed submodule of M such that A and $\frac{M}{A}$ satisfies ACC (respectively, DCC) on y – closed submodules, then M satisfies ACC (respectively, DCC) on y – closed submodules.

Proof :

Assume that $B_1 \subseteq B_2 \subseteq \dots$, is an ascending chain of y – closed submodules of M , then $B_i \cap A$ is an y – closed submodule of A , $\forall i \in Z_+$. Since B_i and A are y – closed submodules of M , then by our assumption $B_i + A$ is an y – closed submodule of M and hence $\frac{B_i + A}{A}$ is an y – closed submodule of $\frac{M}{A}$.

Now consider the following two ascending chains of y – closed submodules of A and $\frac{M}{A}$ respectively : $B_1 \cap A \subseteq B_2 \cap A \subseteq \dots$, and $\frac{B_1 + A}{A} \subseteq \frac{B_2 + A}{A} \subseteq \dots$. But A and satisfies ACC on y – closed submodules .Therefore there exists $k_1, k_2 \in Z_+$ such that $B_n \cap A = B_{k_1} \cap A$, $\forall n \geq k_1$ and $\frac{B_n + A}{A} = \frac{B_{k_2} + A}{A}$, $\forall n \geq k_2$and hence $B_n + A = B_{k_2} + A$, $\forall n \geq k_2$.

Let $k = max\{k_1, k_2\}$, so $B_n \cap A = B_k \cap A$, $\forall n \geq k$ and $B_n + A = B_k + A$,$\forall n \geq k$. Now, $\forall n \geq k B_n = B_n \cap (B_n + A) = B_n \cap (B_k + A) = B_k + (B_n \cap A) =$

$B_k + (B_k \cap A) = B_k$. Thus M satisfies ACC on y – closed submodules .
By the same way we can prove the proposition for DCC on y – closed submodules .

Proposition (2.10)
Let M be an R – module and let $A_1, A_2, \ldots A_n$ be y – closed submodules of M, if $\frac{M}{A_i}$ is satisfies DCC on y – closed submodules, for each $i = 1,2,\ldots,n$, then $\frac{M}{A_1 \cap A_2 \cap \ldots \cap A_n}$ satisfies DCC on y – closed submodules .

Proof :
Let $\frac{M}{A_i}$ is satisfies DCC on y – closed submodules . Since A_i is an y – closed submodules of M ,then $\frac{M}{A_i}$ is nonsingular for each i = 1,2,…,n .so $\frac{M}{A_i}$ satisfies DCC on y – closed submodules. Thus $\frac{M}{A_i}$ satisfies DCC on closed submodules. Thus $\frac{M}{A_1 \cap A_2 \cap \ldots \cap A_n}$ satisfies DCC on closed submodules, by theorem (3.2.9) . Hence $\frac{M}{A_1 \cap A_2 \cap \ldots \cap A_n}$ satisfies DCC on y – closed submodules .

Proposition (2.11)
Let $M = Rm_1 + Rm_2 + \ldots + Rm_n$ be an R – module such that Rm_i is an y – closed submodule of M , for each i = 1,2,…,n . If M satisfies DCC on y – closed submodule, then $\frac{R}{Ann(M)}$ satisfies DCC on y – closed submodule.

Proof :
Let $M = Rm_1 + Rm_2 + \ldots + Rm_n$, where $m_1, m_2, \ldots, m_n \in M$. For each $i = 1,2,\ldots,n$. Let $\Phi_i : R \to Rm_i$ be a map define as follows $\Phi_i(r) = rm_i$, $\forall r \in R$. It is clear that Φ_i is an epimorphism. By the first isomorphism $\frac{R}{ker\Phi_i} \cong Rm_i, \forall i = 1,2,\ldots,n$.
But $ker\Phi_i = \{ r \in R : \Phi_i(r) = 0\} = ann\ (m_i)$, So $\frac{R}{ann\ (m_i)} \cong Rm_i$. Since M satisfies DCC on y –closed submodule, then Rm_i satisfiesDCC on y – closed submodules, $\forall i = 1,2,\ldots,n$, by (2.5).

Since by [9],proposition(2.3-4), p.38]$ann(M) = ann(m_1) \cap ann(m_2) \cap \ \ldots \ \cap ann(m_n)$, then $\frac{R}{ann\ (M)}$ satisfies DCC on y–closed submodule, by proposition (2.10) .
The following proposition gives a characterization of rings with chain condition on y – closed ideals.

Proposition (2.12)
Let R be a ring, then the following statements are equivalent.
(1) R satisfies ACC on y – closed ideals .
(2) $\frac{R}{A}$ Satisfies ACC on y – closed ideals, for every y – closed ideal A of R .

Proof :
(1) $\Rightarrow$(2) It is clear by proposition (3.2.6) .
(2) $\Leftarrow$(1)
Let $A_1 \subseteq A_2 \subseteq \ldots$, be ascending chain of y–closed ideals of R . Since $A_1 \subseteq A_i$and A_i is an y – closed ideals in R for each $i \in Z_+$, then $\frac{A_i}{A_1}$ is an y – closed in $\frac{R}{A_1}$ for each $i \in Z_+$.[8].
Thus we have the following ascending chain of y – closed ideals of $\frac{R}{A_1}$:
$\frac{A_1}{A_1} \subseteq \frac{A_2}{A_1} \subseteq \ldots$. Since $\frac{R}{A_1}$ satisfies ACC on$y$ – closed ideals (by our assumption) , then there exists $k \in Z_+$ such that $\frac{A_n}{A_1} = \frac{A_k}{A_1}$, for each $n \geq k$. It is follow that $A_n = A_k$ for each $n \geq k$.
The following proposition gives a condition under which a direct sum of two modules satisfies ACC is again satisfies ACC.

Proposition (2.13)
Let M_1 and M_2 be R – modules such that $ann\ M_1 + ann\ M_2 = R$, if M_1 and M_2 satisfies ACC on y – closed submodule, then $M_1 \oplus M_2$ satisfies ACC on y – closed submodules .

Proof :
Let $A_1 \subseteq A_2 \subseteq \ldots$, be ascending chain of y – closed submodules of $M_1 \oplus M_2$. Since $annM_1 + annM_2 = R$, then by the same way of the prove [1,prop.4.2,CH.1], $A_i = C_i \oplus D_i$, where C_i is a submodule of M_1 and $D_i$is a submodule of M_2 . Since $A_i = C_i \oplus D_i$ be an y – closed submodule of $M_1 \oplus M_2$, then C_i and D_i are y – closed submodule in M_1 and M_2 respectively, by [3]. So we have two ascending

chain of y – closed submodule of M_1 and M_2 respectively :$C_1 \subseteq C_2 \subseteq \dots$, be ascending chain of y – closed submodules of $M_1$and $D_1 \subseteq D_2 \subseteq \dots$, be ascending chain of y – closed submodules of M_2.Thus there exists $k_1, k_2 \in Z_+$ such that $C_n = C_{k1} \forall n \geq k_1$ and $D_n = D_{k2} \forall n \geq k_2$. Let $k = max\{ k_1 , k_2\}$. To show $A_n= A_k$, let $A_n = C_n + D_n = C_{k1}+D_{k2}$. But $\forall n \geq k C_{k1}=C_k$ and $\forall n \geq k D_k=D_{k2}$, Therefore $A_n = C_k + D_k = A_k \forall n \geq k$.

Proposition (2.14)

Let $M =\bigoplus_{i\in I} M_i$ be an R – module where I is a finite index set.If M satisfies *ACC(*respectively *DCC)* on y – closed submodule, then M_i satisfies *ACC(*respectively *DCC)* on y – closed submodules, for each $i \in I$. The converse is true if every y – closed submodule of M is fully invariant .

Proof :

$\Rightarrow$) Clear by the proposition (3.2.4) .

$\Leftarrow$) suppose that $A_1 \subseteq A_2 \subseteq \dots$, is an ascending chain of y – closed submodules of M and let $\pi_i : M \to M_i$ be the projection maps, for each $j \in J$ claim that $A_j =\bigoplus_{i\in I}(A_j \cap M_i)$, to verify this , let $x \in A_j$, then $x =\sum_{i\in I} m_i$, $m_i \in M_i$. Since A_j is any– closed submodule of M , then by our assumption , A_j is fully invarient and hence $\pi_i(x) = m_i \in A_j \cap M_i$. So $x \in \bigoplus_{i\in I}(A_j \cap M_i)$. Thus $A_j \subseteq \bigoplus_{i\in I}(A_j \cap M_i)$. But $\bigoplus_{i\in I}(A_j \cap M_i) \subseteq A_j$, therefore $A_j= \bigoplus_{i\in I}(A_j \cap M_i)$.

Since $\frac{M}{A_j}$ is nonsingular and $\frac{M}{A_j} = \frac{\bigoplus_{i\in I} M_i}{\bigoplus_{i\in I}(A_j \cap M_i)} \cong \bigoplus_{i\in I}(\frac{M_i}{(A_j \cap M_i)})$, then $A_j \cap M_i$ is an y – closed submodule of M_i , for each $i \in I$.

For each $i \in I$ we have the following ascending chain of y – closed submodule of M_i:$(A_1 \cap M_i) \subseteq (A_2 \cap M_i) \subseteq \dots$, . But M_i satisfies *ACC* on y – closed submodules. So for each $i \in I$, there exists $k_i \in Z_+$ such that $A_n \cap M_i = A_{ki} \cap M_i$, $\forall n \geq k_i$ .Let $k = max\{k_i : i \in I \}$. So $A_n= \bigoplus_{i\in I}(A_n \cap M_i) = \bigoplus_{i\in I}(A_k \cap M_i) = A_k$,$\forall n \geq k$. Thus M satisfies *ACC* on y – closed submodules .

By the same way we can prove the proposition for *DCC* on y – closed submodules .

Proposition (2.15)

Let M be an R – module such that $M = A_1 +A_2 +\dots +A_n$, where A_i is an y – closed submodule of M , $\forall i = 1,2,\dots,n$ if A_i satisfies *ACC* (respectively, *DCC*) on y – closed submodules$\forall i = 1,2,\dots,n$, then M satisfies *ACC* (respectively , *DCC*) on y – closed submodules.

Proof :

By induction . If $k = 1$, then $M = A_1$ and hence M satisfies *ACC* on y – closed submodules .

Now, assume that is true when $k \leq n - 1$. Now let $k = n$ and let $B = \sum_{i=1}^{k-1} A_i$. So B satisfies *ACC* on y – closed submodules . By the second isomorphism theorem $\frac{M}{A_n} = \frac{B+A_n}{A_n} \cong \frac{B}{B\cap A_n}$.

Since B satisfies *ACC* on y – closed submodules and $B\cap A_n$ be an y – closed submodule of B, then $\frac{B}{B\cap A_n}$ satisfies *ACC* on y – closed submodules, by (2.6). Thus M as R – module satisfies *ACC* on y – closed submodule

Before we give our next result. We give the following lemma.

Lemma (2.16)

Let M be an R – module and $\bar{R}= \frac{R}{ann (M)}$. If A is an y – closed submodule of M as R – module, then A is an y – closed submodule of M as $\bar{R}$ – module .

Proof :

Assume that A be an y – closed submodule of M as R – module and hence $\frac{M}{A}$ is nonsingular as R – module . Now consider $\frac{M}{A}$ as $\bar{R}$ – module.

Let $m +A \in Z$ ($\frac{M}{A}$), then $ann_{\bar{R}}(m+A) \subseteq_e \bar{R}$.

Claim that ann_R (m+A) $\subseteq_e$ R To verify this , let $0 \neq r \in R$, we want to show that there exists $r_1 \in R$ such that $0\neq rr_1 \in ann_R(m+A)$.

if $r \in ann(M)$,then $0 \neq r = r .1 \in ann_R (m+A)$

Now assume that $r \notin ann(M)$, then $r + ann(M) \neq ann(M)$ .But $ann_{\bar{R}} (m+A) \not\subset_e \bar{R}$, therefore there exists $r_1 + ann(M) \in \bar{R}$ such that $ann (M) \neq r r_1 + ann(M) \in ann_{\bar{R}} (m+A)$ and hence $0 \neq r r_1 \in ann_R (m+A)$. But $\frac{M}{A}$ is nonsingular as R – module , then $m \in A$. Thus $\frac{M}{A}$ is nonsingular as $\bar{R}$ – module .

Proposition (2.17)

Let M be an R – module. If M satisfies ACC (respectively , DCC) on y –closed submodule as $\bar{R} = \frac{R}{ann(M)}$, then M satisfies ACC (respectively , DCC) on y – closed submodule as R – module .

Proof :

Assume that M as $\bar{R}$ – module satisfies ACC on y – closed submodule . We want to prove that M as$R$ – module satisfies ACC on y – closed submodule . Let $A_1 \subseteq A_2 \subseteq \dots$, be ascending chain of y – closed submodules of M as an R – module, so by previous lemma A_i is an y – closed submodule as an$\bar{R}$ – module . Since M as $\bar{R}$ – module satisfies ACC on y – closed submodule, then there exists $k \in Z_+$ such that $A_n = A_k$ for each $n \geq k$ .Thus M as R – module satisfies ACC on y – closed submodule.

Proposition (2.18)

Let M be a faitfull and multiplication R – module, if R satisfies ACC(respectively DCC) on y – closed ideals, then M satisfies ACC (respectively, DCC) on y – closed submodules.

Proof :

Let $A_1 \subseteq A_2 \subseteq \dots$, be descending chain of y – closed submodules.Since M is multiplication module , then $A_i = (A_i : M)M$, $\forall i = 1,2,\dots,n$. Clearly $(A_1 : M) \subseteq (A_2 : M) \subseteq \dots$ Since A_i is an y – closed in M, then $(A_i : M)$ is an y – closed in R, for each $i \in Z_+$, by (2.1.21) But R satisfies ACC on y – closed ideals, therefore there exist $k \in Z_+$ such that $(A_n : M) = (A_k : M)$ for each $n \geq k$. Thus $A_n = A_k$,$\forall n \geq k$.

By the same way we can prove the theorem for ACCon y – closed submodule .

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