DOI: 10.24996/ijs.2019.60.10.18

ISSN: 0067-2904

On a Class of Meromorphic Multi valent Functiions Convoluted with Higher Derivat ives of Fractional Calculus Operator

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Abstract
The main goal of this paper i s to study and discuss a new class of meromorphic "func ions[which are multivalent defined by [fractional calculus operators. Coefficients estimates , radiis of satarlikeness , convexity and closed-to- convexity are studied. Also distortion and closure theorems for the class $\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ are considered.

Keywords: "Meromorphic Functions , Fractional calculus, Radius of convexity, starlikeness", convexity and c losed-to- convexity , distortion and closure theorems .

حول صنف من الدوال الميرومورفية المتعددة التكافؤ المرتبطة مع المشتقات العليا لمؤثر التفاضل الكسري

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الخلاصة:
الهدف الرئيسي من هذا البحث هو دراسة ومناقشة لصنف جديد من الدوال الميرومورفية متعددة التكافؤ المعرفة بواسطة مؤثر التفاضل الكسري, تم دراسة المعاملات التخمينية,انصاف اقطار النجمية, التحدبية و القريبة من التحدبية. ايضا تم دراسة نظرية التشوه ونظرية الانغلاق لهذا الصنف.

1.Introduction

Let $\sum_b$ "denotes the c lass of meromophic funct ion s defined by:

$$f(w) = w^{-b} + \sum_{i=b}^{\infty} a_i w^i , (b \in \mathbb{N}) \quad (1)$$

which are anal ytic and p-valent in the punctured unit d isk
$U^*=\{w \in \mathbb{C} :(0 < |w| <1)\}$.

A function $f \in \sum_b$ is "said to be in the cl ass $\sum_b^*(\alpha)$ of meromorphic p-valenty starlike function(see Duren[1]) of order α if" :

$$-\text{Re}\left\{\frac{wf\prime(w)}{f(w)}\right\} > \alpha, (w \in U^*, (0 \leq \alpha < b), b \in \mathbb{N}) . \quad (2)$$

A function $f \in \sum_b$ is "sa id to be in the c lass $\sum_b^k(\alpha)$ of meromorphic b-valently convex funct ion of order α if" :

$$-Re\left\{1 + \frac{w(f''(w))}{f'(w)}\right\} > \alpha, (w \in U^*, (0 \leq \alpha < b), b \in \mathbb{N}). \quad (3)$$

In this paper , we discuss and st udy a new cl ass of meromorphic multivalent functions" b- valently conv ex functions by using of the fractional" calculus oper ators contained in

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Definition (1):

$$M_{0,w}^{\lambda,\mu,v,\eta}f(w)=\begin{cases}\frac{\Gamma(\mu+v+\eta-\lambda)\Gamma(\eta)}{\Gamma(\mu+\eta)\Gamma(v+\eta)}w^{-b+\eta+1}J_{0,w}^{\lambda,\mu,v,\eta}\,[w^{\mu+b}f(w)](0\le\lambda<1),\\ \frac{\Gamma(\mu+v+\eta-\lambda)\Gamma(\eta)}{\Gamma(\mu+\eta)\Gamma(v+\eta)}w^{-b-\eta+1}I_{0,w}^{-\lambda,\mu,v,\eta}\,[w^{\mu+b}f(w)](-\infty\le\lambda<0)\end{cases} \quad (4)$$

where $J_{0,W}^{\lambda,\mu,v,\eta}$ "is the generalized fractional derivative operator of order α normalized by"

$$J_{0,w}^{\lambda,\mu,v,\eta}\, f\,(w)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dw}\left\{w^{\lambda-\mu}\int_0^w t^{\eta-1}(w-t)^{-\lambda}\ {}_2F_1\left(\mu-\lambda,1-v;1-\lambda;1-\frac{t}{2}\right)f(t)dt\right\} \quad (5)$$

(r ∈ R^+ ,r>(max{o,μ}-ŋ) and(0≤ λ<1),μ,ŋ ∈ R),

where f is analytic function in a simply– connected region of the w-plane containing the origin and the multiplicity of $(w-t)^{-\lambda}$is removed by requiring log (w-t)to be real when (w-t)>0 , provided further that :

$$("f\,(w)) = 0(|w|^r) \qquad (w\to 0) , \quad (6)$$

and $I_{0,w}^{-\lambda,\mu,,v,\eta}$ is the generalized fractional integral operator of order - λ $(-\infty<\lambda<0)$ normalized by

$$I_{0,w}^{\lambda,\mu,v,\eta}\, f\,(w)=\frac{w^{-\lambda-\mu}}{\Gamma(\lambda)}\int_0^w t^{\eta-1}\,(w-t)^{-(\lambda+1)}\ {}_2F_1\left(\lambda+\mu,\ -v;\lambda;1-\frac{t}{2}\right)f(t)dt \quad (7)$$

(λ >0,μ,ŋ∈R,ir∈R^+ and r > (max{0,μ}-ŋ)),

Where f is constrained and the multiplicity of $(w\text{-t})^{\lambda-1}$ is removed as above and r [is given by the order estimates (6).

by using (5) and (7) it follows from

$$J_{0,w}^{\lambda,\mu,v,1}\ (f(w))=\ J_{0,w}^{\lambda,,\mu,v}(f(w)), \quad (8)$$

and

$$I_{0,w}^{\lambda,\mu,v,1}(f(w))=I_{0,w}^{\lambda,\mu,v}\ (f(w)), \quad (9)$$

where $J_{0,w}^{\lambda,\mu,v}$ and $I_{0,w}^{\lambda,\mu,v}$ are the familiar Owa –Saigo -Srivastava generalized fractional derivative and integral operators(see, e.g. ,[2] and[3] see also [4]).

Also

$$J_{0,w}^{\lambda,\lambda,v,1}\ (f(w))=D_w^{\lambda}(f(w)), \qquad (0\le\lambda<1) \quad (10)$$

and

$$I_{0,w}^{\lambda,-\lambda,v,1}(f(w))=D_w^{-\lambda}\ (f(w)), \qquad (\lambda>0) \quad (11)$$

where D_w^{λ} and $D_w^{-\lambda}$ are the familiar Owa- Saigo-*Srivastana* fractional derivative and integral of order λ (cf.Owa[5]; see also Srivastava and Owa [6]).

in the terms of Gamma function , we have

$$J_{0,w}^{\lambda,\mu,v,\eta}w^k=\frac{\Gamma(k+\eta)\Gamma(k+\eta-\mu+v)}{\Gamma(k+\eta-\mu)\Gamma(k+\eta-\lambda+v)}w^{k+\eta-\mu-1} \quad (12)$$

$0\le\lambda\ <1),\mu,\eta\in R, v\in R^+\ and\ k>(max\{0,\mu\}-\eta)),(($

and

$$I_{0,w}^{\lambda,\mu,v,\eta}w^k=\frac{\Gamma(k+\eta)\Gamma(k+\eta-\mu+v)}{\Gamma(k+\eta-\mu)\Gamma(k+\eta+\lambda+v)}w^{k+\eta-\mu-1} \quad (13)$$

$(\lambda>0),\mu,\eta\in R, v\in R^+\ and\ k>(\max\{0,\mu\}-\eta)).($

By using (1),(12) and (13) in (4), we find

$$M_{o,W}^{\lambda,\mu,v,\eta}f(w)=w^{-b}+\sum_{i=b}^{\infty}\Gamma_i^{\lambda,\mu,v,\eta}a_iw^i \quad (14)$$

Provided that

$(\mu+v+\eta>\lambda),(-\infty<\lambda<1)(\mu>-\eta),(\eta>0),(b\in\mathbb{N}),f\in\sum_b$

and

$$\Gamma_i^{\lambda,\mu,v,\eta}=\frac{(\mu+\eta)_{i+b}(v+\eta)_{i+b}}{(\mu+v+\eta-\lambda)_{i+b}(\eta)_{i+b}}. \quad (15)$$

The operator $M_{0,w}^{\lambda,\mu,v,\eta}f(w)$ reduces to the well -known Ruscheweyh derivative $D^{\lambda}f(w)$ for meromorphich univalent functions [3]

We are study a subclass of (1)def ine below

Definition (2) : A function $f \in \sum_b$ is in the class $\sum_b$ $(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ if it satisfies the condition :

$$\left|\frac{\gamma\left(\frac{w(M_{0,w}^{\lambda,\mu,v,\eta}f(w))^{q+1}}{(M_{0,w}^{\lambda,\mu,v,\eta}f(w))^{q}}+(b+q)\right)}{\frac{w(M_{0,w}^{\lambda,\mu,v,\eta}f(w))^{q+1}}{(M_{0,w}^{\lambda,\mu,v,\eta}f(w))^{q}}+(2\alpha-\gamma)}\right| < \beta, \qquad (16)$$

for som α $(o \le \alpha < 1)$, $\beta(0 \le \beta < 1), \gamma(0 \le \gamma \le 1), q \in \mathbb{N}\cup\{0\}, q < b, b \in \mathbb{N}, (-\infty < \lambda$ $1), (\mu + v + \eta > \lambda), (\mu > -\eta), (\eta > 0) and (v > -\eta)$.

The class $\sum_b$ $(\lambda, , \mu, v, \eta, \gamma, \alpha, \beta)$ reduces to the cl ass studied rece ntly by Darus [7].

Definition(3): Let $\sum_b^+$ denote the subclass $\sum_b$ normalized by

$$\left(M_{0,w}^{\lambda,\mu,v,\eta}f(w)\right)^q = \left(\frac{(b+q-1)!}{(b-1)!}\right)(-1)^q w^{-b-q} + \sum_{i=b}^{\infty}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q)!}a_i w^{i-q}, (a_i \ge 0; (b \in \mathbb{N}))$$

and

$$\left(M_{0,w}^{\lambda,\mu,v,\eta}f(w)\right)^{q+1} = \frac{(b+q)!}{(b-1)!}(-1)^{q+1}w^{-b-q-1} + \sum_{i=b}^{\infty}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_i w^{i-q-1} \qquad (17)$$

we define a new subclass $\sum_b^+$ $(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ by

$\sum_b^+$ $(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ $=\sum_b^+ \cap \sum_b$ $(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.

2. Coefficient Estimates:

Theorem (1): Suppose that $f \in \sum_p$ and

$$\sum_{i=b}^{\infty}\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_i$$

$$\le \beta\ (b+q+\gamma-2\alpha))\frac{(b+q-1)!}{(b-1)!}, \qquad (18)$$

Where $\Gamma_i^{\lambda,\mu,v,\eta}$ is normalized by (15) and the cond itions with (16) hold .

Th en $f \in \sum_b$ $(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.

Proof: "Let use suppose that inequality (18) is true.

Further assume that"

$$\Omega(f) = \left|\gamma(w(M_{0,w}^{\lambda,\mu,v,\eta}f(w))^{q+1}) + \gamma(b+q)(M_{0,w}^{\lambda,\mu,v,\eta}f(w))^q\right|$$
$$-\beta\left|w(M_{0,w}^{\lambda,\mu,v,\eta}f(w))^{q+1} + (2\alpha-\gamma)(M_{0,w}^{\lambda,\mu,v,\eta}f(w))^q\right| \le 0.$$

By using (14),we find that

$$\Omega(f) = \left|\frac{\gamma(b+q)!}{(b-1)!}(-1)^{q+1}w^{-b-q} + \sum_{i=b}^{\infty}\gamma\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_i w^{i-q}\right.$$
$$\left.+\gamma(b+q)\frac{(b+q-1)!}{(b-1)!}(-1)^q w^{-b-q} + \sum_{i=b}^{\infty}\gamma(b+q)\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q)!}a_i w^{i-q}\right|$$
$$-\beta\left|\frac{(b+q)!}{(b-1)!}(-1)^{q+1}w^{-b-q} + \sum_{i=b}^{\infty}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_i w^{i-q}\right.$$
$$\left.+(2\alpha-\gamma)\frac{(b+q-1)!}{(b-1)!}(-1)^q w^{-b-q} + \sum_{i=b}^{\infty}(2\alpha-\gamma)\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q)!}a_i w^{i-q}\right| \le 0$$
$$= \left|\frac{\gamma(b+q)\ (b+q-1)!}{(b-1)!}(-1)^q(-1)^1 w^{-b-q} + \sum_{i=b}^{\infty}\gamma\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_i w^{i-q}\right.$$
$$\left.+\gamma(b+q)\frac{(b+q-1)!}{(b-1)!}(-1)^q w^{-b-q} + \sum_{i=b}^{\infty}\gamma(b+q)\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q)(i-q-1)!}a_i w^{i-q}\right|$$
$$-\beta\left|\frac{(b+q)(b+q-1)!}{(b-1)!}(-1)^q(-1)^1 w^{-b-q} + \sum_{i=b}^{\infty}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_i w^{i-q}\right.$$
$$\left.+(2\alpha-\gamma)\frac{(b+q-1)!}{(b-1)!}(-1)^q w^{-b-q} + \sum_{i=b}^{\infty}(2\alpha-\gamma)\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q)(i-q-1)!}a_i w^{i-q}\right| \le 0$$

$$= \left[\left| \sum_{\mathrm{i=b}}^{\infty} (\gamma + \frac{\gamma(b+q)}{(i-q)}) \Gamma_i^{\lambda,\mu,v,\eta} \frac{i!}{(i-q-1)!} a_i w^{i-q} \right| - \beta \left(\left| -(b+q+\gamma-2\alpha) \frac{(b+q-1)!}{(b-1)!} \, w^{-b-q} + \sum_{i=b}^{\infty} (1 + \frac{2\alpha-\gamma}{i-q}) \Gamma_i^{\lambda,\mu,v,\eta} \frac{i!}{(i-q-1)!} a_i w^{i-q} \right| \right) \right] \leq 0$$

$$\sum_{\mathrm{i=b}}^{\infty} (\gamma + \frac{\gamma(b+q)}{(i-q)}) \Gamma_i^{\lambda,\mu,v,\eta} \frac{i!}{(i-q-1)!} |a_i| r^i - \beta \left(\left| -(b+q+\gamma-2\alpha) \frac{(b+q-1)!}{(b-1)!} w^{-b} \right| - \left| \sum_{i=b}^{\infty} \left(\frac{i-q+2\alpha-\gamma}{i-q} \right) \Gamma_i^{\lambda,\mu,v,\eta} \frac{i!}{(i-q-1)!} a_i w^i \right| \right)$$

$$\sum_{\mathrm{i=b}}^{\infty} (\gamma + \frac{\gamma(b+q)}{(i-q)}) \Gamma_i^{\lambda,\mu,v,\eta} \frac{i!}{(i-q-1)!} |a_i| r^i - \beta(b+q+\gamma-2\alpha) \frac{(b+q-1)!}{(b-1)!} r^{-b} + \sum_{i=b}^{\infty} \beta \left(\frac{i-q+2\alpha-\gamma}{i-q} \right) \Gamma_i^{\lambda,\mu,v,\eta} \frac{i!}{(i-q-1)!} \, |a_i| \mathrm{r}^{\mathrm{i}} \leq 0$$

$$\sum_{\mathrm{i=b}}^{\infty} \frac{\gamma(\mathrm{i-q}) + \gamma(\mathrm{b+q})}{\mathrm{i-q}} \Gamma_i^{\lambda,\mu,v,\eta} \frac{i!}{(i-q-1)!} a_i - \beta \, (b+q+\gamma-2\alpha) \frac{(b+q-1)!}{(b-1)!} + \sum_{i=b}^{\infty} \beta \left(\frac{i-q+2\alpha-\gamma}{i-q} \right) \Gamma_i^{\lambda,\mu,v,\eta} \frac{i!}{(i-q-1)!} a_i \leq 0$$

$$\sum_{i=b}^{\infty} (\frac{i(\gamma+\beta) + \gamma(b-\beta) + \beta(2\alpha-q)}{i-q}) \Gamma_i^{\lambda,\mu,v,\eta} \frac{i!}{(i-q-1)!} a_i \leq (\beta(b+q+\gamma-2\alpha)) \frac{(b+q-1)!}{(b-1)!} \,. \tag{19}$$

Since the above ineguality holdy for every r,(0< r< 1). Letting $r \to 1^-$ in the(19) we obtain that $\Omega(f) \leq 0$, hence $f \in \sum_b(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.

Theorem (2): If $f \in \sum_b^+$, then $f \in \sum_p^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ if and only

$$\sum_{i=b}^{\infty} (\frac{i(\gamma+\beta) + \gamma(b-\beta) + \beta(2\alpha-q)}{i-q}) \Gamma_i^{\lambda,\mu,v,\eta} \frac{i!}{(i-q-1)!} a_i \leq (\beta(b+q+\gamma-2\alpha)) \frac{(b+q-1)!}{(b-1)!} \,, \tag{20}$$

where $\Gamma_i^{\lambda,\mu,v,\eta}$ is normalized by (15) and every the parameters are constrained as in (1) Theorem **:** "In the Theorem (1) , it is sufficient to prove" the "on ly if " part. **proof**

Let us suppose that $f \in \sum_b^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.

Then

$$\left| \frac{\gamma(\frac{w(M_{0,w}^{\lambda,\mu,v,\eta} f(w))^{q+1}}{(M_{0,w}^{\lambda,\mu,v,\eta} f(w))^q} + (b+q))}{\frac{w(M_{0,w}^{\lambda,\mu,v,\eta} f(w))^{q+1}}{(M_{0,w}^{\lambda,\mu,v,\eta} f(w))^q} + (2\alpha-\gamma)} \right| < \beta,$$

$$= \left| \frac{\sum_{i=b}^{\infty} (\frac{\gamma(i+b)}{i-q}) \Gamma_i^{\lambda,\mu,v,\eta} \frac{i!}{(i-q-1)!} a_i w^{i+b}}{(2\alpha-\gamma-b-q)\frac{(b+q-1)!}{(b-1)!} + \sum_{i=b}^{\infty} \, (\frac{i-q+2\alpha-\gamma}{i-q}) \Gamma_i^{\lambda,\mu,v,\eta} \frac{i!}{(i-q-1)!} a_i w^{i+b}} \right| < \beta.$$

Since Re(w)≤|w| for every w, it follows that

$$\operatorname{Re}\left\{\frac{\sum_{i=b}^{\infty}\left(\frac{\gamma(i+b)}{i-q}\right)\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_i w^{i+b}}{(b+q+\gamma-2\alpha)\frac{(b+q-1)!}{(b-1)!}-\sum_{i=b}^{\infty}\left(\frac{i-q+2\alpha-\gamma}{i-q}\right)\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_i w^{i+b}}\right\}<\beta.$$

letting r→1^-, through real values , we get the result (20).

3. Distortion Theorems

"The distortion property for the functions in the class $\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ is contained in

Theorem (3): Let$f\in\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha\beta,)$.Then

$$\frac{1}{|w|^b}-\frac{\left(\beta(b+q+\gamma-2\alpha)\right)\frac{(b+q-1)!}{(b-1)!}}{\frac{b(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{b-q}\left(\frac{b!}{(b-q-1)!}\right)}|w|^b\le\left|M_{0,w}^{\lambda,\mu,v,\eta}(f(w))\right|$$

$$\le\frac{1}{|w|^b}+\frac{\left(\beta(b+q+\gamma-2\alpha)\right)\frac{(b+q-1)!}{(b-1)!}}{\frac{b(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{b-q}\left(\frac{b!}{(b-q-1)!}\right)}|w|^b,$$

For every the parameters are constrianed in the Theorem (1).

Proof: Since $f\in\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$.

In the of Theorem (2),we have

$$\sum_{i=b}^{\infty}a_i\,\Gamma_i^{\lambda,\mu,v,\eta}\le\frac{\left(\beta(b+q+\gamma-2\alpha)\right)\frac{(b+q-1)!}{(b-1)!}}{\frac{b(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{b-q}\left(\frac{b!}{(b-q-1)!}\right)}.\qquad(21)$$

Now

$$\left|M_{0,w}^{\lambda,\mu,v,\eta}f(w)\right|\le\frac{1}{|w|^b}+\sum_{i=b}^{\infty}a_i\,\Gamma_i^{\lambda,\mu,v,\eta}|w|^i\le\frac{1}{|w|^b}+|w|^b\sum_{i=b}^{\infty}a_i\,\Gamma_i^{\lambda,\mu,v,\eta}.$$

By using (21), we get

$$\left|M_{0,w}^{\lambda,\mu,v,\eta}f(w)\right|\le\frac{1}{|w|^b}+\frac{\left(\beta(b+q+\gamma-2\alpha)\right)\frac{(b+q-1)!}{(b-1)!}}{\frac{b(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{b-q}\left(\frac{b!}{(b-q-1)!}\right)}|w|^b.$$

Also

$$\left|M_{0,w}^{\lambda,\mu,v,\eta}f(w)\right|\ge\frac{1}{|w|^b}-\sum_{i=b}^{\infty}a_i\,\Gamma_i^{\lambda,\mu,v,\eta}|w|^i\ge\frac{1}{|w|^b}-|w|^b\sum_{i=b}^{\infty}a_i\,\Gamma_i^{\lambda,\mu,v,\eta}.$$

Also use of (21), we obtain

$$\left|M_{0,w}^{\lambda,\mu,v,\eta}f(w)\right|\ge\frac{1}{|w|^b}-\frac{\left(\beta(b+q+\gamma-2\alpha)\right)\frac{(b+q-1)!}{(b-1)!}}{\frac{b(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{b-q}\left(\frac{b!}{(b-q-1)!}\right)}|w|^b.$$

The proof is complete.

4. Radii of Starlikeness, Convexity,close -to-convex for the class $\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$:

Theorem (4): If $f\in\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$,then $f$is meromorphically p-valent starlike of order ψ $(0\le\psi<b)$ in |w|<R_1, where

$$R_1=\inf_i\left\{\frac{(b-\psi)\left(\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\right)\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}}{(i+2b-\psi)\left(\beta(b+q+\gamma-2\alpha)\right)\frac{(b+q-1)!}{(b-1)!}}\right\}^{\frac{1}{i+b}},\qquad(22)$$

For every the parameters are constrained in the Theorem (1)

Proof: It is sufficient to show that, For $(0\le\psi<b)$:

$$\left|\frac{w(f'(w))}{f(w)}+b\right|\le(b-\psi).\qquad(23)$$

That is

$$\left|\frac{w(f'(w))+bf(w)}{f(w)}\right|$$

$$= \left|\frac{-bw^{-b} + \sum_{i=b}^{\infty} ia_i w^i + bw^{-b} + \sum_{i=b}^{\infty} ba_i w^i}{w^{-b} + \sum_{i=b}^{\infty} a_i w^i}\right| = \left|\frac{\sum_{i=b}^{\infty}(i+b)a_i w^{i+b}}{1+\sum_{i=b}^{\infty} a_i w^{i+b}}\right|$$

$$\leq \frac{\sum_{i=b}^{\infty}(i+b)\ a_i|w|^{i+b}}{1-\sum_{i=b}^{\infty} a_i|w|^{i+b}} \leq (b-\psi),$$

or equivalently

$$\sum_{i=b}^{\infty}\left(\frac{i+2b-\psi}{b-\psi}\right)\ a_i|w|^{b+i} \leq 1.$$

It is enough to consider

$$|w|^{b+i} \leq \left\{\frac{(b-\psi)\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}}{(i+2b-\psi)(\beta(b+q+\gamma-2\alpha))\frac{(b+q-1)!}{(b-1)!}}\right\}.$$

Therefore, $$|w| \leq \left\{\frac{(b-\psi)\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}}{(i+2b-\psi)(\beta(b+q+\gamma-2\alpha))\frac{(b+q-1)!}{(b-1)!}}\right\}^{\frac{1}{i+b}}. \quad (24)$$

Setting $|w|=R_1(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$in (24), we obtain the radius of starlikeness."

Theorem (5): Let$f \in \sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ .Then f is meromrphically p-valently convex of order ψ $(0 \leq \psi < b)$, in $|w|<R_2$, where

$$R_2=\inf_i\left\{\frac{b(b-\psi)\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}}{i(i+2b-\psi)(\beta(b+q+\gamma-2\alpha))\frac{(b+q-1)!}{(b-1)!}}\right\}^{\frac{1}{i+b}}. \quad (25)$$

proof: Let$f \in \sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$. Then by Theorem (1)

$$\sum_{i=b}^{\infty}\frac{\left(\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\right)\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_i}{(\beta(b+q+\gamma-2\alpha))\frac{(b+q-1)!}{(b-1)!}} \leq 1.$$

For $(0 \leq \psi < b)$, we see that"

$$\left|\frac{w(f''(w))}{f'(w)}+(1+b)\right| \leq (b-\psi).$$

That is

$$\left|\frac{-b(-b-1)w^{-(b+1)} + \sum_{i=b}^{\infty} i(i-1)a_i w^{i-1} - b(1+b)w^{-(b+1)} + \sum_{i=b}^{\infty} i(1+b)a_i w^{i-1}}{-bw^{-(b+1)} + \sum_{i=b}^{\infty} ia_i w^{i-1}}\right|$$

$$= \left|\frac{\sum_{i=b}^{\infty} i(i+b)a_i w^{i-1}}{-bw^{-(b+1)}+\sum_{i=b}^{\infty} ia_i w^{i-1}}\right| \leq \frac{\sum_{i=b}^{\infty} i(i+b)a_i|w|^{i+b}}{b-\sum_{i=b}^{\infty} i\, a_i|w|^{i+b}} \leq (b-\psi),$$

or equivalently

$$\sum_{i=b}^{\infty}\frac{i(i+2b-\psi)}{b(b-\psi)}a_i\ |w|^{i+b} \leq 1.$$

It is enough to consider

$$|w|^{i+b} \leq \left\{\frac{b(b-\psi)\left(\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}\right)}{i(i+2b-\psi)(\beta(b+q+\gamma-2\alpha))\frac{(b+q-1)!}{(b-1)!}}\right\}.$$

Therefore,

$$|w| \leq \left\{\frac{b(b-\psi)\left(\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}\right)}{i(i+2b-\psi)(\beta(b+q+\gamma-2\alpha))\frac{(b+q-1)!}{(b-1)!}}\right\}^{\frac{1}{i+b}}. \quad (26)$$

Setting $|w|=R_2(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ in (26), we obtian the radius of convexity.

Theorem (6): Let$f \in \sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ .Then f is meromorphically p-valently close-to-convex of order ψ $(0 \leq \psi < b)$, in $|w|<R_3$, where

$$R_3=\inf_i\left\{\frac{(b-\psi)\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}}{i(\beta(b+q+\gamma-2\alpha))\frac{(b+q-1)!}{(b-1)!}}\right\}^{\frac{1}{i+b}}. \quad (27)$$

Proof: Let$f \in \sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$. Then by Theorem (1)

$$\sum_{i=b}^{\infty}\frac{\left(\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\right)\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_n}{(\beta(b+q+\gamma-2\alpha))\frac{(b+q-1)!}{(b-1)!}}\le 1.$$

For $(0 \le \psi < b)$ ",we see that"

$$\left|\frac{f'(w)}{w^{-b-1}}+b\right|\le(b-\psi).$$

That is

$$\left|\frac{-bw^{-b-1}+\sum_{i=b}^{\infty}ia_iw^{i-1}+bw^{-b-1}}{w^{-b-1}}\right|\le(b-\psi)$$

$$\sum_{n=p}^{\infty}ia_i|w|^{i+b}\le(b-\psi)$$

or equivalently

$\sum_{i=b}^{\infty}\left(\frac{i}{b-\psi}\right)a_i|w|^{i+b}\le 1.$

It is enough to consider

$$|w|^{i+b}\le\left\{\frac{(b-\psi)\left(\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}\right)}{i(\beta(b+\gamma+q-2\alpha))\frac{(b+q-1)!}{(b-1)!}}\right\}.$$

Therefore,

$$|w|\le\left\{\frac{(b-\psi)\left(\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}\right)}{i(\beta(b+q+\gamma-2\alpha))\frac{(b+q-1)!}{(b-1)!}}\right\}^{\frac{1}{i+b}}. \quad (28)$$

Setting $|w|=R_3(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ in (28), we obtain the radius of close-to- convexity.

5. Closure Theorems

Let the functions $f_k(w),(k=1,2,\dots,s)$, is defined by:

$$f_k(w) = w^{-(b)}+\sum_{i=b}^{\infty}a_{i,k}w^i,\left(w\in U^*,a_{i,k}\ge 0\right). \quad (29)$$

"We shall prove the following closure theorems

Theorem (7): If the function $f_k(w),(k=1,2,\dots,s)$, in the form (29),by in the class $\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ .Then the function $F\in\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$,where

$F(w)=\sum_{k=1}^{s}c_kf_k(w);(c_k\ge 0$ and$\sum_{k=1}^{s}c_k=1)$. (30)

Proof: By using (30), we can write

$$F(w)= w^{-(b)}+\sum_{i=b}^{\infty}(\sum_{k=1}^{s}c_ka_{i,k})w^i. \quad (31)$$

Since $f_k\in\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)(k=1,2,\dots,s)$, therefore

$\sum_{i=b}^{\infty}\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}(\sum_{k=1}^{s}c_ka_{i,k})w^i$

$$=\sum_{k=1}^{s}c_k\,(\sum_{i=b}^{\infty}\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_{i,k})$$

$$\le\sum_{k=1}^{s}c_k\left(\beta(b+q+\gamma-2\alpha)\right)\left(\frac{(b+q-1)!}{(b-1)!}\right)=\left(\beta(b+q+\gamma-2\alpha)\right)\frac{(b+q-1)!}{(b-1)!}.$$

By using Theorem (2), we have $F\in\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$.

The proof is complete.

Theorem (8): The class $\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ is closed under convex linear combination .

Proof: If the functionf_k (k =1,2) givn by (30) be in the class $\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$,then the function

$$g(w)=\sigma f_1(w)+(1-\sigma)f_2(w),\quad(0\le\sigma\le 1), \quad (32)$$

is also in the class $\sum_b^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$.

Since, for $(0 \leq \sigma \leq 1)$,

$$g(w) = w^{-(b)} + \sum_{i=b}^{\infty}[\sigma a_{i,1} + (1-\sigma)a_{i,2}]w^{i},$$

We observe that

$$\sum_{i=b}^{\infty}\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}\{\sigma a_{i,1} + (1-\sigma)a_{i,2}\}$$
$$=\sigma\sum_{i=b}^{\infty}\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_{i,1}$$
$$+(1-\sigma)\sum_{i=b}^{\infty}\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}a_{i,2}$$
$$\leq (\beta(b+q+\gamma-2\alpha)\frac{(b+q-1)!}{(b-1)!}.$$

By using Theorem (2) , we have $g \in \sum_b^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.

Theorem (9): Let $f_{b-1}(w) = w^{-q}$,

$$f_b(w) = w^{-(b)} + \frac{\beta(b+q+\gamma-2\alpha))\frac{(b+q-1)!}{(b-1)!}}{\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}}w^{i}, \tag{33}$$

for every parameters are constrained as in Theorem (1) .

Then

$f \in \sum_b^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ if and only if f ca n be expressed in the form

$$f(w) = \sigma_{b-1}f_{b-1}(w) + \sum_{b}^{\infty}\sigma_i f_i(w), \tag{34}$$

where $\sigma_{b-1} \geq 0, \sigma_i \geq 0$ and $\sigma_{b-1} + \sum_{i=b}^{\infty}\sigma_i = 1$.

Proof: Let

$$f(w) = \sigma_{b-1}f_{b-1}(w) + \sum_{i=b}^{\infty}\sigma_i f_i(w)$$

$$=w^{-(b)} + \sum_{i=b}^{\infty}\frac{\beta(b+\gamma+q-2\alpha)\frac{(b+q-1)!}{(b-1)!}}{\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}}\sigma_i w^{i}.$$

Then

$$\sum_{i=b}^{\infty}\frac{(\beta(b+\gamma+q-2\alpha)\frac{(b+q-1)!}{(b-1)!}(\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!})}{(\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!})(\beta(b+\gamma+q-2\alpha)\frac{(b+q-1)!}{(b-1)!})}\sigma_n$$
$$= \sum_{i=b}^{\infty}\sigma_i = 1-\sigma_{b-1} \leq 1.$$

By using Theorem (2) , we have $f \in \sum_b^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.

Conversely , let $f \in \sum_b^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.

Since

$$a_i \leq \frac{\beta(b+q+\gamma-2\alpha)\frac{(b+q-1)!}{(b-1)!}}{\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}}, \text{for } i \geq b.$$

We may take

$$\sigma_i = \frac{\frac{i(\gamma+\beta)+\gamma(b-\beta)+\beta(2\alpha-q)}{i-q}\Gamma_i^{\lambda,\mu,v,\eta}\frac{i!}{(i-q-1)!}}{\beta(b+q+\gamma-2\alpha)\frac{(b+q-1)!}{(b-1)!}}a_i, \text{ for } i \geq b$$

and $\sigma_{b-1} = 1 - \sum_{i=b}^{\infty}\sigma_i$. Then

$$f(w) = \sigma_{b-1}f_{b-1}(w) + \sum_{i=b}^{\infty}\sigma_i f_i(w).$$

proof is complete.

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