ON CLS- MODULES

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Abstract
Let R be a commutative ring with identity and let M be a unital left R-module. A.Tercan introduced the following concept.An R-module M is called a CLS-module if every y-closed submodule is a direct summand .The main purpose of this work is to develop the properties of y-closed submodules.

Keywords: CLS-modules , Y-closed submodules.

حول المقاسات من النمط –CLS

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الخلاصة
لتكن R حلقة ابدالية ذات عنصر محايد و M مقاس احادي ايسر معرف عليها.تركان قدم مفهوم المقاس من النمط CLS يدعى المقاس M مقاس من النمط CLS اذا كل مقاس جزئي مغلق من النمط–y يكون مركبة مجموع مباشر.الغرض الرئيسي من هذا البحث هو تطوير خواص كل من المقاسات الجزئية المغلقة من النمط–y والمقاسات من النمط CLS .

1.Introduction

About thirty years ago,M.Harada and B.Muller introduced the following concept.An R-module M is called extending (briefly CS-module)if every submodule is essential in a direct summand of M, see[1].Equivalently,M is an extending module if and only if every closed submodule of M is a direct summand .Extending modules has been studied recently by several authors.Among them P.F.Smith and Clark and Mohamed see [1, 2] .
Now recall that, a submodule N of an R-module M is a y-closed submodule if $\frac{M}{N}$ is non-singular, see[3].It is easily seen that every y-closed is closed.

A.Tercan generalizes the extending modules as follow:An R-module M is called a CLS-module if every y-closed submodule of M is a direct summand, see[4]. Note that ,Tercan used the concepts complement(closed submodule) in the sense of closed submodule (y-closed submodule).CLS-modules also have been studied by YongduoWang ,see [5].
In this paper,we give some results on y-closed submodules and CLS-modules.
In section one ,we study the properties of y-closed submodules.We prove that if $f:M \longrightarrow N$ is an epimorphisim and $B\subseteq_y N$,then for every singular submodule A of M ,$f(A)\subseteq B$,see proposition 1.4.
In section two of the paper, we give characterizations of CLS-modules.For example , we show that an R-module M is CLS if and only

if for every y-closed submodule A of M, there is a decomposition $M=M_1 \oplus M_2$ such that $A \subseteq M_1$ and M_2 is a complement of A in M.

Y-closed Submodule

Proposition(1.1) :

Let M be an R-module and let $A \subseteq B \subseteq M$,then

1- If $A \subseteq_y M$, then $A \subseteq_y B$.
2- Let $A \subseteq B \subseteq M$,then $B \subseteq_y M$ if and only if $\frac{B}{A} \subseteq_y \frac{M}{A}$.
3- $A \subseteq_y A+B$ if and only if $A \cap B \subseteq_y B$.
4- If $A \cap_y M$ and $B \cap_y M$,then $A \cap B \subseteq_y M$.

Proof:

1- Clear .
2- Clear by the third isomorphism theorem.
3- Clear by the second isomrphisim theorem.
4- Assume that $A \subseteq_y M$ and $B \subseteq_y M$ to show that $A \cap B \subseteq_y M$,let $m \in M$ such that $m+(A \cap B) \in Z(\frac{M}{A \cap B})$.Thus $ann(m+A \cap B) \subseteq_e R$.Since $ann(m+A \cap B) \subseteq ann(m+A)$,then $ann(m+A) \subseteq_e R$,by[3].But $Z(\frac{M}{A})=0$, therefor m+A=A.By the same way m+B=B.So $m \in A \cap B$ and hence $Z(\frac{M}{A \cap B})=0$.

Proposition(1.2) :

Let A and B be a submodule of an R-module M if $A \subseteq_y B$ and $B \subseteq_y M$, then $A \subseteq_y M$.

Proof:

Let $A \subseteq_y B$ and $B \subseteq_y M$. Now consider the following short exact sequence:

$$0 \rightarrow \frac{B}{A} \xrightarrow{i} \frac{M}{A} \xrightarrow{\pi} \frac{M/A}{B/A} \rightarrow 0$$

Where i is the inclusion map and π is the natural epimorphism. Since$A \subseteq B$ and $B \subseteq_y M$,then $\frac{B}{A} \subseteq_y \frac{M}{A}$ by proposition {1.1-2}.

Since $\frac{B}{A}$ and $\frac{M/A}{B/A}$ are non-singular, then $\frac{M}{A}$ is non-singular by [3].

Note :

Let M be an R-module and $A \subseteq B \subseteq M$,then

1- If $B \subseteq_y M$,then A need not be y-closed submoduleof M ,for example :-
Consider Z as Z-module and $2Z \subseteq Z$ it is clear that $Z \subseteq_y Z$.But $Z(\frac{Z}{2Z})=Z(Z_2)=Z_2$ is singular.
2- If $A \subseteq_y M$, then B need not be y-closed in M,for example:-
$0 \subseteq 2Z \subseteq Z$. Clearly $0 \subseteq_y Z$.But $Z(\frac{Z}{2Z})=Z(Z_2)=Z_2$ is singular.

Note :

An epimorphic image of an y-closed submodule need not be y-closed submodule as the following example show:-

Let $\pi:Z \rightarrow \frac{Z}{4Z}$ be the natural epimorphism .Clearly $0 \subseteq_y Z$,but f(0)=0 is not y-closed in $\frac{Z}{4Z}$ {because $\frac{Z}{4Z} \cong Z4$ is singular}.

Proposition(1.3) :

Let $f:M \rightarrow N$ be an epimorphism and $A \subseteq_y M$. If $\ker f \subseteq A$,then $f(A) \subseteq_y N$.

Proof:

Assume that $A \subseteq_y M$. To show that $f(A) \subseteq_y N$ .let $n \rightarrow N$ such that $ann(n+f(A)) \subseteq_e R$. Since f is an epimorphism ,then n=f(m),for some $m \in M$.Since $\ker f \subseteq A$, then $ann(n+f(A)) \subseteq ann(m+A)$ and hence $ann(n+f(A)) \subseteq_e R$,by[3].But $A \subseteq_y M$, therefore $m \in A$. Thus $n=f(m) \in f(A)$.

Proposition(1.4):

Let $f:M \rightarrow N$ be an R-homomorphism and $B \subseteq_y N$, then for every singular submodule A of M ,$f(A) \subseteq B$.

Proof:

Let π:N$\to \frac{N}{B}$ be the natural epimorphism.

Consider $\pi\, o\, f : M \to \frac{N}{B}$.

Now $\pi\, o\, f|_A$:A $\to \frac{N}{B}$

But A is singular and $\frac{N}{B}$ is non-singular,

Therefor $\pi\, o\, f|_A$ =0,by[3].Thus π(f(A))=0

And hence f(A)$\subseteq$kerπ=B.

The following corollary follows immediately fromproposition{ 1.4}.

Corollary(1.5):

Let N be an R-module and B$\subseteq_y$N.Then (HOM(M,N))(M)$\subseteq$B,for every singular R-module M.

Proposition(1.6):

Let M be an R-module and A$\subseteq_y$M,then Z(M)=Z(A).

Proof:

It is enough to show that Z(M)$\subseteq$Z(A)

Let i:Z(M) $\to$ M be the inclusionmap and π:M $\to \frac{M}{A}$ be the natural epimorphism.

Consider the map $\pi\, o$ i:Z(M) $\to \frac{M}{A}$

Since Z(M) is singular and $\frac{M}{A}$ is non-singular ,then $\pi\, o$ i=0,by [3].So $\pi\, o$ i: (Z(M))=π (Z(M))=0.Thus Z(M)$\subseteq$kerπ=A.But Z(A)=Z(M)$\cap$A, Therefor Z(A)=Z(M).

Proposition(1.7):

Let M be an R-module and let A$\subseteq$B$\subseteq$M and A$\subseteq_y$M, then $\frac{M}{B}$ is singular if and only B$\subseteq_e$M.

Proof:

Let A$\subseteq_y$M and $\frac{M}{B}$ is singular.By the third isomorphism theorem $\frac{M}{B} \cong \frac{M/A}{B/A}$.Since $\frac{M}{A}$ is non-singular ,then by [3] $\frac{B}{A} \subseteq_e \frac{M}{A}$.

Letπ :M $\to \frac{M}{A}$ be the natural epimorphism.By [3],

B=$\pi^{-1}(\frac{B}{A}) \subseteq_e \pi^{-1}(\frac{M}{A})$ = M

The converse is clear by [3].

Proposition(1.8):

Let M be an R-module and B be a maximal and y-closed submodule of M .Then $\frac{M}{B}$ is projective and B is a direct summand of M.

Proof:

Since B is maximal submodule of M ,then $\frac{M}{B}$ is simple and hence semisimple .But $\frac{M}{B}$ is non-singular,therefor $\frac{M}{B}$ is projective,by[3].

Now consider the following short exact sequence:-

$$0 \to B \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{B} \to 0$$

Where i is the inclusion map and π is the natural epimorphism.

Since $\frac{M}{B}$ is projective, then the sequence is splits, by [6].Thus B is a direct summand of M.

Let M be an R-module and N$\subseteq$M.Recall that the resdual of M in N{denoted by [N:M]} is defined as follows:-

[N:M]={r$\in$R,rM$\subseteq$N},see[7]

Proposition(1.9):

Let M be an R-module and N$\subseteq_y$M,then [N:M]$\subseteq_y$R.

Proof:
Let $N \subseteq_y M$ .Assume that [N:M] is not y-closed in R .So there exists $r \in R$ such that $[N:M] \neq r+[N:M] \in Z(\frac{N}{[N:M]})$.Thus $rM \not\subset N$ and hence $\exists\, m_0 \in M$ such that $rm_0 \notin N$.One can easily show that $ann(r+[N:M]) \subseteq ann(rm_0+N)$.Since $ann(r+[N:M]) \subseteq_e R$, then $ann(rm_0+N) \subseteq_e R$
But $\frac{M}{N}$ is non-singular ,thereforr $m_o+N=N$
Which is contradiction.

Proposition(1.10):
Let M be an R-module and let $\{B\alpha, \in \wedge\}$be an independent family of submodules of M and$A\alpha \subseteq B\alpha$, $\forall\, \alpha \in \wedge$.Then $\oplus A\alpha \subseteq_y \oplus B\alpha$ if and only if $A\alpha \subseteq_y B\alpha$, $\forall\, \alpha \in \wedge$.

Proof:
Suppose that $\oplus A\alpha \subseteq_y \oplus B\alpha$
By [8] $\frac{\oplus B\alpha}{\oplus A\alpha} \cong \frac{B\alpha}{A\alpha}$.Then by [3]
$A_\alpha \subseteq_y B_{\alpha}$, $\forall\, \alpha \in \wedge$.
Conversely, $A_\alpha \subseteq_y B_{\alpha}$, $\forall\, \alpha \in \wedge$.
Then $\frac{B\alpha}{A\alpha}$ is non-singular , $\forall\, \alpha \in \wedge$.
and hence
$\oplus \frac{B\alpha}{A\alpha}$ is non-singular by [3]
But $\oplus \frac{B\alpha}{A\alpha} \cong \frac{\oplus B\alpha}{\oplus A\alpha}$, by [8],so
$\oplus A\alpha \subseteq_y \oplus B\alpha$.

1. Characterizations of CLS-modules

Following [4],we say that an R-module M is a CLS-module if every y-closed submodule is a direct summand .
It is know that a direct summand of a CLS-module is CLS ,see[4].
We prove the following:-

Proposition(2.1):
Every y-closed submodule of CLS-module is CLS.

Proof:
Let M be a CLS-module and $A \subseteq_y M$.We want to show that A is a CLS-module .Let $K \subseteq_y A$,then by proposition{ 1.2 }$K \subseteq_y M$.But M is CLS,therefor K is a direct summand of M and hence K is a direct summand of A.

Proposition(2.2):
Let M be a CLS-module and N be a submodule of M, then $\frac{M}{N}$ is a CLS-module.

Proof:
Let $\frac{B}{N} \subseteq_y \frac{M}{N}$ .Then by proposition 1.1-2 $B \subseteq_y M$.But M is CLS,therefor $M=B \oplus K, K \subseteq M$.Since$N \subseteq B$,then one can easily show that $\frac{M}{N} = \frac{B}{N} \oplus \frac{K+N}{N}$ .Thus $\frac{M}{N}$ is CLS-module.
Recall that a module M is called generalized extending if for any submodule N of M , there is a direct summand K of M such that $N \subseteq K$ and $\frac{K}{N}$ is singular, see[9].

Proposition(2.3):
Let M be a generalized extending R-module ,then M is CLS.

Proof:
Let $N \subseteq_y M$.SinceM is generalized extending,then there exists a direct summand K of M such that $N \subseteq K$ and $\frac{K}{N}$ is singular .But $\frac{K}{N} \subseteq \frac{M}{N}$,so is non- singular .Thus K=N .

Proposition(2.4):
An R-module M is a CLS-module if and only if for every y-closed submodule A of M ,there is a decomposition $M=M_1 \oplus M_2$ such that $A \subseteq M_1$ and M_2 is a complement of A in M.

Proof:
$\Rightarrow$) Clear
$\Leftarrow$) let $A \subseteq_y M$,then by our assumption ,there exists a decomposition $M=M_1 \oplus M_2$ such that $A \subseteq M_1$ and M_2 is a complement of A in M.So $A \oplus M_2 \subseteq_e M$,by [3] .Thus$A \subseteq_e M_1$ by[3] and hence $\frac{M1}{N}$ is singular.But $A \subseteq M_1$ and $A \subseteq_y M$,therefor$A \subseteq_y M_1$,by Proposition {1.1-1} .Thus $A=M_1$

Proposition(2.5):
An R-module M is CLS-module if and only if every y-closed submodule of M is essential in a direct summand.

Proof:
$\Rightarrow$) Clear .

$\Leftarrow$) let A$\subseteq_y$M, we want to show that A is a direct summand of M. Since A$\subseteq_y$M, then by our assumption A$\subseteq_e$D, where D is a direct summand of M. Thus $\frac{D}{A}$ is singular.
But $\frac{D}{A} \subseteq \frac{M}{A}$, therefore $\frac{D}{A}$ is non-singular.
Thus A=D and hence M is CLS.

Proposition(2.6):
An R-module M is CLS-module if and only if for every y-closed submodule A of M there exists a decomposition M=$M_1 \oplus M_2$ such that A$\subseteq$ M_1 andA$\oplus$$M_2$$\subseteq_e$M.
Proof:
$\Rightarrow$) Clear .
$\Leftarrow$)let A$\subseteq_y$M, we want to show that A is a direct summand of M.SinceA$\subseteq_y$M,then by assumption there exists a decomposition M=$M_1 \oplus M_2$ such that A$\subseteq$ M_1 andA$\subseteq$$M_2$$\subseteq_e$M.
So $\frac{M}{A \oplus M2}$ is singular by [3].But A$\subseteq$$M_1$and A$\subseteq_y$M, therefore by proposition (1.1-1) A$\subseteq_y$$M_1$.Since $M_2 \subseteq_y M_2$,then by proposition(1.10) A$\oplus$$M_2$$\subseteq_y$ $M_1 \oplus M_2$=M.So $\frac{M}{A \oplus M2}$ is non-singular .Thus M= A$\oplus$$M_2$.

Proposition(2.7):
An R-module M is CLS-module if and only if for every direct summand A of the injective hull E(M) of M such that A$\cap$M$\subseteq_y$M, then A$\cap$M is a direct summand of M.
Proof:
$\Rightarrow$) Clear .
$\Leftarrow$) Let A$\subseteq_y$M and let B be a relative complement of A,then by [3] A$\oplus$B$\subseteq_e$M.SinceM$\subseteq_e$E(M),then A$\oplus$B$\subseteq_e$E(M).Thus E(A)$\oplus$E(B)=E(A$\oplus$B)=E(M).Since E(A) is a summand of E(M) ,then by our assumption E(A)$\cap$M is a summand of M.NowA$\subseteq_e$E(A) and M$\subseteq_e$M,thus by[3] A=A$\cap$M$\subseteq_e$E(A)$\cap$M.Hence by proposition {2.5}, M is CLS.

Proposition(2.8):
Let R be a ring ,then R is a CLS-module if and only if every cyclic non-singular R-module is projective.

Proof:
Let R be a CLS-ring and M=Ra, a$\in$M be a nonsingular R-module. Now consider the short exact sequence .

$$0 \rightarrow ann(a) \xrightarrow{i} R \xrightarrow{f} Ra \quad o$$

Where i is the inclusion homomorphisim and f is a map defined by f(r)= ra,r$\in$R. Clearly that f is an epimorphisim and ker f =ann(a) . Then by first isomorphisim theorem, $\frac{R}{ann(a)} \cong$ Ra.But Ra is non-singular ,therefor ann(a)$\subseteq_y$ R.
Since R is CLS ,then ann(a) is a direct summand of R ,so the sequence is split.Thus by[6] R$\cong$ann(a)$\oplus$Ra.Since R is projective, then Ra is projective by [6].

Conversely , let A be a y-closed ideal in R , then $\frac{R}{A}$ is non-singular.Since R is cyclic ,then $\frac{R}{A}$ is cyclic.By our assumption $\frac{R}{A}$ is projective.Now
consider the following short exact sequence:

$$0 \rightarrow A \xrightarrow{i} R \xrightarrow{\pi} \frac{R}{A} \rightarrow 0$$

Where i is the incusionhomomorphisim and π is the natural epimorphisim ,since $\frac{R}{A}$ is projective,then the sequence is split by [6].
Thus A is a summand of R.It is well known that a direct sum of CLS-modules need not to be a CLS-modules see[4],so we give some conditions under which this relation is true.

Proposition(2.9):
Let M and N be CLS-modules such that annM+annN=R ,then M$\oplus$N is CLS.
Proof:
Let A be a y-closed submodule of M$\oplus$N. Since annM+annN=R,then by the same way of the prove [11,prop.4.2,CH.1] , A=C$\oplus$D,where C is a submodule of M and D is a submodule of N.
Since A=C$\oplus$D$\subseteq_y$M$\oplus$N,then C and D are y-closedsubmodule in M and N respectively by proposition{1.10}.
But M and N are CLS-modules,therefor C is a summand of M and D is a summand of N
So A=$C \oplus D$ is a summand of M$\oplus N$.
Thus M$\oplus N$ is a CLS-module.

Recall that a submodule N of R-module M is called a fully invarientsubmodule of M ,if for every endomorphism f:M $\rightarrow$ M, f(N)$\subseteq$N,see[11].

Proposition(2.10):
Let M=$\oplus M_i$ be an R-module ,such that every y-closedsubmodule of M is fully invarient ,then M is CLS if and only if M_i is CLS $\forall$ $i \in I$.
Proof:
$\Rightarrow$)Clear .
$\Leftarrow$) let S be a y-closed sub module of M .
For each $i \in I$, let π_i:M$\rightarrow M_i$ be the projection map .Now Let $x \in S$,then x=$\sum_{i \in I} mi$, $m_i \in M_i$ and m_i=0 for all but finite many element of $i \in I$.
$\pi_i(x) = m_i$, $\forall i \in I$
.Since S is y-closed submodule ,then by our assumption ,Sis fully invarient and hence $\pi_i(x)=m_i \in S \cap M_i$So $x \in \otimes$ ($S \cap M_i$).Thus $S \subseteq \oplus (S \cap M_i)$.
But $\oplus (S \cap M_i) \subseteq S$,therefor $S = \oplus (S \cap M_i)$.
Since $S \subseteq_y M$, then by proposition {1.1-1} $(S \cap M_i) \subseteq_y M_i \forall i \in I$. But M_i CLS $\forall i \in I$,therefor $(S \cap M_i)$ is a direct summand of M_i
Thus S is a direct summand on M
Recall that an R-module M is called a distributive module if $A \cap (B+C)=(A \cap B)+(A \cap C)$,for all submodules A,B and C of M,see[12].

Proposition(2.11):
Let M= $M_1 \oplus M_2$ be distributive R-module.Then M is CLS if and only if M_1 and M_2 are CLS.
Proof:
$\Rightarrow$) Clear.
$\Leftarrow$) $K \subseteq_y$ M.Since M=$M_1 \oplus M_2$, then
K = K$\cap$ ($M_1 \oplus M_2$) . But M distributive , therefore K=(K$\cap M_1$)$\oplus$(K$\cap M_2$).by proposition{1.10} $K \cap M_1 \subseteq_y M_1$ and $K \cap M_2 \subseteq_y M_2$.
Since $M_1$and M_2 are CLS,then
(K$\cap$M1)is a direct summand of M1, and
(K$\cap$M2)is a direct summand of M2
Clearly that K=(K$\cap M_1$)$\oplus$(K$\cap M_2$) is a direct summand of M.

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