Derivable Maps of Prime Rings

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Abstract

Our active aim in this paper is to prove the following. Let Ŕ be a ring having an idempotent element $e(e \neq 0, e \neq 1)$. Suppose that R is a subring of Ŕ which satisfies:

(i) $eR \subseteq R$ and $Re \subseteq R$.

(ii) $xR = 0$ implies $x = 0$.

(iii) $eRx = 0$ implies $x = 0$(and hence $Rx = 0$ implies $x = 0$).

(iv) $exeR(1-e) = 0$ implies $exe = 0$.

If D is a derivable map of R satisfying $D(R_{ij}) \subseteq R_{ij}; i, j = 1,2$. Then D is additive. This extend Daif's result to the case R need not contain any non-zero idempotent element.

Keywords: Prime ring, Idempotent element, Derivable map, Additive map.

المشتقات الضربية على الحلقات الاوليه

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الخلاصة

هدفنا الاساسي في هذا البحث هو برهان الاتي. لتكن Ŕ حلقة تمتلك عنصر متحايد $e(e \neq 0, e \neq 1)$. نفرض بأن R حلقة جزئية من Ŕ تحقق:

(i) $eR \subseteq R$ و $eR \subseteq R$

(ii) $xR = 0$ يؤدي إلى $x = 0$.

(iii) $eRx = 0$ يؤدي إلى $x = 0$ (ولذلك $Rx = 0$ يؤدي إلى $x = 0$).

(iv) $exeR(1-e) = 0$ يؤدي إلى $exe = 0$.

إذا كانت D مشتقه ضربية على R تحقق $D(R_{ij}) \subseteq R_{ij}$;$i, j = 1,2$ فأن D تجميعية. وهذه النتيجة هي توسيع لنتيجة الباحث ضيف في حاله كون R لا تحتوي على اي عنصر متحايد غير صفري.

1. Introduction

Let R be an associative ring (not necessarily with identity element) and let $x, y \in R$. Recall that R is prime if $xRy = 0$, then either $x = 0$ or $y = 0$. A mapping $D: R \to R$ is derivable (multiplicative derivation) if $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. A mapping φ of R onto arbitrary associative ring S is called a multiplicative isomorphism if φ is bjiective and satisfies $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in R$. The study of relationship between the multiplicative and the additive structures of a ring has become an interesting and active topic in ring theory and operator theory recently. For operator algebra, one often studies the additivity of bijective Jordan (semitriple) multiplicative map from a standard operator algebra on a Banach space of dimension at least 2 onto another algebra [1, 2, 3]. For rings, the study of the question when a multiplicative isomorphism is additive was first proved by Rickart [4] and also by Johnson [5], under some conditions were imposed on R. Later on, Martindale [6] obtained a quite surprising and famous theorem which generalize the main theorem of Rickart's. Also, Lu and Xie [7], extended Martindale's result to ring without idempotent. More precisely, Martindale proved, Theorem M. Let R be a ring containing a family of idempotents $\{ e_i ; i \in \Omega \}$ which satisfies:

(i) $xR = 0$ implies $x = 0$.

(ii) If $e_i Rx = 0$ for each $i \in \Omega$ then $x = 0$ (and hence $Rx = 0$ implies $x = 0$).

(iii) For each $i \in \Omega$, $e_i x e_i R(1 - e_i) = 0$ implies $e_i x e_i = 0$.

Then any multiplicative isomorphism of R onto an arbitrary ring is additive. In particular every multiplicative bijective map from a prime ring containing a nontrivial idempotent element onto an arbitrary ring is necessarily additive. Recently Martindale's results extended to elementary maps of rings [8]. In 2010 Lu [9], defined the following notations. A mapping $D: R \to R$ is called a Jordan derivable (resp. Jordan semitriple derivable) if $D(xy + yx) = D(x)y + xD(y) + D(y)x + yD(x)$ (resp. $D(xyx) = D(x)yx + xD(y)x + xyD(x)$) for all $x, y \in R$. He showed that every Jordan derivable (Jordan Semitriple derivable) of a 2-torsion free until Prime ring having a nontrivial idempotent element is additive. The notion of derivable map of a ring is due to Daif [10], who proved that every derivable map of a ring satisfies Martindale's conditions on his theorem is additive. We notice that Martindale's condition requires that R possess idempotents, and many rings do not have idempotents, as in strictly upper matrix ring.

The aim of the present paper is extend Daif's result to ring need not contain any non-zero idempotent element. It should be mentioned that the idea of the method comes from [6, 7,10].

2. Derivable Maps

The main result in this section reads as follows.

Theorem 2.1. Let Ŕ be a ring having an idempotent element $e (e \neq 0, e \neq 1)$. Suppose that R is a subring of Ŕ which satisfies:

(i) $eR \subseteq R$ and $Re \subseteq R$.

(ii) $xR = 0$ implies $x = 0$.

(iii) $eRx = 0$ implies $x = 0$(and hence $Rx = 0$ implies $x = 0$).

(iv) $exeR(1 - e) = 0$ implies $exe = 0$.

If D is a derivable map of R satisfying $D(R_{ij}) \subseteq R_{ij}; i, j = 1,2$. Then D is additive.

Moreover, D is not only derivation but also covers the concepts of another types of derivation.

In this sequel we will need the following lemmas which are necessary for our proof, in which we call this idempotent e_1 and set formally $e_2 = 1 - e_1$ (R need not have an identity). By Condition (i) we may write R in its Peirce decomposition relative to idempotent element,

$$R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$$

Where $R_{ij} = e_i \operatorname{Re}_j$; $i, j = 1,2$ and x_{ij} will denote an element of R_{ij}.

Let's begin with

Lemma 2.2 [10]. For any x_{mn} in R_{mn} and x_{pq} in R_{pq} with $p \neq q$, we have $D(x_{mn} + x_{pq}) = D(x_{mn}) + D(x_{pq})$.

Lemma 2.3. Let $1 \leq k, i \neq j \leq 2$. Then $D(x_{ii} y_{ik} + x_{ij} y_{jk}) = D(x_{ii} y_{ik}) + D(x_{ij} y_{jk})$.

Proof.

It easy to verify that,

$x_{ii}y_{ik} + x_{ij}y_{jk} = (x_{ii} + x_{ij})(y_{ik} + y_{jk})$.

Then making use of Lemma 2.2 the following equation

$D(x_{ii}y_{ik} + x_{ij}y_{jk}) = D[(x_{ii} + x_{ij})(y_{ik} + y_{jk})]$

$= D(x_{ii} + x_{ij})(y_{ik} + y_{jk}) + (x_{ii} + x_{ij})D(y_{ik} + y_{jk})$

$= D(x_{ii})(y_{ik} + y_{jk}) + D(x_{ij})(y_{ik} + y_{jk}) + x_{ii} D(y_{ik} + y_{jk}) + x_{ij} D(y_{ik} + y_{jk}) = D(x_{ii}(y_{ik} + y_{jk})) + D(x_{ij}(y_{ik} + y_{jk})) = D(x_{ii} y_{ik})$

$+ D(x_{ij} y_{jk})$, hold true.

The following are auxiliary lemmas in our Proof.

Lemma 2.4. D is additive on R_{12}, i.e.

$D(x_{12} + y_{12}) = D(x_{12}) + D(y_{12})$.

Proof.

Let x_{12} and y_{12} be two elements in subring R_{12} We consider the sum $D(x_{12}) + D(y_{12})$.

For any $s_{1j} \in R_{1j}$ and any $t_{i2} \in R_{i2}$ from our assumption we get

$[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]s_{1j} = 0$. (1)

$t_{i2}\ [D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]$ (2)

Now let $s_{2j} \in R_{2j}$ be arbitrary. For $t_{11} \in R_{11}$ making use of Lemma 2.3 together with the fact every derivable is Jordan semitriple derivable, we see that

$t_{11}[D(x_{12}) + D(y_{12})]s_{2j} = t_{11} D(x_{12})s_{2j} + t_{11} D(y_{12})s_{2j} = D((t_{11}(x_{12} s_{2j})) + D((t_{11} y_{12})s_{2j}) - D(t_{11})x_{12} s_{2j} - t_{11} x_{12} D(s_{2j}) - D(t_{11})y_{12} s_{2j} - t_{11} y_{12} D(s_{2j})$

$= D(t_{11}(x_{12} + y_{12})s_{2j}) - D(t_{11})(x_{12} + y_{12})s_{2j} - t_{11}(x_{12} + y_{12})D(s_{2j}) = D(t_{11})(x_{12} + y_{12})s_{2j} + t_{11} D(x_{12} + y_{12})s_{2j} + t_{11}(x_{12} + y_{12})D(s_{2j}) - D(t_{11})(x_{12} + y_{12})s_{2j} - t_{11}(x_{12} + y_{12})D(s_{2j})$

$= t_{11} D(x_{12} + y_{12})s_{2j}$.

This imply that

$t_{11}[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]s_{2j} = 0$.

Left multiplying Equation (1) by t_{11}, we obtain

$t_{11}[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]s_{1j} = 0$.

Comparing those two equations, we arrive at

$t_{11}[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]R = 0$.

Then by Condition (ii), we get

$t_{11}[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})] = 0$ (3)

In a similar fashion as above, for $t_{21} \in R_{21}$ one shows that

$t_{21}[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})] = 0$.

This together with (2) and (3) gives us

$R[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})] = 0$.

Therefore,

$D(x_{12} + y_{12}) = D(x_{12}) + D(y_{12})$ in view of condition (iii).

Lemma 2.5. D is additive on R_{11} i.e.

$D(x_{11} + y_{11}) = D(x_{11}) + D(y_{11})$.

Proof.

Let x_{11}, y_{11} be arbitrary elements in R_{11}.

For $t_{12} \in R_{12}$, we have

$[D(x_{11}) + D(y_{11})]t_{12} = D(x_{11} t_{12}) + D(y_{11} t_{12}) - (x_{11} + y_{11})D(t_{12})$

But $x_{11} t_{12}$ and $y_{11} t_{12}$ are in R_{12} and D is additive on R_{12} by Lemma 2.4, hence

$[D(x_{11}) + D(y_{11})]t_{12} = D((x_{11} + y_{11})t_{12}) - (x_{11} + y_{11})D(t_{12}) = D(x_{11} + y_{11})t_{12} + (x_{11} + y_{11}) D(t_{12}) - (x_{11} + y_{11})D(t_{12})$.

Therefore,

$[D(x_{11}) + D(y_{11}) - D(x_{11} + y_{11})]t_{12} = 0$.

In other words

$[D(x_{11}) + D(y_{11}) - D(x_{11} + y_{11})]R_{12} = 0$.

Since

$D(x_{11}) + D(y_{11}) - D(x_{11} + y_{11})$ is an element in R_{11} by assumption, and our previous conclusion that

$[D(x_{11}) + D(y_{11}) - D(x_{11} + y_{11})]R_{12} = 0$, forces $D(x_{11} + y_{11}) = D(x_{11}) + D(y_{11})$, because of condition (iv).

In light of these lemmas we can prove

Lemma 2.6. D is additive on $R_{11} + R_{12} = e_1 R$, i.e.

$D((x_{11} + x_{12}) + (y_{11} + y_{12})) = D(x_{11} + x_{12}) + D(y_{11} + y_{12})$.

Proof

Let x_{11}, y_{11} be in R_{11} and x_{12}, y_{12} be in R_{12}. Then taking use of Lemmas 2.2, 2.4-2.5, we have

$D((x_{11}+x_{12})+(y_{11}+y_{12}))$
$=D(x_{11}+y_{11}+x_{12}+y_{12})$
$=D(x_{11}+y_{11})+D(x_{12}+y_{12})$
$=D(x_{11})+D(y_{11})+D(x_{12})+D(y_{12})$
$=D(x_{11}+x_{12})+D(y_{11}+y_{12})$. □

Now we are in a position to show that D preserves addition.

Proof of main Theorem.

Let x, y be any elements of R and let t be in eR Thus tx and ty are elements of eR. Hence in light of Lemma 2.6, the equations
$t[D(x)+D(y)]=t\,D(x)+t\,D(y)=D(tx)+D(ty)-D(t)(x+y)=D(t(x+y))-D(t)(x+y)=D(t)(x+y)+t\,D(x+y)-D(t)(x+y)=t\,D(x+y)$, hold true.
Therefore,

$$t[D(x)+D(y)]=t\,D(x+y)$$

Since t is arbitrary in eR, we can deduce that

$$eR[D(x)+D(y)-D(x+y)]=0.$$

By Condition (iii), we see that

$$D(x+y)=D(x)+D(y).$$

Therefore D is derivation.

Obviously, Theorem 2.1 has the following

Corollary 2.7. Let Ŕ be a ring having an idempotent element $e(e \neq 0, e \neq 1)$. Suppose that R (R need not have an identity) is a prime subring of Ŕ which satisfies $eR \subseteq R$ and $Re \subseteq R$. If D is a derivable map of R satisfying $D(R_{ij}) \subseteq R_{ij}; i, j = 1,2.$ Then D is additive.

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