Lie and Jordan Structure in Prime Γ- rings with Γ-centralizing Derivations

Abdulrahman H. Majeed*, Aliaa Aqeel Majeed
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq
*ahmajeed6@yahoo.com

Abstract

Let M be a prime Γ-ring satisfying $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma$ with center *Z*, and *U* be a Lie (Jordan) ideal. A mapping $d: M \to M$ is called Γ- centralizing if $[u,d(u)]_\alpha \in Z$ for all $u \in U$ and $\alpha \in \Gamma$.In this paper , we studied Lie and Jordan ideal in a prime Γ - ring M together with Γ - centralizing derivations on U.

Keywords: Prime Γ-ring, Lie ideal, Jordan ideal, Γ- centralizing, Derivation.

تركيبه لي و جوردان في الحلقات الاوليه من النمط –Γ مع المشتقات المركزيه من النمط –Γ

عبدالرحمن حميد مجيد, علياء عقيل مجيد
قسم الرياضيات, كلية العلوم, جامعة بغداد, بغداد, العراق

الخلاصه

لتكن M حلقه اوليه من النمط – Γ وتحقق الشرط $a\alpha b\beta c = a\beta b\alpha c$ لكل $a,b,c \ni M$ و $\alpha,\beta \ni \Gamma$ مع مركز Z وليكن U مثالي لي (جوردان) . الداله $d: M \to M$ تدعى مركزيه من النمط –Γ اذا كان $[u,d(u)]_\alpha \ni Z$ لكل $u \ni U$ و $\alpha \ni \Gamma$. في هذا البحث درسنا المثالي لي و جوردان للحلقه الاوليه من النمط –Γ مع داله المشتقات المركزيه من النمط –Γ على U.

1. Introduction

N. Nobusawa [1] introduced the notion of Γ -ring, more general than a ring.W. E. Barnes [2]weakened slightly the conditions in the definition of Γ -ring in the sense of Nobusawa after these two papers were published, number of modern algebraists have determined a lot of fundamental properties of Γ-ring and extended numerous significant results in classical ring theory to gamma ring theory see [3, 4, 5 and 6] for partial references.

In classical ring the theory of centralizing mapping on prime ring was initiated by Posner [7] who proved that the existence of a nonzero derivation on a prime ring forces the ring to be commutative. In [8] R. Awtar considered centralizing derivations on Lie and Jordan ideals generalized Posner's theorem. A lot of work has been done during the last decades in this field see [9, 10, 11, and 12] where further reference can be found.

By the same motivation as in the classical ring theories we proved the following results.

Let *M* be a prime Γ -ring, satisfying, $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma$ and it will represented by (*)

i) If characteristic of M is different from 2 and 3and *U* be Lie ideal then if d is Γ -centralizing

on U then U is central in M.
ii) If M has characteristic 3 and U is Jordan ideal . then If d is Γ -centralizing then U is central in M further , if U is a Lie ideal with $u\alpha u \in U$ for all $u \in U$ and, $\alpha \in \Gamma$,then U is central in M. The case when M has characteristic 2 is also studied.

2. Some Basic Definitions

Definition 2.1 [2]**:** Let M and Γ be two additive abelian groups If there exists a mapping $(a,\alpha,b) \to a\alpha b$ of $M \times \Gamma \times M \to M$ Which satisfies for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma$: 1)

$i)(a+b)\alpha c = a\alpha c + b\alpha c,$

$ii)a(\alpha+\beta)b = a\alpha b + a\beta b,$

$iii)a\alpha(b+c) = a\alpha b + a\alpha c.$

2) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Then M is called Γ -ring in the sense of Barnes.

Definition2.2[3]**:** An additive subgroup S of a Γ -ring M is called subring if $S\Gamma S \subset S$.

Definition2.3[3]**:** An additive subgroup I of M is said to be a left (or right) ideal of M if $M\Gamma I \subset I$ (or $I\Gamma M \subset I$), if I is both a right and left ideal, then we say that I is an ideal.

Definition2.4[3]**:** Let M be a Γ -ring then M is called prime if $a\Gamma M\Gamma b = 0$ implies either $a = 0$ or $b = 0$ where $a,b \in M$.

Definition2.5[3]**:** Asubset S if a Γ -ring M is called stronglyilpotent if there exists a positive integer n such that $(S\Gamma)^n S = (0)$.

Remark:

1)For any $a,b \in M$ $a\alpha b - b\alpha a$ are denoted by $[a,b]_\alpha$ .Then one has the basic identities, $[a\beta b,c]_\alpha = [a,c]_\alpha \beta b + a\beta[b,c]_\alpha + a[\beta,\alpha]_c b$

And,

$[a,b\beta c]_\alpha = b\beta[a,c]_\alpha + [a,b]_\alpha + b[\beta,\alpha]_a c,$

for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma$. Using the assumption (*) the above identities reduce to, $[a\beta b,c]_\alpha = [a,c]_\alpha \beta b + a\beta[b,c]_\alpha$

And, $[a,b\beta c]_\alpha = b\beta[a,c]_\alpha + [a,b]_\alpha$.

2) Let M be Γ -ring, the center of M is defined as, Z= $\{a \in M : a\alpha m = m\alpha a$ for all $m \in M, \alpha \in \Gamma\}$.

Definition2.6 [13]**:** An additive subgroup U of a Γ -ring M is said to be a Lie ideal of M if $[u,m]_\alpha \in U,$ for all $u \in U, m \in M$ and $\alpha \in \Gamma.$ And U is said to be Jordan ideal if $u\alpha m + m\alpha u \in U,$ for all $u \in U, m \in M$ and $\alpha \in \Gamma.$

Definition2.8[14]**:** An additive mapping $d : M \to M$ is called a derivation of M if , $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$, holds for all $x,y \in M$ and $\alpha \in \Gamma.$

For a fixed $a \in M$ and $\alpha \in \Gamma$ the mapping, $I_a^\alpha : M \to M$ given by $I_a^\alpha = [m,a]_\alpha,$ is said to be inner derivation of M [15].

Definition2.9[16]**:** Let M be a Γ ring with center Z and U be lie (Jordan) ideal of M .A mapping $d : M \to M$ is called *Γ*-centralizing (resp. *Γ*-commuting) if $[u,d(u)]_\alpha \in Z$ (resp. $[u,d(u)]_\alpha = 0,$ for all $u \in U,$ and $\alpha \in \Gamma.$

3. Basic Lemmas

For proving our main results, we need some important results which we have proved here as lemmas. So, we start as follows:

Lemma3.1: Let M be a prime Γ-ring, d a nonzero derivation of M and a be an element of M if $a\alpha d(m) = 0,$ for all $m \in M$ and $\alpha \in \Gamma$. Then either $a = 0$ or d is zero.

Proof:

We have $a\alpha d(m) = 0,$ for all $m \in M$ and $\alpha \in \Gamma$. Replace m by $m\alpha x$ where $x \in M$,then

$$a\alpha d(m\alpha x) = a\alpha d(m)\alpha x + a\alpha m\alpha d(x)$$
$$= a\alpha m\alpha d(x).$$

For all $x \in M$ and $\alpha \in \Gamma$. That is

$a\Gamma M\Gamma d(x) = 0,$ for all $x \in M$.

Since M is prime, either $a = 0$ or d is zero

Lemma3.2: Let M be a prime Γ-ring of characteristic not 2 and d_1, d_2 be a derivation of M such that the iterate $d_1 d_2$ is also a derivation, Then one at least of d_1, d_2 is zero.

Proof:

We have $d_1 d_2$ is a derivation of M that is,

$$d_1 d_2(a\alpha b) = d_1 d_2(a)\alpha b + a\alpha d_1 d_2(b),$$

for all $a,b \in M$ and $\alpha \in \Gamma$.

But d_1, d_2 are each derivation so,

$$d_1d_2(a\alpha b) = d_1d_2(a)\alpha b + d_2(a)\alpha d_1(b) + d_1(a)\alpha d_2(b) + a\alpha d_1(d_2(b)).$$

But,

$$d_1d_2(a\alpha b) = d_1d_2(a)\alpha b + a\alpha d_1(d_2(b))$$

So,

$$d_2(a)\alpha d_1(b) + d_1(a)\alpha d_2(b) = 0,$$

for all for all $a,b \in M$ and $\alpha \in \Gamma$ ...(1)

Replace a in the last equation by $a\alpha d_1(c)$

$d_2(a\alpha d_1(c))\alpha d_1(b) + d_1(a\alpha d_1(c))\alpha d_2(b) = 0,$ for all $a,b,c \in M$ and $\alpha \in \Gamma$.

That is

$$a\alpha(d_2(d_1(c)\alpha d_1(b)) + d_1(d_1(c))\alpha d_2(b) = 0$$

for all $a,b,c \in M$ and $\alpha \in \Gamma$.

Which is merely equation (1) with a replaced by $d_1(c)$, then we are left with

$d_2(a)\alpha d_1(c)\alpha d_1(b) + d_1(a)\alpha d_1(c)\alpha d_2(b) = 0$, for all $a,b,c \in M$ and $\alpha \in \Gamma$.

But,

$d_1(a)\alpha d_2(b) = -d_2(a)\alpha d_1(b)$ by replacing a by c the last equation becomes,

$$d_2(a)\alpha d_1(c)\alpha d_1(b) - d_1(a)\alpha d_2(c)\alpha d_1(b) = 0$$

Factoring out $a\alpha d_1(b)$ on the right, we have

$$(d_2(a)\alpha d_1(c) - d_1(a)\alpha d_2(c))\alpha d_1(b) = 0,$$

for all $a,b,c \in M$ and $\alpha \in \Gamma$.

And by Lemma 3.1 unless $d_1 = 0$ we have,

$$(d_2(a)\alpha d_1(c) - d_1(a)\alpha d_2(c)) = 0,$$

for all $a,c \in M$ and $\alpha \in \Gamma$.

Replace b by c in (1) then,

$$(d_2(a)\alpha d_1(c) + d_1(a)\alpha d_2(c)) = 0,$$

for all $a,c \in M$ and $\alpha \in \Gamma$.

Adding these last two equations, we get

$2d_2(a)\alpha d_1(c) = 0,$ for all $a,b,c \in M$ and $\alpha \in \Gamma$.

Since characteristic of M not equal 2, then $d_2(a)\alpha d_1(c) = 0,$ or else $d_1 = 0$ using Lemma 3.1 again with a replacing $d_2(a)$ we get, either $d_1 = 0$ or $d_2 = 0$

Lemma3.3: Let M be a prime Γ -ring of characteristic different from 2, U be Lie ideal of M and d be anon zero derivation of M. Then if d is Γ-centralizing on U and $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$, then M is Γ-commuting on U.

Proof:

We have d is Γ-centralizing on U i.e.

$[u, d(u)]_\alpha \in Z$, for all $u \in U$, and $\alpha \in \Gamma$.

Linearizing the above relation on, $u = u + u\alpha u$, we get

$$[u\alpha u, d(u)]_\alpha + [u, u\alpha d(u) + d(u)\alpha u]_\alpha \in Z,$$

for all $u \in U$, and $\alpha \in \Gamma$.

That is,

$4[u, d(u)]_\alpha \alpha u \in Z,$ for all $u \in U$, and $\alpha \in \Gamma$.

Since characteristic of M not equal 2 and $[u, d(u)]_\alpha \in Z$ then we get

$$[u, d(u)]_\alpha \alpha [u, m]_\beta = 0, \text{ for all}$$

$m \in M, u \in U$ and $\alpha, \beta \in \Gamma$.

If for some $u \in U$, $[u, d(u)]_\alpha \neq 0$ then we get $[u, m]_\beta = 0$, in particular $[u, d(u)]_\alpha = 0$

Hence,

$[u, d(u)]_\alpha = 0$,for all $u \in U$, and $\alpha \in \Gamma$.

Lemma3.4: Let M be a prime Γ -ring, U be a Lie ideal of M and d a nonzero derivation of M. If d is Γ -centralizing on U then

$$[[d(m), u]_\beta, u]_\alpha \in Z,$$

for all $m \in M, u \in U$ and $\alpha, \beta \in \Gamma$.

Further, if d is Γ-commuting on U then,

$[[d(m), u]_\beta, u]_\alpha = 0,$

for all $m \in M, u \in U$ and $\alpha, \beta \in \Gamma$.

Proof:

Since U is Lie ideal then,

$$[u, m]_\alpha \in U,$$

for all $u \in U, m \in M$ and $\alpha \in \Gamma$.

So that, $[u + [u, m]_\beta, d(u + [u, m]_\beta)]_\alpha \in Z.$

That is,

$$[[u, m]_\beta, d(u)]_\alpha + [u, [d(u), m]_\beta]_\alpha + [u, [u, d(m)]_\beta]_\alpha \in Z,$$

for all $m \in M, u \in U$ and $\alpha, \beta \in \Gamma$.

Now since, for any for all $m \in M, u \in U$, $\alpha, \beta \in \Gamma$ and by (*) we have

$$[[u, m]_\beta, d(u)]_\alpha + [u, [d(u), m]_\beta]_\alpha = [m, [d(u), u]_\beta]_\alpha \in Z.$$

By Γ-centralizing of d we get ,

$$[[u, m]_\beta, d(u)]_\alpha + [u, [d(u), m]_\beta]_\alpha = 0.$$

Hence,

$$[[d(m),u]_\beta,u]_\alpha \in Z,$$

for all $m \in M, u \in U$ and $\alpha,\beta \in \Gamma$.

The last part can be obtained similarly.

Lemma3.5: Let M be a prime Γ -ring of characteristic not equal 2 and 3, and let U be a Lie ideal of , if d is Γ -centralizing on U then d is Γ-commuting on U.

Proof:

Since d is Γ -centralizing then, by Lemma 3.4, we have

$$[[d(m),u]_\phi,u]_\alpha \in Z,$$

for all $m \in M, u \in U$ and $\alpha,\beta \in \Gamma$.

By using the assumption (*) we get

$$u\beta u\alpha d(m) + d(m)\alpha u\beta u - 2u\beta d(m)\alpha u \in Z,$$

for all $m \in M, u \in U$ and $\alpha,\beta \in \Gamma$. ...(2)

Commuting with u, we have

$$3u\beta u\alpha d(m)\delta u + u\beta u\alpha u\delta d(m) = 3u\delta d(m)\alpha u\beta u + d(m)\delta u\alpha u\beta u \quad ...(3)$$

In (3) replace m by u and using d is Γ-centralizing,

$$u\beta u\alpha u\delta d(u) - d(u)\delta u\alpha u\beta u = 3(u\alpha d(u) - d(u)\alpha u)\delta u\beta u \quad ...(4)$$

Furthermore,

$$2(u\alpha d(u) - d(u)\alpha u)\beta u = u\beta u\alpha d(u) - d(u)\alpha u\beta u. \quad ...(5)$$

Write $d(m) = m'$ and then by replacing m by $u\alpha m'$ in (4) , we get

$$3u\delta u\alpha m''\alpha u\beta u + u\beta u\alpha u\delta u\alpha m'' - 3u\beta u\alpha u\alpha m''\delta u - u\alpha m''\delta u\alpha u\beta u + 3u\delta d(u)\alpha m'\alpha u\beta u + u\beta u\alpha u\delta m'\alpha d(u) - 3u\beta u\alpha d(u)\alpha m'\delta u - d(u)\alpha m'\delta u\alpha u\beta u = 0,$$

for all $m \in M, u \in U$ and $\alpha,\beta,\delta \in \Gamma$. ...(6)

However, by assumption (*) and (4), we have

$$3u\delta u\alpha m''\alpha u\beta u + u\beta u\alpha u\delta u\alpha m'' - 3u\beta u\alpha u\alpha m''\delta u - u\alpha m''\delta u\alpha u\beta u = u\alpha(3u\delta m''\alpha u\beta u + u\beta u\alpha u\delta m'' - 3u\beta u\alpha m''\delta u - m''\delta u\alpha u\beta u) = 0.$$

Then equation (6) becomes,

$$3u\delta d(u)\alpha m'\alpha u\beta u + u\beta u\alpha u\delta m'\alpha d(u) - 3u\beta u\alpha d(u)\alpha m'\delta u - d(u)\alpha m'\delta u\alpha u\beta u = 0,$$

for all $m \in M, u \in U$ and $\alpha,\beta,\delta \in \Gamma$. ...(7)

Multiply (4) on the left by $d(u)\alpha$ and then subtract the results from (7) to get,

$$3(u\alpha d(u) - d(u)\alpha u)\delta m'\alpha u\beta u + u\beta u\alpha u\alpha d(u) - d(u)\alpha u\beta u\alpha u)\delta m' - (u\beta u\alpha d(u) - d(u)\alpha u\beta u)\alpha m'\delta u = 0 \quad ...(8)$$

Using (5) and (6), we arrive at after dividing by 3,

$$(u\alpha(u) - d(u)\alpha u)\alpha(m'\delta u\beta u + u\beta u\delta m' - 2u\beta m'\delta u) = 0,$$ for all $m \in M, u \in U$ and $\alpha,\beta,\delta \in \Gamma$.

If , $(u\alpha d(u) - d(u)\alpha u \neq 0,$ for some $u \in U$ and $\alpha \in \Gamma$. Then we have

$$m'\delta u\beta u + u\beta u\delta m' - 2u\beta m'\delta u = 0 \quad ...(9)$$

Replace m by $u\beta m$ in (9) and using (*) we get,

$$u\beta m'\delta u\beta u + u\beta u\beta u\delta m' - 2u\beta u\delta m'\beta u + d(u)\beta m\delta u\beta u + u\beta u\beta d(u)\delta m - 2u\beta d(u)\delta m\beta u = 0 \quad ...(10)$$

By using (9) we get,

$$u\beta(m'\delta u\beta u + u\beta u\delta m' - 2u\delta m'\beta u) = 0,$$

Then equation (10) becomes,

$$d(u)\beta m\delta u\beta u + u\beta u\beta d(u)\delta m - 2u\beta d(u)\delta m\beta u = 0. \quad ...(11)$$

Now in (9) replace m by u , and multiply this on the right by βm ,

$$d(u)\delta u\beta u\beta m + u\beta u\delta d(u)\beta m - 2u\beta d(u)\delta u\beta m = 0. \quad ...(12)$$

Subtract (12) from (11),

$$d(u)\beta(m\delta u\beta u - u\delta u\beta m) - 2u\beta d(u)\delta(m\beta u - u\beta m) = 0. \quad ...(13)$$

Replace m by $u\beta m$ and using assumption (*)

$$d(u)\beta u\beta(m\delta u\beta u - u\beta u\delta m) - 2u\beta d(u)\delta u\beta(m\beta u - u\beta m) = 0. \quad ...(14)$$

Multiply (13) by $u\beta$ from left and then subtract the results from (14),

$$(u\beta d(u) - d(u)\beta u)\beta(m\delta u\beta u - u\beta u\delta m) - 2u\beta(u\beta d(u) - d(u)\beta u)\delta(m\beta u - u\beta m) = 0.$$

Since, $u\alpha d(u) - d(u)\alpha u \neq 0,$ for all $u \in U$ and $\alpha \in \Gamma$. Then,

$$m\delta u\beta u - u\beta u\delta m - 2u(m\beta u - u\beta m) = 0,$$

for all $m \in M$

So, $m\delta u\beta u - u\beta u\delta m - 2u\beta m\beta u = 0,$ that is

$$u\beta(m\delta u - u\delta m) = (m\delta u - u\delta m)\beta u,$$

That is u in the center by Lemma 3.2 or else

$$u\alpha d(u) - d(u)\alpha u = 0,$$

Which in both cases

$[u,d(u)]_\alpha = 0$ for all $u \in U$ and $\alpha \in \Gamma$.

The following lemma may have some independent interest.

Lemma3.6: Let M be a prime Γ-ring of characteristic not 2, U be Jordan ideal of M and d be a nonzero derivation of M. If $u\alpha d(u) = d(u)\alpha u = 0,$ for all $u \in U$, $\alpha \in \Gamma.$ Then $U = 0.$

Proof:

Linearizing the relation $u\alpha d(u) = 0$ on $u = u + w$ where $w \in U$ to get,

$$u\alpha d(w) + w\alpha d(u) = 0,$$

for all $u, w \in U$ and $\alpha \in \Gamma.$ ...(15)

For $u \in U$ and any $m \in M, \alpha \in \Gamma,$

$$u\alpha(u\alpha m - m\alpha u) + (u\alpha m - m\alpha u)\alpha u \in U.$$

But, $2(m\alpha u\alpha u - u\alpha u\alpha m) =$
$\{u\alpha(m\alpha u - u\alpha m) + (m\alpha u - u\alpha m)\alpha u\} -$
$\{(m\alpha u - u\alpha m)\alpha u + u\alpha(m\alpha u - u\alpha m)\}$

As the first and second term on the right hand side are in U,

$$2(m\alpha u\alpha u - u\alpha u\alpha m) \in U.$$

Now since,
$2u\alpha u \in U$ and $2(m\alpha u\alpha u - u\alpha u\alpha m) \in U.$
Then, $4u\alpha u\alpha m$ and $4m\alpha u\alpha u$ are in U.
Replacing w by $4m\alpha u\alpha u$ in (15) and using the hypothesis, we get

$$u\alpha d(m)\alpha u\alpha u = 0,$$

for all $m \in M, u \in U$ and $\alpha \in \Gamma.$ ...(16)

Replace w by $m\beta u + u\beta m$ and using the hypothesis, we get
$u\beta u\alpha d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u) +$
$u\beta m\alpha d(u) = 0,$ for all $m \in M, u \in U$ and $\alpha, \beta \in \Gamma.$

Multiply by αu on the right and using the assumption (*) together with equation (16) we obtain

$$u\beta u\alpha d(m)\alpha u = 0,$$

for all $m \in M, u \in U$ and $\alpha, \beta \in \Gamma.$ ...(17)

Again replace w by $4u\alpha u\alpha m$ in (15), we get

$u\alpha u\alpha u\alpha d(m) = 0,$ for all $m \in M, u \in U$ and $\alpha \in \Gamma.$

Then by Lemma 3.1, we have
$u\alpha u\alpha u = 0,$ for all $u \in U$ and $\alpha \in \Gamma.$
For $m \in M, u \in U$ and $\alpha \in \Gamma,$
$2(u\alpha u\alpha m + m\alpha u\alpha u) \in U.$
That is,
$2^3[(u\alpha u\alpha m + m\alpha u\alpha u)\alpha]^2(u\alpha u\alpha m +$
$m\alpha u\alpha u) = 0,$ for all $m \in M, u \in U$ and $\alpha \in \Gamma.$

Multiply from the right side by $u\alpha u\alpha u = 0$ we get

$$2^3[(u\alpha u\alpha m)\alpha]^3(u\alpha u\alpha m) = 0,$$

for all $m \in M, u \in U$ and $\alpha \in \Gamma.$

If for some $u \in U$ and $\alpha \in \Gamma$, $u\alpha u \neq 0$ then $u\alpha u\alpha M$ is a nonzero right ideal of M, then by Levitzki's Theorem [13] M would have a nilpotent ideal; which is impossible for prime Γ-ring, hence
$u\alpha u = 0,$ for all $u \in U$ and $\alpha \in \Gamma.$
By repeating the above argument we can show that $u = 0,$ for all $u \in U$

4. The Main Theorems

Theorem 4.1: Let M be a prime Γ-ring of characteristic different from 2 and 3. Let d be a nonzero derivation of M and U be a Lie ideal of M .If d is Γ-centralizing on U then $U \subset Z.$

Proof:

Since d is Γ -centralizing on U, then by using Lemma 3.5, we have
$[u, d(u)]_\alpha = 0,$ for all $u \in U$ and $\alpha \in \Gamma.$
Then by Lemma 3.4, we get

$$[[d(m), u]_\beta, u]_\alpha = 0,$$

for all $m \in M, u \in U$ and $\alpha \in \Gamma.$ ...(1)

In (1) replace u by $u + w$ where $w \in U$,
$[[d(m), u]_\beta, w]_\alpha + [[d(m), w]_\beta, u]_\alpha = 0,$
for all $m \in M, u, w \in U$ and $\alpha, \beta \in \Gamma.$ ...(2)

Suppose now, $u, w \in U$ are such that $w\alpha v$.
Then by replacing w by $w\alpha v$ in (2) we get after using (*),
$w\alpha[[d(m), u]_\beta, v]_\alpha + [[d(m), u]_\beta, w]_\alpha \alpha v +$
$[d(m), w]_\alpha \beta[u, v]_\alpha + [[d(m), w]_\beta, u]\alpha v +$
$w\alpha[[d(m), v]_{\beta,}, u]_\alpha + [w, u]_\alpha \alpha[d(m), v]_\beta = 0.$
In view of (2) the last equation reduces to ,
$[d(m), w]_\alpha \beta[u, v]_\alpha + [w, u]_\alpha \alpha[d(m), v]_\beta$ =0.
Replace v by $[t, w]_\alpha$ where $t \in M$ in above equation , we have

$$[d(m), w]_\beta \alpha[[t, w]_\alpha, u]_\alpha + [w, u]_\alpha [d(m), [t, w]_\alpha]_\beta = 0, \quad ...(3)$$

for all $t, m \in M, u, w \in U$ and $\alpha, \beta \in \Gamma.$
Putting $u = w$ in (3), we have
$[d(m), w]_\beta \alpha[[t, w]_\alpha, w]_\alpha = 0$ ...(4)
Replace t by $t\alpha d(a)$ in (4) where $a \in M$ yields on expansion and (*),

$[d(m),w]_\beta \alpha\{2[t,w]_\alpha \alpha[d(a),w]_\alpha +$
$[[t,w]_\alpha, w]_\alpha \alpha d(a) + t\alpha[[d(a),w]_\alpha, w]_\alpha\}$ =0.
By (4) the second term is zero, while by (1) the third term is zero .Hence
$[d(m),w]_\beta \alpha[t,w]_\alpha \alpha[d(a),w]_\alpha = 0,$
for all $m,t,a \in M, w \in U$ and $\alpha \in \Gamma$. ...(5)
Put $u = [t,w]_\alpha$ in (3), and linearization it s on $t = t + d(a)$ where $a \in M$ together with (1) yields $[[t,w]_\alpha, w]_\alpha \alpha[d(a),w]_\alpha, d(m)]_\beta = 0,$
for all $m,t,a \in M, w \in U$ and $\alpha \in \Gamma$. ...(6)
Replace t by $d(t)\alpha p$ where $p \in M$ in (6) then by expanding we get ,

$$\{2[d(t),w]_\alpha \alpha[p,w]_\alpha + d(t)\alpha [[p,w]_\alpha, w]_\alpha + [[d(t),w]_\alpha, w]_\alpha \alpha p\}\gamma + [[d(a),w]_\alpha, d(m)]_\beta = 0.$$

By (6) the second term is zero, while by (1) the third term is zero .Hence
$[d(t),w]_\alpha \alpha[p,w]_\alpha \gamma[[d(a),w]_\alpha, d(m)]_\beta$ =0.In view of (5), the last equation reduces to,
$[d(t),w]_\alpha \alpha[p,w]_\alpha \gamma d(m)\alpha[d(a),w]_\beta = 0,$
for all $p,a \in M, w \in U$ and $\alpha,\gamma \in \Gamma$.
In (5) replace t by $t\alpha d(a)$ where $p \in M$ then by using the last equation, we get
$[d(m),w]_\beta \Gamma M \Gamma [d(p),w]_\alpha [d(a),w]_\alpha = 0,$
for all $m,a \in M, w \in U$ and $\alpha,\beta \in \Gamma$.
Since M is prime either $[d(m),w]_\beta = 0$ or
$[d(p),w]_\alpha \alpha[d(a),w]_\alpha = 0.$
If for all $m \in M, w \in U$ and $\beta \in \Gamma$,
$[d(m),w]_\beta = 0.$ That is, $I_w^\beta(d(m)) = 0.$
Then by Lemma 3.1, $w \in Z,$ for all $w \in U$
Thus assume there exists a $w \in U$ such that for some $m \in M, [d(m),w]_\beta \neq 0.$ That is $w \notin Z.$ Then for all $a,p \in M,$

$$[d(p),w]_\alpha \alpha[d(a),w]_\alpha = 0. \qquad ...(7)$$

Replace a by $b\beta c$ where $b,c \in M$ then by expanding, we get

$$[d(p),w]_\alpha \alpha[d(b),w]_\alpha \beta c + [d(p),w]_\alpha \alpha d(b)\beta[c,w]_\alpha + [d(p),w]_\alpha \alpha b\beta[d(c),w]_\alpha + [d(p),w]_\alpha \alpha[b,w]_\alpha \beta d(c) = 0.$$

Replace b by $[t,w]_\alpha$ where $t \in M$. Then by (7) the first term is zero, by (5) the third term is zero and by (4) the fourth term is zero, thus
$[d(p),w]_\alpha \alpha d([t,w]_\alpha)\beta[w,c]_\alpha = 0.$
Since, $d([t,w]_\alpha) = [d(t),w]_\alpha + [t,d(w)]_{\alpha,}$ and using (3), we get
$[d(p),w]_\alpha \alpha[t,d(w)]_\alpha \beta[w,c]_\alpha = 0,$
for all $c,t,p \in M, w \in U$ and $\alpha,\beta \in \Gamma$.
Replace c by $m\alpha c$ where $m \in M$, then

$$[d(p),w]_\alpha \alpha[t,d(w)]_\alpha \Gamma M \Gamma [w,c]_\alpha = 0.$$

Since M is prime and $w \notin Z$, we get

$$[d(p),w]_\alpha \alpha[t,d(w)]_\alpha = 0,$$

for all $t,p \in M, w \in U$ and $\alpha \in \Gamma$. Thus

$$[d(p),w]_\alpha \Gamma M \Gamma [t,d(w)]_\alpha = 0,$$

for all $t,p \in M, w \in U$ and $\alpha \in \Gamma$.
Which in both cases $d(w) \in Z$.
Now suppose that $u \in U$ and $u \in Z$ then

$$0 = d([u,a]_\alpha) = [d(u),a]_\alpha + [u,d(a)]_\alpha$$

and hence $d(u) \in Z$. Therefore, $d(u) \in Z$ for all $u \in U$. So that, $d([w,a]_\alpha) \in Z$ for all $a \in M$,that is
thus $[w,d(a)]_\alpha \in Z$ .In particular,
$[w,d(a\beta w)]_\alpha = [w,d(a)]_\alpha \beta w + [w,a]_\alpha \beta d(w) \in Z$ ...(8).
By commuting (6) with w, we get

$$[w,[w,a]_\alpha]_\alpha \beta d(w) = 0,$$

for all $a \in M, w \in U$ and $\alpha,\beta \in \Gamma$.
If $d(w) \neq 0$ and as its in the center Z,
$[w,[w,a]_\alpha]_\alpha = 0,$ for all $a \in M$ and $\alpha \in \Gamma$.
By sub- Lemma [14] $w \in Z$ a contradiction**.**
Hence, $d(w) = 0.$ Thus by (8), we have
$[w,d(a)]_\alpha \beta w \in Z,$ for all $a \in M$ and $\alpha \in \Gamma$.
That is, $[w.d(a)]_\alpha \beta[w,b]_\alpha = 0,$
for all $a,b \in M$ and $\alpha,\beta \in \Gamma$.
Replace b by $c\alpha b$ where $c \in M$, then

$$[d(a),w]_\alpha \Gamma M \Gamma [w,b]_\alpha = 0.$$

By primness of M we get, either $w \in Z$ or $[d(a),w]_\alpha = 0,$ for all $a \in M$ and $\alpha \in \Gamma$.
Which us in both cases a contradiction Hence, $w \in Z$ for all $w \in U$.

Now we should like to settle the problem when M has characteristic 3 .Hence we get the following result.

Theorem4.2: Let M be a prime Γ -ring of characteristic 3, and d be a nonzero derivation of M. if d is Γ-centralizing on U and $u\alpha u \in U$ then $U \subset Z$.

Proof:

Since d is Γ-centralizing on U then,
By Lemma 3.3 we get d is Γ-commuting on U .Therefore , by similar way of the proof in Theorem 4.1 we can get $U \subset Z$.

Now we show that the conclusion of Theorem 4.1 and Theorem 4.2 holds even if U is Jordan ideal of M.

Theorem4.3: Let M be a prime Γ-ring of characteristic not 2. Let d be a nonzero derivation of M and U be a Jordan ideal of M if d is Γ-centralizing then $U \subset Z$.

Proof:

Since $2u\alpha u \in U$, then by Lemma 3.3, $[u,d(u)]_\alpha = 0,$ for all $u \in U$ and $\alpha \in \Gamma.$

Linearizing the relation $[u,d(u)]_\alpha = 0,$ on $u = u + v$ where $v \in U$, we get

$$[u,d(v)]_\alpha + [v,d(u)]_\alpha = 0,$$

for all $u,v \in U$ and $\alpha \in \Gamma.$ ...(9)

In (9) ,replace v by $u\beta m + m\beta u$ where $m \in M$ then by expanding, we get

$$u\beta[u,d(m)]_\alpha + [u,d(m)\beta u + d(u)\beta[u,m]_\alpha + [u,m]_\alpha \beta d(u) + u\beta[m,d(u)]_\alpha + [m.d(u)]_\alpha \beta u = 0.$$ i.e.

$$2u\beta m\alpha d(u) - 2d(u)\alpha m\beta u + u\beta u\alpha d(m) - d(m)\alpha u\beta u = 0 \quad ...(10)$$

Replace m by $u\alpha m$ in (10) , we get

$$d(u)\alpha(u\beta u\alpha m - m\alpha u\beta u) = 0,$$

for all $m \in M.u \in U$ and $\alpha,\beta \in \Gamma$ ....(11)

That is, $d(u)\alpha I^{u}_{u\beta u}(m) = 0,$

for all $m \in M.u \in U$ and $\alpha,\beta \in \Gamma$.

Hence by Lemma 3.1 we have, either $u\beta u \in Z$ or $d(u) = 0,$ for all $u \in U$ and $\alpha,\beta \in \Gamma$.

For $u \in U$ and any $m \in M, \alpha \in \Gamma$, we have $u\alpha m + m\alpha u \in U.$ But,

$$4u\alpha m\alpha u = 2\{u\alpha(u\alpha m + m\alpha u) + (u\alpha m + m\alpha u)\alpha u\} - \{2u\alpha u\alpha m + m\alpha 2u\alpha u\}.$$

The first and second term on the right are in U then, $4u\alpha m\alpha u \in U$ .Replace v by $4u\alpha m\alpha u$ in (9), we get

$$u\alpha u\alpha m\alpha d(u) - d(u)\alpha m\alpha u\alpha u + u\alpha u\alpha m\alpha d(m)\alpha u - u\alpha d(m)\alpha u\alpha u = 0 \quad ...(12)$$

Replace m by $u\alpha m$ in (12) and then by using (12) we get,

$$u\alpha d(u)\alpha(u\alpha m\alpha u - m\alpha u\alpha u) = 0.$$

In view of (11) the last equation reduces to

$$u\alpha d(u)\alpha u\alpha(u\alpha m - m\alpha u) = 0.$$

That is , $u\alpha d(u)\alpha u\alpha I^{\alpha}_{u}(m) = 0.$

Then by Lemma 3.1, we have either $u\alpha d(u)\alpha u = 0$ or $U \subset Z,$ for all $u \in U$ and $\alpha \in \Gamma.$ ...(13)

In (11), replace u by $u+v$ where $v \in U$ then by using (11) , we get

$$\{d(u) + d(v)\}\alpha[v\beta u + v\beta u, m]_\alpha + d(u)\alpha[v\beta u, m]_\alpha + d(v)\alpha[u\beta u, m]_\alpha = 0.$$

Replace u by $-u$ then,

$$\{-d(u) + d(v)\}\alpha[-v\beta u - v\beta u, m]_\alpha - d(u)\alpha[v\beta u, m]_\alpha + d(v)\alpha[u\beta u, m]_\alpha = 0.$$

Adding the last two equations and dividing by 2, we have

$$d(u)\alpha[v\beta u + v\beta u, m]_\alpha + d(v)\alpha[u\beta u, m]_\alpha = 0$$

for all $m \in M, u,v \in U$ and $\alpha,\beta \in \Gamma.$

By lemma 3.6 we get $u\alpha d(u)\alpha u \neq 0,$ for some $u \in U, \alpha \in \Gamma; d(u) \neq 0.$

Hence by (12), $u\beta u \in Z.$ The net results of this is $d(u)\alpha[v\beta u + v\beta u, m]_\alpha = 0,$

for all $m \in M, u,v \in U$ and $\alpha,\beta \in \Gamma.$

That is , $d(u)\alpha I^{\alpha}_{u\beta v + v\beta u}(m) = 0,$

for all $m \in M, u,v \in U$ and $\alpha,\beta \in \Gamma.$

By Lemma 3.1, $v\beta u + v\beta u \in Z,$ for all $u,v \in U$ and $\alpha,\beta \in \Gamma.$

If $u\alpha u = 0,$ then

$$0 = d(u\alpha u) = u\alpha d(u) + d(u)\alpha u = 2u\alpha d(u).$$

That is, $u\alpha d(u) = 0$ a contradiction hence $u\alpha u \neq 0,$ Now suppose that $u\alpha d(u)\alpha u = 0,$ then $u\alpha u\alpha d(u) = 0$ that is, $d(u) = 0$ a contradiction hence $u\alpha d(u)\alpha u \neq 0,$

So by (13) $U \subset Z$ hence $2u\alpha v \in Z;$ that is $2u\alpha v \in Z$ for all $v \in U$ and $\alpha \in \Gamma.$

As $u \neq 0$ we have $v \in Z$ for all $v \in U$.

Hence $U \subset Z$

We should like to settle the problem even when M has characteristic 2 .In this case Lie and

Jordan ideals will coincide.

Theorem 4.4: Let M be a prime Γ-ring of characteristic 2 ,and let d be a nonzero derivation of M .Let U be Lie (Jordan)ideal and subring of M .If d is Γ-centralizing on U then U is commutative

Proof:

Since d is Γ-centralizing on U then by Lemma 3.4

$$d(m)\beta u\alpha u+u\alpha u\beta d(m)\in Z \qquad ...(14)$$

Commute(14) with $d(m)$ and $u\alpha u$ respectively we get ,

$$u\alpha u\beta d(m)\gamma d(m)=d(m)\gamma d(m)\beta u\alpha u \quad (15a)$$

And ,

$$d(m)\beta u\alpha u\delta u\alpha u=u\alpha u\delta u\alpha u\beta d(m) \quad (15b)$$

in (15a) replace m by $v+u\alpha u\beta v$ and by using (15 a) we get ,

$$u\alpha u\beta d(v+u\alpha u\beta v)\gamma d(v+u\alpha u\beta v)$$
$$=d(v+u\alpha u\beta v)\gamma d(v+u\alpha u\beta v)\beta u\alpha u.$$

For $u\in U,\alpha\in\Gamma$,

$$d(u\alpha u)=u\alpha d(u)+d(u)\alpha u\in Z.$$

So in view of (15b) the last equation reduces to

$$u\alpha u\beta d(v)\gamma u\alpha u\beta d(v)+d(v)\gamma u\alpha u\beta d(v)\beta u\alpha u=0,$$ for all $u,v\in U,\alpha\in\Gamma$.

Since M is prime, and by using (14) we get,

$$u\alpha u\beta d(v)=d(v)\beta u\alpha u,$$ for all $u,v\in U$, and $\alpha\in\Gamma$ ...(16)

Replace u by $u+w$ where $w\in U$ we get,

$$(u\alpha w+w\alpha u)\beta d(v)=d(v)\beta(u\alpha w+w\alpha u)$$

Replace v by $v\alpha w$ and by using (*) we have,

$$(u\alpha w+w\alpha u)\beta(u\alpha d(v)+d(v)\alpha u)=0,$$
for all $u,v,w\in U,\alpha,\beta\in\Gamma$. ...(17)

Linearize the last equation on $u=u+v\alpha v$ where $v\in U$ and put $v=u$ then using (16) we get,

$$(v\alpha v\alpha w+w\alpha v\alpha v)\beta(u\alpha d(u)+d(u)\alpha u)=0$$
for all $u,v,w\in U,\alpha,\beta\in\Gamma$.

If $[u,d(u)]_\alpha\neq 0$, for some $u\in U$ and $\alpha\in\Gamma$. Then,

$(v\alpha v\alpha w+w\alpha v\alpha v)=0$, for all $v,w\in U$ and $\alpha\in\Gamma$. So that,

$$u\alpha u\alpha(w\alpha m+m\alpha w)=(w\alpha m+m\alpha w)\alpha u\alpha u$$

That is

$$w\alpha(u\alpha u\alpha m+m\alpha u\alpha u)=(u\alpha u\alpha m+m\alpha u\alpha u)\alpha v.$$ Replace m by $m\alpha u$ then

$$(u\alpha u\alpha m+m\alpha u\alpha u)\alpha(w\alpha u+u\alpha w)=0,$$
for all $m\in M,u,w\in U$ and $\alpha\in\Gamma$.

Replace w by $[u,t]_\alpha$ we get,

$$(u\alpha u\alpha m+m\alpha u\alpha u)\alpha(u\alpha u\alpha t+t\alpha u\alpha u)=0,$$
for all $m,t\in M,u,w\in U$ and $\alpha\in\Gamma$.

Replace t by $p\alpha t$ where $p\in M$, then

$$(u\alpha u\alpha m+m\alpha u\alpha u)\Gamma M\Gamma(u\alpha u\alpha t+t\alpha u\alpha u)$$
=0. By primness of M we have,

$u\alpha u\in Z$, for all $u\in U$. Thus assume that

$[u,d(u)]_\alpha=0$, for all $u\in U,\alpha\in\Gamma$.

Then by lemma 3.4 we have,

$$u\alpha u\beta d(m)=d(m)\beta u\alpha u.$$

Replace m by $m\alpha a$ where $a\in M$ and using (*) we get,

$$d(m)\alpha(u\alpha u\beta a+a\beta u\alpha u)+$$
$$(u\alpha u\beta m+m\beta u\alpha u)\alpha d(a)=0.$$

For $v\in U,\alpha\in\Gamma$,

$$d(v\alpha v)=v\alpha d(v)+d(v)\alpha v=0.$$

Hence the last equation becomes,

$$d(m)\alpha(u\alpha u\beta v\alpha v+v\alpha v\beta u\alpha u)+$$
$$(u\alpha u\beta m+m\beta u\alpha u)\alpha d(v\alpha v)=0.$$

Thus by lemma 3.4 we have,

$u\alpha u\beta v\alpha v=v\alpha v\alpha u\alpha u$ .Therefore,

$$u\alpha u\beta(v\alpha w+w\alpha v)=(v\alpha w+w\alpha v)\beta u\alpha u$$
for all $u,v,w\in U,\alpha,\beta\in\Gamma$.

Replace v by $[w,m]_\alpha$ then we have,

$$I^{\alpha}_{w\alpha w}(m)\beta(u\alpha u\alpha w+w\alpha u\alpha u)=0,$$

By using Lemma 3.1 we get,

$w\alpha w\notin Z$, for some $w\in U$ and $\alpha\in\Gamma$.

So that, $u\alpha u\alpha w=w\alpha u\alpha u$ That is,

$[[u,v]_\alpha,w]_\alpha=0$, for all $u,w\in U$ and $\alpha\in\Gamma$.

Since, $[[v,w]_\alpha,u]_\alpha+[[w,u]_\alpha,v]_\alpha=$
$[[u,v]_\alpha,w]_\alpha$.

Replace in above equation v by $v\alpha w$ and expanding we get,

$$[v,w]_\alpha\alpha[w,u]_\alpha=0,$$
for all $u,w\in U$ and $\alpha\in\Gamma$.

Replace v by $[w,m]_\alpha$ and u by $[w,t]_\alpha$ we get,

$$((w\alpha w\alpha m+m\alpha w\alpha w)\alpha(w\alpha w\alpha t+t\alpha w\alpha w)=0.$$

Replace t by $p\alpha t$ where $p\in M$, then

$[w\alpha w, m]_\alpha \Gamma M \Gamma [w\alpha w, t]_\alpha = 0.$

By primness of M we have, $w\alpha w \in Z$ a contradiction .Hence the conclusion is that, So in all possible cases,

$w\alpha w \in Z,$ for all $u \in U, \alpha \in \Gamma.$ So that,

$(u\alpha v + v\alpha u) \in Z$ and $(u\alpha v + v\alpha u)\alpha u \in Z$

If $u \notin Z(U)$ where $Z(U)$ denotes the center of, then $(u\alpha v + v\alpha u = 0,$ for all $v \in U$ and $u \in Z(U)$

Hence U is commutative.

References

1. Nobusawa N., **1964,** *On a Generalization of the Ring Theory, Osaka J. Math.* **1**, 81-89.
2. Barnes W. E., **1966**, *On the Γ-Rings of Nobusawa, Pacific J. Math.*, **18**, 411- 422.
3. Kyuno S.,**1977**, *On the Semi-simple Gamma rings, Tohoku Math. J.*, **29**, 217-225.
4. Luh J., **1969**,*On the theory of simple Γ-rings, Michigan Math. J.*, **16**, 65-75.
5. Booth G. L., **1987**, *On the radicals of Γ_N-rings, Math. Japonica*, **32(3),** 357-372.
6. S. Kyuno,**1978**, *On prime gamma rings, Pacific J. Math.*, **75(1),** 185-190.
7. Posner, E. C, **1957**. *Derivations in prime rings,* Proc. *Amer. Soc.*, **8**, 1093-1100.
8. Awtar, R, **1973**. *Lie and Jordan structure in prime rings with derivations*, Proc. Amer. Math. Soc., **41**, 67-74.
9. Mayne J ., **1984** , *Centralizing mappings of prime rings, Canad. Math.Bull.***27**, 122-126.
10. Bell H. E. and Martindale W. S., **1987**. *Centralizing mappings of semiprime rings, Canad. Math. Bull.* **30**, 92–101
11. Vukman J.,**1990,** *Commuting and centralizing mappings in prime rings, Proc. Amer. Math. Soc.* **109**, 47-52.
12. Bresar M., **1993** , *Centralizing mappings and derivations in prime rings, J. Algebra* **156** ,385-394.
13. Paul A. C. and Sabur Uddin, **2010**, *Lie and Jordan Structure in Simple Gamma Rings, Journal of Physical Sciences*, **14**, 77-86.
14. Sapanc, M. and Nakajima, A, **1997,** *Jordan derivations on completely prime gamma rings*, Math.Japonica, **46**, 1, 47-51.
15. Dey K.K. and Paul A.C. , **2012**,*Generlized Derivations Acting as Homomorphisms and Anti - Homomorphisms of Gamma Rings ,Journal of scientific research* , **4 (1),** 33-37.
16. Motashar S.K., **2011**, *Γ-centralizing mappings on prime and semi-prime Γ-rings*, M.Sc. thesis Baghdad University.