Complex of Lascoux in Partition (4,4,4)

Haytham R. Hassan
Department of mathematics, College of science, University of Al-Mustansiriya, Baghdad, Iraq
haythamhassaan@yahoo.com

Abstract

In this paper the diagrams and divided power of the place polarization $\partial_{ij}^{(k)}$, with its capelli identities have been used, to study the complex of Lascoux in case of the partition (4,4,4).

Keywords: Divided Power, Complex of Lascoux, Capelli Identity

سلسلة لاسكو في التجزئة (4,4,4)

هيثم رزوقي حسن
قسم الرياضيات, كلية العلوم, جامعةالمتنصرية ,بغداد, العراق

الخلاصة

في هذا البحث تم استخدام المخططات و القوى المقسمة لاستقطاب مكان $\partial_{ij}^{(k)}$ مع مشخصات كابلي لدراسة سلسلة لاسكو في حالة التجزئة (4,4,4).

1.Introduction

Let R be commutative ring with 1, F be free R-module and D_nF be the divided power of degree n [1]. Buchsbaum in 2004 modified the boundary method [2], then he and author [3], studied the complex of Lascoux (characteristic zero) in the partition (2,2,2), (3,3,3) respectively, also the author in [4] studied the complex of Lascoux (characteristic zero) in the partition (4,4,3) using this modified method and the technique of Letter place methods [4] obtain characteristic zero.

Consider the map

$D_{p+k}F \otimes D_{q-k}F \to D_pF \otimes D_qF$, this map is the place polarization from place one to place two, where place one and place two are denoted by $D_{p+k}F$ $D_{q-k}F$ respectively, and the map

$\partial_{32}^{(k)} : D_{p+k}F \otimes D_{q+k}F \otimes D_{r-k}F \to$ $D_pF \otimes D_qF \otimes D_rF$ is the place polarization from place two to place three [5,6,7 and 8] . In this point we can also ask for the identities in case such that $\partial_{21}\partial_{32}$ where three places have been looked, so we have to use the following equation

$$\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{21}^{(1)} \circ \partial_{32}^{(1)} = \partial_{31}^{(1)} \qquad (1.1)$$

This is a typical Capelli identity [2,3 and5], more than

$$\partial_{32}^{(k)} \circ \partial_{21}^{(\ell)} = \sum_{\alpha \geq 0} \partial_{21}^{(\ell-\alpha)} \circ \partial_{32}^{(k-\alpha)} \circ \partial_{31}^{(\alpha)} \quad \text{where} \quad \partial_{ij}^{(0)} = I \qquad (1.2)$$

In general the divided power of the place polarizations satisfy the following identities in case of $k \neq i$

$$\partial_{ij}^{(r)} \circ \partial_{jk}^{(s)} = \sum_{\alpha \geq 0} \partial_{jk}^{(s-\alpha)} \circ \partial_{ij}^{(r-\alpha)} \circ \partial_{ik}^{(\alpha)} \qquad (1.3)$$

$$\partial_{jk}^{(s)} \circ \partial_{ij}^{(r)} = \sum_{\alpha \geq 0} (-1)^{\alpha} \partial_{ij}^{(r-\alpha)} \circ \partial_{jk}^{(s-\alpha)} \circ \partial_{ik}^{(\alpha)} \quad (1.4)$$

$$\partial_{21}^{(\ell)} \circ \partial_{31} = \partial_{31} \circ \partial_{21}^{(k)} \quad (1.5)$$

$$\partial_{32}^{(k)} \circ \partial_{31} = \partial_{31} \circ \partial_{32}^{(k)} \quad (1.6)$$

In this paper we study the complex of Lascoux in case of the partition (4,4,4), in particular, in section two of this work we find the terms of the complex of Lascoux in case of the partition (4,4,4), in section three, we study this complex as a diagram by using the divided power of the place polarization and it's Capelli identities.

2.The Terms of Lascoux Complex in The Case of Partition (4,4,4)

The terms of the Lascoux complex are obtained from the determinantal expansion of the Jacobi-Trudi matrix of the partition. The positions of the terms of the complex are determined by the length of the permutation to which they correspond [2, 3]. Now in the case of the partition $\lambda = (4,4,4)$, we have the following matrix :

$$\begin{pmatrix} D_4F & D_5F & D_6F \\ D_3F & D_4F & D_5F \\ D_2F & D_3F & D_4F \end{pmatrix}$$

Then the Lascoux complex has the correspondence between it's terms as follows

$D_4F \otimes D_4F \otimes D_4F \longleftrightarrow$ identity

$D_5F \otimes D_3F \otimes D_4F \longleftrightarrow (12)$

$D_4F \otimes D_5F \otimes D_3F \longleftrightarrow (23)$

$D_6F \otimes D_3F \otimes D_3F \longleftrightarrow (132)$

$D_5F \otimes D_5F \otimes D_2F \longleftrightarrow (123)$

$D_6F \otimes D_4F \otimes D_2F \longleftrightarrow (13)$

So, the complex of Lascoux in the case of the partition $\lambda = (4,4,4)$ has the form :

$$D_6F \otimes D_4F \otimes D_2F \to \begin{matrix} D_6F \otimes D_3F \otimes D_3F \\ \oplus \\ D_5F \otimes D_5F \otimes D_2F \end{matrix} \to$$

$$\begin{matrix} D_5F \otimes D_3F \otimes D_4F \\ \oplus \\ D_4F \otimes D_5F \otimes D_3F \end{matrix} \to D_4F \otimes D_4F \otimes D_4F$$

3.The Complex of Lascoux as a Diagram

Consider the following diagram :

$$\begin{tikzcd} D_6F \otimes D_4F \otimes D_2F \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\mathrm{A}"] & D_6F \otimes D_3F \oplus D_3F \arrow[r, "f_2"] \arrow[d, "2"'] \arrow[dr, phantom, "\mathrm{B}"] & D_5F \otimes D_3F \otimes D_4F \arrow[d, "g_3"] \\ D_5F \otimes D_5F \otimes D_2F \arrow[r, "h_1"] & D_4F \otimes D_5F \otimes D_3F \arrow[r, "h_2"] & D_4F \otimes D_4F \otimes D_4F \end{tikzcd}$$

So; if we define

$f_1 : D_6F \otimes D_4F \otimes D_2F \longrightarrow D_6F \otimes D_3F \otimes D_3F$

as

$f_1(v) = \partial_{32}^{(1)}(v)$ where $v \in D_6F \otimes D_4F \otimes D_2F$

$g_1 : D_6F \otimes D_4F \otimes D_2F \longrightarrow D_5F \otimes D_5F \otimes D_2F$

as

$g_1(v) = \partial_{21}^{(1)}(v)$ where $v \in D_6F \otimes D_4F \otimes \oplus D_2F$

and

$g_2 : D_6F \otimes D_3F \otimes D_3F \longrightarrow D_4F \otimes D_5F \otimes D_3F$

as

$g_2(v) = \partial_{21}^{(2)}(v)$ where $v \in D_6F \otimes D_3F \otimes D_3F$

Now, we have to define

$h_1 : D_5F \otimes D_5F \otimes D_2F \longrightarrow D_4F \otimes D_5F \otimes D_3F$

which makes the diagram A commutative i.e $h_1 \circ g_1 = g_2 \circ f_1$ which means

$h_1 \circ \partial_{21}^{(1)} = \partial_{21}^{(2)} \circ \partial_{32}^{(1)}$

Then by using Capelli identities (1.2) we have:

$$\partial_{21}^{(2)} \circ \partial_{32}^{(1)} = \partial_{32}^{(1)} \circ \partial_{21}^{(2)} - \partial_{21}^{(1)} \circ \partial_{31}^{(1)}$$

$$= \partial_{32}^{(1)} \circ \partial_{21}^{(2)} - \partial_{31}^{(1)} \circ \partial_{21}^{(1)}$$

$$= (\frac{1}{2} \partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)}) \circ \partial_{21}^{(1)}$$

Thus $h_1 = \frac{1}{2} \partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)}$

On the other hand, if we define

$h_2 : D_4F \otimes D_5F \otimes D_3F \longrightarrow D_4F \otimes D_4F \otimes D_4F$

as

$h_2(v) = \partial_{32}^{(1)}(v)$; where $v \in D_4F \otimes D_5F \otimes D_3F$

and

$g_3 : D_5F \otimes D_3F \otimes D_4F \longrightarrow D_4F \otimes D_4F \otimes D_4F$

as

$g_3(v) = \partial_{21}^{(1)}(v)$ where $v \in D_5F \otimes D_3F \otimes D_4F$

Now, to make the diagram B commute, we have to define

$f_2 : D_6F \otimes D_3F \otimes D_3F \longrightarrow D_5F \otimes D_3F \otimes D_4F$

such that $g_3 \circ f_2 = h_2 \circ g_2$ i.e

$\partial_{21}^{(1)} \circ f_2 = \partial_{32}^{(1)} \circ \partial_{21}^{(2)}$

Again by using Caplli identities we get

$$\partial_{32}^{(1)} \circ \partial_{21}^{(2)} = \partial_{21}^{(2)} \circ \partial_{32}^{(1)} + \partial_{21}^{(1)} \circ \partial_{31}^{(1)}$$
$$= \partial_{21}^{(2)} \circ (\frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)})$$

Then $f_2 = \frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)}$

Now consider the following diagram

$$\begin{tikzcd}
D_6F \otimes D_4F \otimes D_2F \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "C"] & D_6F \otimes D_3F \otimes D_3F \arrow[r, "f_2"] \arrow[dr, phantom, "D"] & D_5F \otimes D_3F \otimes D_4F \arrow[d, "g_3"] \\
D_5F \otimes D_5F \otimes D_2F \arrow[r, "h_1"] \arrow[urr, dotted, "\varphi"] & D_4F \otimes D_5F \otimes D_3F \arrow[r, "h_2"] & D_4F \otimes D_4F \otimes D_4F
\end{tikzcd}$$

Define

$\varphi : D_5F \otimes D_5F \otimes D_2F \longrightarrow D_5F \otimes D_3F \otimes D_4F$

by $\varphi(v) = \partial_{32}^{(2)}(v)$; where

$v \in D_5F \otimes D_5F \otimes D_2F$

Proposition 3.1

The diagram C is commutative.

Proof:

$$f_2 \circ f_1 = (\frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)}) \circ \partial_{31}^{(1)}$$
$$= \partial_{21}^{(1)} \circ \partial_{32}^{(2)} + \partial_{31}^{(1)} \circ \partial_{31}^{(1)}$$

But from (1.2) we have

$$\partial_{21}^{(1)} \circ \partial_{32}^{(2)} = \partial_{32}^{(2)} \circ \partial_{21}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)}$$

Then

$$f_2 \circ f_1 = \partial_{32}^{(2)} \circ \partial_{21}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)} + \partial_{31}^{(1)} \circ \partial_{32}^{(1)}$$
$$= \partial_{32}^{(2)} \circ \partial_{21}^{(1)} = \varphi \circ g_1$$

Proposition3.2

The diagram D is commutative.

Proof:

$$\partial_{32}^{(1)} \circ h_1 = \partial_{32}^{(1)} \circ (\frac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)})$$
$$= \partial_{32}^{(2)} \circ \partial_{21}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)}$$

But from (1.2)

$$\partial_{32}^{(2)} \circ \partial_{21}^{(1)} = \partial_{21}^{(1)} \circ \partial_{32}^{(2)} + \partial_{32}^{(1)} \circ \partial_{31}^{(1)}$$

Then

$$\partial_{32}^{(1)} \circ h_1 = \partial_{21}^{(1)} \circ \partial_{32}^{(2)} + \partial_{32}^{(1)} \circ \partial_{31}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)}$$
$$= \partial_{21}^{(1)} \circ \partial_{32}^{(2)} = g_3 \circ \varphi$$

Finally, we can define the maps σ_1, σ_2 and σ_3 where

$$D_6F \otimes D_4F \otimes D_2F \xrightarrow{\sigma_3} \begin{matrix} D_6F \otimes D_3F \otimes D_3F \\ \oplus \\ D_5F \otimes D_5F \otimes D_2F \end{matrix} \xrightarrow{\sigma_2}$$

$$\begin{matrix} D_5F \otimes D_3F \otimes D_4F \\ \oplus \\ D_4F \otimes D_5F \otimes D_3F \end{matrix} \xrightarrow{\sigma_1} D_4F \otimes D_4F \otimes D_4F$$

by

- $\sigma_3(x) = (f_1(x), g_1(x)); \quad \forall v \in D_6F \otimes D_4F \otimes D_2F$

  $\sigma_2((x_1, x_2) = (f_2(x) - \varphi(x_2)), h_1(x_2) - g_2(x_1);$

  $\forall (x_1, x_2) \in \begin{matrix} D_6F \otimes D_3F \otimes D_3F \\ \oplus \\ D_5F \otimes D_5F \otimes D_2F \end{matrix}$

  and
- $\sigma_1((x_1, x_2)) = (g_3(x_1), h_1(x_2));$

  $\forall (x_1, x_2) \in \begin{matrix} D_5F \otimes D_3F \otimes D_4F \\ \oplus \\ D_4F \otimes D_5F \otimes D_3F \end{matrix}$

Proposition 3.3

$$0 \longrightarrow D_6F \otimes D_4F \otimes D_2F \xrightarrow{\sigma_3} \begin{matrix} D_6F \otimes D_3F \otimes D_3F \\ \oplus \\ D_5F \otimes D_5F \otimes D_2F \end{matrix} \xrightarrow{\sigma_2}$$

$$\begin{matrix} D_5F \otimes D_3F \otimes D_4F \\ \oplus \\ D_4F \otimes D_5F \otimes D_3F \end{matrix} \xrightarrow{\sigma_1} D_4F \otimes D_4F \otimes D_4F$$

is complex.

Proof: Since $\partial_{32}^{(1)}$ and $\partial_{21}^{(1)}$ are injective then we get σ_3 is injective. Now

$$\sigma_2 \circ \sigma_3(x) = \sigma_2(f_1(x), g_1(x))$$
$$= \sigma_2(\partial_{32}^{(1)}(x), \partial_{21}^{(1)}(x))$$
$$= (f_2(\partial_{32}^{(1)}(x)) - \varphi(\partial_{21}^{(1)}(x)), h_1(\partial_{21}^{(1)}(x)) - g_2(x)),$$

thus

$$f_2(\partial_{32}^{(1)}(x) - \varphi(\partial_{21}^{(1)}(x))$$
$$= (\frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)}) \circ \partial_{32}^{(1)}(x) - \partial_{32}^{(2)} \circ \partial_{21}^{(1)}(x)$$
$$= (\partial_{21}^{(1)} \circ \partial_{32}^{(2)} + \partial_{31}^{(1)} \circ \partial_{32}^{(1)} - \partial_{32}^{(2)} \circ \partial_{21}^{(1)})(x)$$
$$= \partial_{32}^{(2)} \circ \partial_{21}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)} + \partial_{32}^{(1)} \circ \partial_{31}^{(1)} - \partial_{32}^{(2)}\partial_{21}^{(1)} = 0$$

and

$$h_1(\partial_{21}^{(1)}(x)) - g_2(x))$$
$$= (\frac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)}) \circ \partial_{21}^{(1)}(x) - \partial_{21}^{(2)} \circ \partial_{32}^{(1)}(x)$$
$$= (\partial_{32}^{(1)} \circ \partial_{21}^{(2)} - \partial_{31}^{(1)} \circ \partial_{21}^{(1)} - \partial_{21}^{(2)} \circ \partial_{32}^{(1)})(x)$$
$$= (\partial_{21}^{(2)} \circ \partial_{32}^{(1)} + \partial_{21}^{(1)} \circ \partial_{31}^{(1)} - \partial_{21}^{(1)} \circ \partial_{31}^{(1)} - \partial_{21}^{(2)} \circ \partial_{32}^{(1)})(x) = 0$$

So we get $(\sigma_2 \circ \sigma_3)(x) = 0$

and

$$(\sigma_2 \circ \sigma_3)(x_1, x_2) = \sigma_1(f_2(x_1) - \varphi(x_2), h_1(x_2) - g_2(x_1))$$
$$= \sigma_1((\frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)})(x_1) - \partial_{32}^{(2)}(x_2),$$
$$(\frac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)})(x_2) - \partial_{21}^{(2)}(x_1))$$
$$= \partial_{21}^{(1)}(\frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)})(x_1) - \partial_{21}^{(1)} \circ \partial_{32}^{(2)}(x_2)$$
$$+ \partial_{32}^{(1)}(\frac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)})(x_2) - \partial_{32}^{(1)} \circ \partial_{21}^{(2)}(x_1)$$
$$= (\partial_{21}^{(2)} \circ \partial_{32}^{(1)} + \partial_{21}^{(1)} \circ \partial_{31}^{(1)} - \partial_{32}^{(1)} \circ \partial_{21}^{(2)})(x_1)$$
$$+ (\partial_{32}^{(2)} \circ \partial_{21}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)} - \partial_{21}^{(1)} \circ \partial_{32}^{(2)})(x_2).$$

But $\partial_{21}^{(2)} \circ \partial_{32}^{(1)} = \partial_{32}^{(1)} \circ \partial_{21}^{(2)} - \partial_{21}^{(1)} \circ \partial_{31}^{(1)}$ and $\partial_{32}^{(2)} \circ \partial_{21}^{(1)} = \partial_{21}^{(1)} \circ \partial_{32}^{(2)} + \partial_{32}^{(1)} \circ \partial_{31}^{(1)}$ which implies that $(\sigma_2 \circ \sigma_3)(x_1, x_2) = 0$.

References

1. Akin K., Buchsbaum D. A. and Weyman J., **1982**, Schur functors and Schur complexes, Adv. Math. **44**, 207-278.
2. Buchsbaum D. A., **2004**, Characteristic free example of Lascoux resolution and letter place mathods for intertwining numbers, European Journal of Gombinatorics, Vol. **25**, 1169-1179.
3. Haytham R. **2006**,,Hassan, Application of the characteristic-free resolution of Weyl module to the Lascoux resolution in the case (3,3,3). Ph. D. thesis, universita di Roma "Tor Vergata", **2012**.
4. Haytham R. Hassan, The Reduction of Resolution of Weyl Module from Characteristic-Free Resolution in Case (4,4,3), to appear, J. Ibn Al-Haitham for pur and applied sience.
5. Buchsbaum D. A. and Rota G. C. , **2001**, , Approaches to resolution of Weyl modules, Adv. In applied math. **27**, 82-191, **1973**.
6. Boffi G. and Buchsbaum D. A., Threading homology through algebra : Selected patterns, clarendon press. Oxford.
7. Buchsbaum D. A. and Rota G. C., **1994**, A new construction in homological algebra, Natl. acad. Sci, USA **91**, 4115-4119.
8. Rotman J.J., **1979** , Introduction to homo-logical algebra, A cademic press. INC.