On The Dynamics of Discrete-Time Prey-Predator System with Ratio-Dependent Functional Response

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Abstract:
In this paper, a discrete- time ratio-dependent prey- predator model is proposed and analyzed. All possible fixed points have been obtained. The local stability conditions for these fixed points have been established. The global stability of the proposed system is investigated numerically. Bifurcation diagrams as a function of growth rate of the prey species are drawn. It is observed that the proposed system has rich dynamics including chaos.

Keywords: Discrete prey- predator system, Ratio-dependent functional, Stability, Bifurcation diagrams, Chaos.

حول ديناميكية نظام الفريسة–المفترس المتقطع ذو دالة الاستجابة الوظيفية النسبية

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الخلاصة
في هذا البحث , تم اقتراحُ وتحليل نظام الفريسةِ–المفترسِ المتقطع ذو دالة الاستجابة الوظيفية النسبية. تم الحصول على جميع النقاط الثابتة المحتملة, ووجدنا شروط الاستقرار المحليّ لها. كذلك, درسنا الاستقرار الشاملَ للنظامِ المُقتَرَحِ عدديا. كما أن مخططات التشعيبِ كدالة لمعدل النمو في الفريسةَ رسمتُ. لوحظ أن النظامَ المُقتَرَحَ لَهُ ديناميكية متنوعة بضمنها فوضى.

Introduction

Modern mathematical population dynamics started with the famous Lotka-Volterra prey-predator model. Since that time many mathematical models, which describe the population dynamics, have been proposed and analyzed. These models may take many forms depending on the time scale and space structure of the problem [1]. Some of these forms are represented by discrete-time dynamical models. In fact, many researchers have mainly focused on discrete- time prey-predator models and they showed that these models may be produce a much richer sets of patterns than those observed in continuous time models, see [2-6] and the references their in. In particular Danca et al [3] demonstrated the existence of the chaotic dynamics in a simple discrete-time prey-predator model with Holling type - I functional response. Agiza et al [2] proposed and analyzed a discrete-time prey-predator model with Holling type- II functional response, and they observed that the proposed model has a complex dynamic.

In all of the above mentioned studies the functional response (which is known as prey-dependent) depends entirely on the density of prey species ignoring the effect of predator abundance. Although, these prey-dependent prey-predator models are used extensity in literatures, they suffer from two paradoxes problems: the paradox of enrichment and that of biological control.

Later, Ariditi and Ginzburg [7] proposed and studied a new form of continuous time prey-predator model, in which the functional response depends on both the prey as well as predator species, and they called a ratio-dependent prey-predator system. They have shown that the ratio-dependent model can solve the paradoxes problems.

Therefore, in this paper, the discrete-time prey-predator model given by Agiza et al [2] is modified so that it involve the ratio-dependent type of functional response instead of Holling type-II of functional response. The existence of fixed points is discussed. The local, as well as, global stability of the proposed model is investigated analytically and numerically.

Mathematical Model

One of the possible ways to understand the complex dynamical behavior between two interacting species is the use of the discrete-time model formulation. In the present work we study the dynamics of prey-predator model with ratio-dependent functional response that may be describes by the following two difference equations:

$$\begin{aligned} x_{n+1} &= ax_n(1-x_n) - p(x_n, y_n)\, y_n \\ y_{n+1} &= q(x_n, y_n)\, y_n \end{aligned} \qquad (1)$$

where x_n and y_n represents the number of the prey and predator populations at certain iteration n $(n = 0,1,\ldots)$. The terms $p(x_n, y_n) = \frac{bx_n}{y_n + \varepsilon x_n}$ and $q(x_n, y_n) = \frac{dx_n}{y_n + \varepsilon x_n}$ represent the ratio-dependent functional response and ratio-dependent numerical response, respectively.

The positive parameters a and ε stand for intrinsic growth rate of prey species and the limitation of the growth velocity of the predator species with increase in number of prey while the positive parameters b and d denoted to maximum attack rate and conversion rate of predator, respectively.

The Dynamical Behavior of The System (1)

In this section, the existence and local stability conditions of all possible fixed points are discussed, and the following results are obtained:

1- The axial fixed point $p_1 = (\frac{a-1}{a}, 0)$ exists if and only if

$$a > 1 \qquad (2)$$

2- The positive fixed point $p_2 = (x^*, y^*)$, where

$$x^* = \frac{d(a-1) - b(d-\varepsilon)}{ad} \qquad (3a)$$

$$y^* = x^*(d-\varepsilon) \qquad (3b)$$

exists under the following conditions:

$$d(a-1) > b(d-\varepsilon) > 0 \qquad (4)$$

Now, in order to study the local stability of the above fixed points the Jacobian matrix of the system (1) is computed at each fixed point and then the eigenvalues for the resulting matrix are determined.

Since the Jacobian matrix of system (1) at the point (x, y) may be written as:

$$J(x,y) = \begin{bmatrix} a(1-2x) - \frac{by^2}{(y+\varepsilon x)^2} & \frac{-b\varepsilon^2}{(y+\varepsilon x)^2} \\ \frac{dy^2}{(y+\varepsilon x)^2} & \frac{d\varepsilon^2}{(y+\varepsilon x)^2} \end{bmatrix} \qquad (5)$$

So, the characteristic equation of $J(x, y)$ is:

$$F(\lambda) = \lambda^2 - A\lambda + B = 0 \qquad (6)$$

Where $A = a(1-2x) + \frac{d\varepsilon^2 - by^2}{(y+\varepsilon x)^2}$ and

$B = \frac{ad\varepsilon d^2(1-2x)}{(y+\varepsilon x)^2}$.

Note that before we go further to discuss the dynamical behavior near the above fixed points, it is well known that the discrete-time two dimensional dynamical system is said to be area contracting (dissipative) dynamical system provided that $|B| < 1$ [8], where B is the determinant of the Jacobian matrix.

Hence system (1) is area contracting if the following condition holds:

$$\left|\frac{ad\varepsilon d^2(1-2x)}{(y+\varepsilon x)^2}\right| < 1 \qquad (7a)$$

.while it is conservative dynamical system under the following condition

$$\left|\frac{ad\varepsilon d^2(1-2x)}{(y+\varepsilon x)^2}\right| = 1 \qquad (7b)$$

Now, the following Theorems describe the local dynamical behavior near the fixed points p_1 and p_2, respectively.

Theorem (1): The nature of the axial fixed point $p_1 = (\frac{a-1}{a}, 0)$ is:

1- Sink if $1 < a < 3$ and $d < \varepsilon$.
2- Source if $a > 3$ and $d > \varepsilon$
3- Non-hyperbolic if $a = 3$ or $d = \varepsilon$
4- Saddle otherwise.

Proof: According to equation (6) it is easy to verify that, $A = \frac{\varepsilon(2-a)+d}{\varepsilon}$ and $B = (2-a)\frac{d}{\varepsilon}$ and hence the eigenvalues of $J(p_1)$ are $\lambda_1 = 2 - a$ and $\lambda_2 = \frac{d}{\varepsilon}$. Therefore, for $1 < a < 3$ and $d < \varepsilon$ then $|\lambda_i| < 1$ for all $i = 1,2$ and hence p_1 is sink. While, for $a > 3$ with $d > \varepsilon$ then $|\lambda_i| > 1$ for all $i = 1,2$ and hence p_1 is a source.
Further, for $a = 3$ and/or $d = \varepsilon$ then at least one of $|\lambda_i| = 1$ for all $i = 1,2$. Hence p_1 is a non-hyperbolic point. Finally, for all other sets of parameters p_1 is a saddle point.

Theorem (2): The nature of the positive fixed point $p_2 = (x^*, y^*)$ is:

1- Sink if $H_2 < a < H_1$ (8a)
2- Source if $a < H_2$ (8b)
3-Non-hyperbolic if $a = H_1$ (8c)
4-Saddle point if $a > H_1$ (8d)

Where
$H_1 = \frac{3d^2+bd^2+2bd\varepsilon+3\varepsilon(d-b\varepsilon)}{d(d+\varepsilon)}$,
and $H_2 = \frac{2bd\varepsilon - 2b\varepsilon^2 + 2d\varepsilon - d^2}{d\varepsilon}$.

Proof: By substituting the point p_2 in the Jacobian matrix of system (1), and then compute the characteristic equation. It is observed that the characteristic equation can be written as:

$$F(\lambda) = \lambda^2 - A^*\lambda + B^* = 0 \quad (9)$$

Where $A^* = \frac{ad^2+2bd(d-\varepsilon)-2d^2(a-1)-b(d-\varepsilon)+d\varepsilon}{d^2}$ and
$B^* = \frac{ad\varepsilon+2b\varepsilon(d-\varepsilon)-2d\varepsilon(a-1)}{d^2}$

Now, according to the stability criterion [9], it is well known that if $F(1) > 0$ then the two roots of $F(\lambda) = 0$ (the eigenvalues of $J(p_2)$ that denoted by λ_1 and λ_2) satisfy the following:

[1] $|\lambda_i| < 1$ for all $i = 1,2$ if and only if $F(-1) > 0$ and $B^* < 1$.

[2] $|\lambda_i| > 1$ for all $i = 1,2$ if and only if $F(-1) > 0$ and $B^* > 1$.

[3] $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $A^* \neq 0,2$.

[4] $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$ and $B^* > 1$.

[5] λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2| = 1$ if and only if $A^{*2} - 4B^* < 0$ and $B^* = 1$.

Consequently by a straight forward computation, it is easy to verity that:

$$F(1) = 1^2 - A^* + B^* = \frac{(d-\varepsilon)[d(a-1)-b(d-\varepsilon)]}{d^2} \quad (10)$$

Clearly, $F(1) > 0$ under the existence condition (4). Furthermore, we have that

$$F(-1) = \frac{3d(d+\varepsilon) - ad(d+\varepsilon) + b(d-\varepsilon)(d+3\varepsilon)}{d^2} \quad (11)$$

and

$$1 - B^* = \frac{d^2 - 2b\varepsilon(d-\varepsilon) + ad\varepsilon - 2d\varepsilon}{d^2} \quad (12)$$

Therefore, according to condition $(8a)$, it is easy to verify that:
$F(-1) > 0$ and $1 - B^* > 0$ and hence $|\lambda_i| < 1$ for all $i = 1,2$. Thus p_2 is sink.
Now, from condition $(8b)$, we have that $F(-1) > 0$ while $1 - B^* < 0$. Thus $|\lambda_i| > 1$ for all $i = 1,2$, and hence p_2 is source.
Further, due to condition $(8c)$, it is observes that $F(-1) = 0$ and $A^* \neq 0,2$. Hence $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ and then p_2 is a non-hyperbolic point.
Finally, for condition $(8d)$, it is observed that $F(-1) < 0$. Then $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$), and hence p_2 is a saddle point and thus the proof is complete.

Numerical Simulations

In this section, the global dynamical behavior of system (1) is investigated numerically. The objectives of such study are: first confirm our analytical results and second investigate the existence of complex dynamics (such as chaos) in system (1).
The asymptotic behavior of the orbit of system (1) is studied for different sets of parameter values and for different sets of initial conditions. In order to detect about the types of attracting sets exist in system (1), numbers of bifurcation diagrams and typical phase portraits with their time-series are drawn as a parameters varying as shown in the following:
Bifurcation diagrams as function of the growth rate parameter of the prey are drawn with using phaser scientific software in the following cases:

1) $b=2$, $\varepsilon=1.25$ and $d=2$ while a in the rang $[4, 4.95]$,

2) $b=3$, $\varepsilon=1.25$ and $d=1.5$ while a in the rang $[3.7, 4.5]$.

In first case, the bifurcation diagram between the maximum value of prey population and its growth rate is drawn in Figure 1.

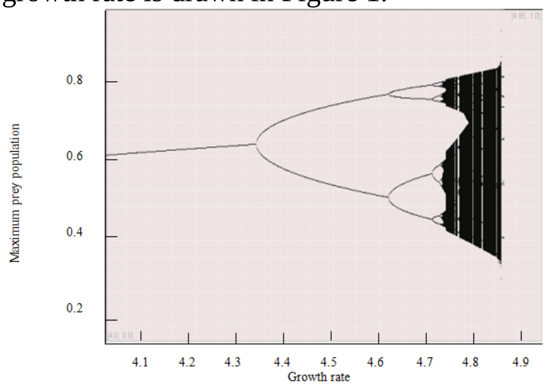
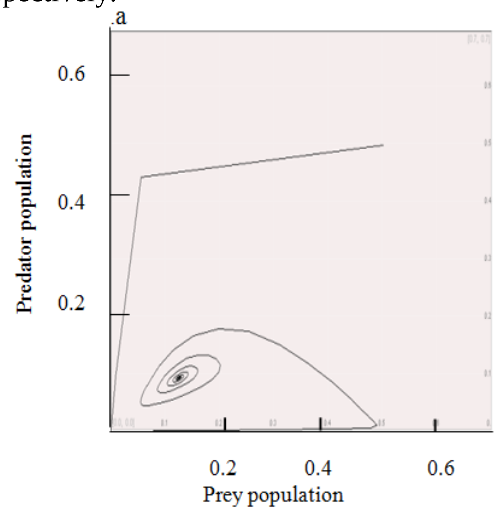


Figure 1- Bifurcation diagram of system [1] as a varied from 4 to 4.95.

Clearly Figure 1 shows the transent from stable point to periodic and then to chaos as the growth rate parameter increase in the range $[4, 4.95]$.

Moreover, the phase portraits of system (1) for parameter values in case (1) with $a=2$, $a=4.5$ and $a=4.85$ are shown in Figures 2a,3a,4a, respectively. While the time-series of the Figures 2a,3a,4a are drawn in Figures 2b,3b,4b, respectively.



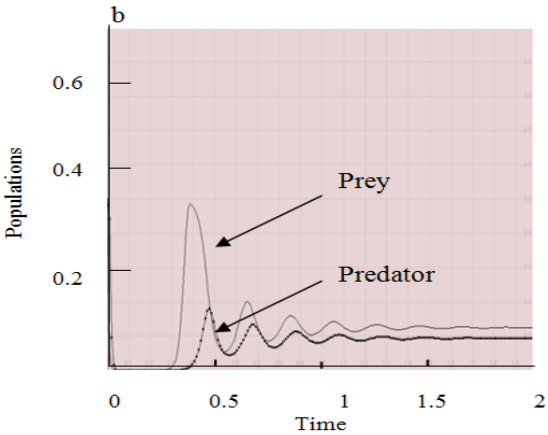
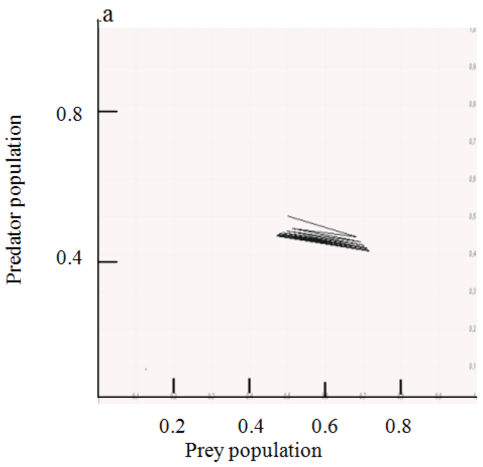


Figure 2- (a) System [1] approaches asymptotically positive fixed point in for the $Int\ R_+^2$ a=2, (b) Time-series diagram of Figure.2a.



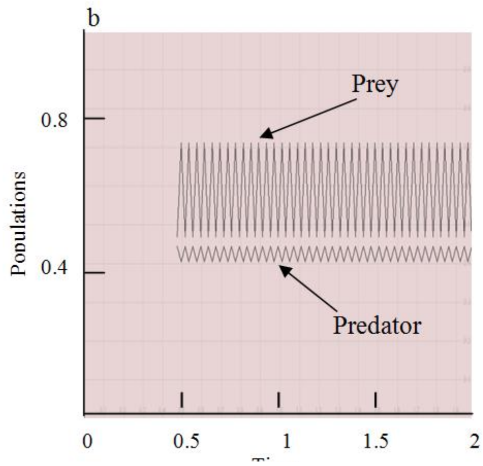


Figure 3- (a) System [1] approaches asymptotically to periodic attractor in $Int\ R_+^2$ for a=4.5, (b) Time-series diagram of Figure.3.

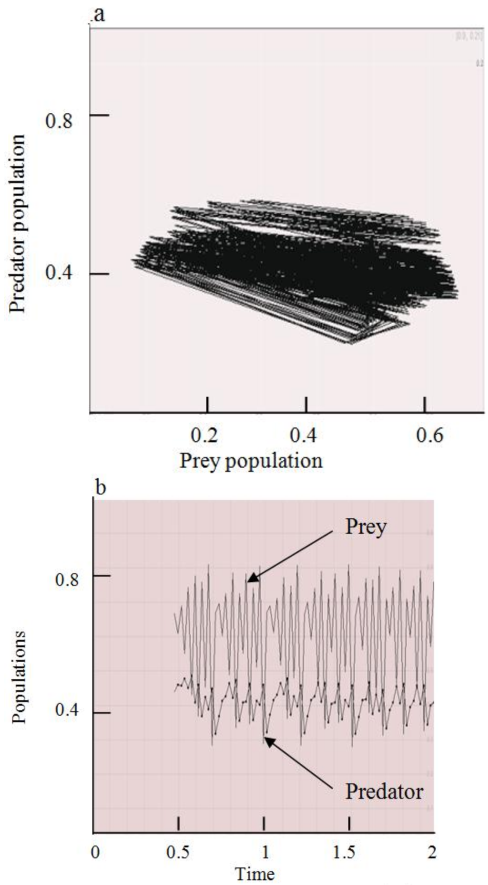


Figure 4- System[1]approaches to chaotic attracter for a=4.85. (b) Time-series diagram of Figure (4a).

According to the above figures, system (1) approaches asymptotically to a positive point $p_2(0.125, 0.093)$ in a spiral pattern as shown in Figure 2a,b for $a = 2$ with the rest of parameter values are fixed as given in case (1).

However, system (1) approaches asymptotically to periodic dynamics as shown in Figure 3a,b for the parameter values given in case (1) with $a = 4.5$ .Therefore, as a increases the system loses its stability and approaches to periodic dynamics. In fact, as shown in Figure1, system (1) has a flip bifurcation occurs at the bifurcation point $a = 4.32$. Finally, for $a = 4.85$ with the rest of parameters fixed as in case (1), system (1) has a chaotic attractor as given in Figure 4a,b. Indeed, the trasition from stable case to chaos in system (1) is occurring through cascade of periodic doubling as shown in Figure1.

In second case, the bifurcation diagram of system (1) between the maximum value of prey population and its growth rate is drawn in the following figures.

Clearly Figure 5 shows the transent from stable point to chaos through cascade of periodic doubling as the growth rate parameter ncreasethe range [3.7, 4.5].

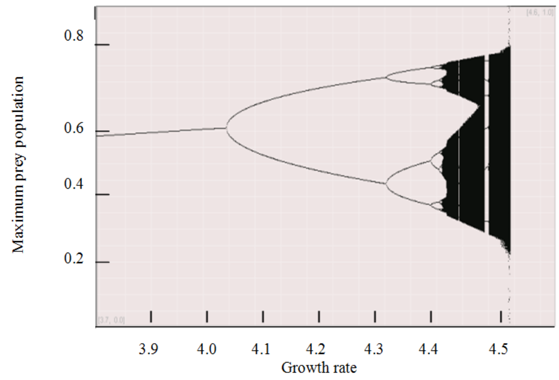


Figure 5- Bifurcation diagram of system (1)as a varied from 3.7 to 4.5.

Moreover, the phase portraits of system (1) along with their time series for parameter values in case (2) with $a = 3$, $a = 4.2$ and $a = 4.502$ are drawn in Figures 6,7,8, respectively

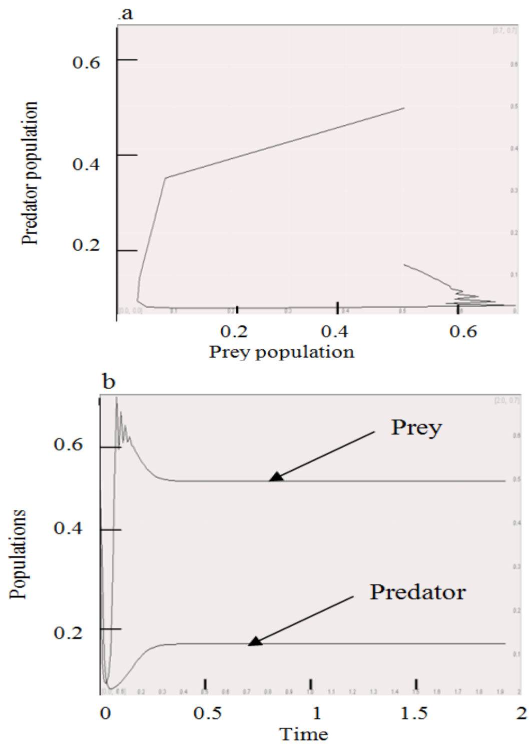
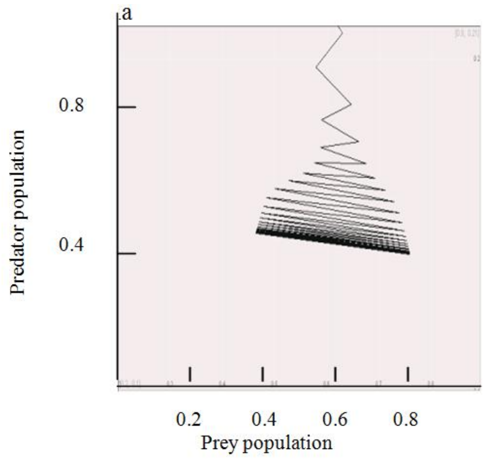


Figure 6- (a) System [1] approaches asymptotical to positive fixed point in the $IntR_+^2$ for a=3, (b) Time-series diagram of



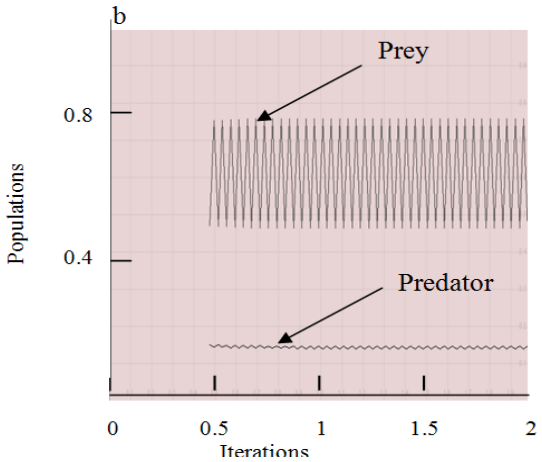
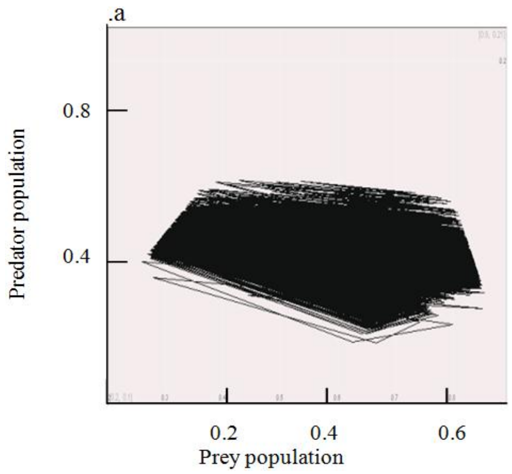


Figure 7- (a) System [1] approaches asymptot periodic attractor in $Int\ R_+^2$ for a=4.2, (b) Time-series diagram of Figure.7a.



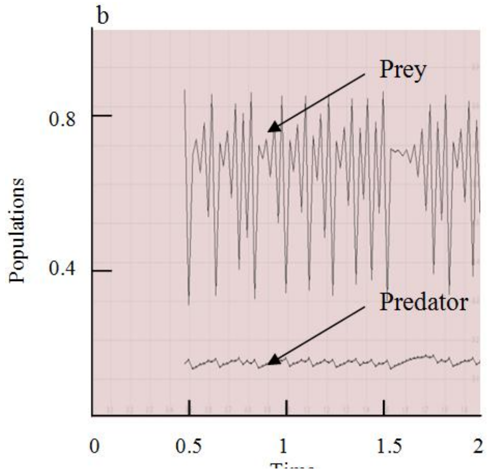


Figure 8-System [1] approaches to chaotic attracter for a=4.502. (b) Time-series diagram of Figure.8a.

Again, due to the above figures, for the parameter values given in case (2) with $a=3$, system (1) approaches asymptotically to a positive point p_2 as shown in Figure 6a,b. While for the parameter values given in case (2) with $a=4.2$, system (1) approaches asymptotically to periodic dynamics as shown in Figure 7a,b. Moreover, as in case (1), system (1) has a flip bifurcation that occurs at the bifurcation point $a=3.95455$ with the rest of parameter values as given in case (2), see Figure 5. Finally, for $a=4.502$ with the rest of parameters as in case (2) the system (1) has a chaotic attractor as given in Figure 8a,b

Sensetive on Initial Conditions

It is well known that, for discrete-time dynamical systems, if the successive iterates approach a fixed point or limit cycle then the difference between any two solutions, which start with two initial conditions that differ from each other by a small amount, will on average grow smaller, with each iteration. However, if the solution is chaotic the difference will tend to grow larger, with each iteration. Therefore in this section, sensitivity to initial conditions criterion for detecting of chaotic dynamics in system (1) is applied. Now, since the chaotic attractors of system (1) given in Figure 4 and Figure 8 are drawn with initial point (0.5,0.5), then we choose the point (0.51,0.5) as another initial point for chaotic attractor of system (1) and then the time-series of system (1) starting at these two different initial points are drawn in the following figures

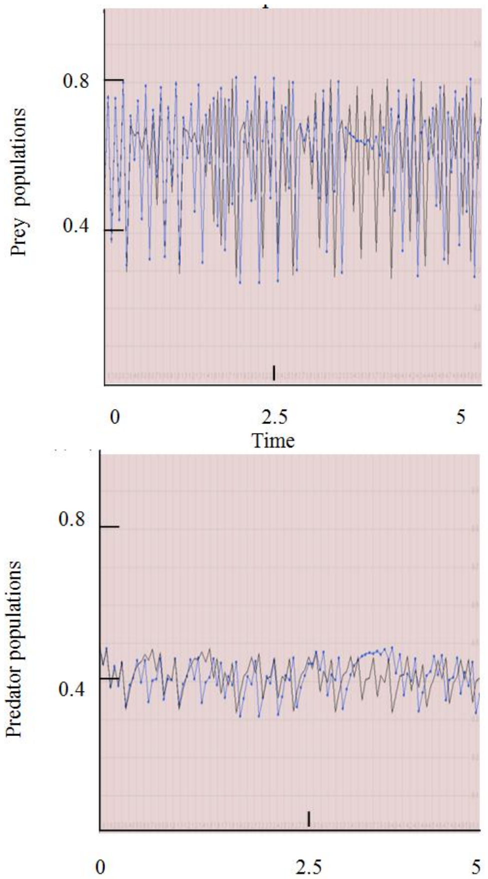
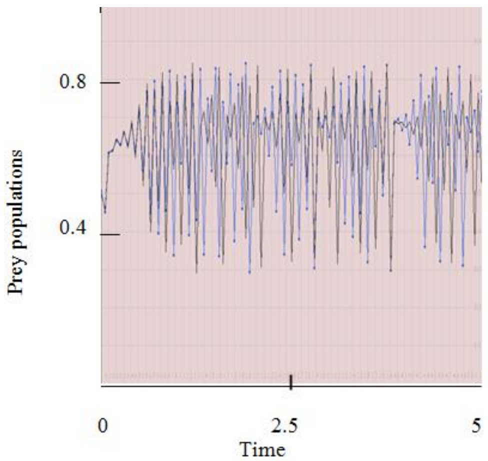


Figure 9-Sensitive dependence on Initial condition



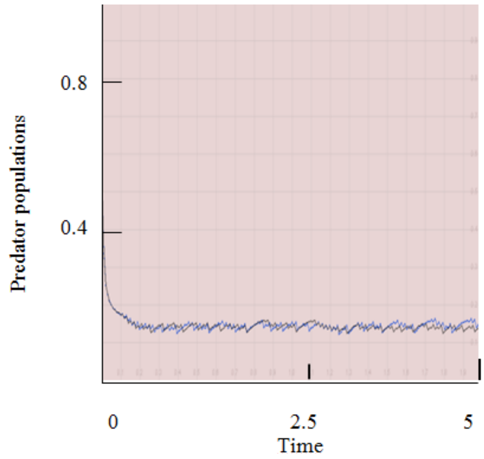


Figure 10- Sensitive dependence on initial conditions.

According to the above two figures it is clear that the solution of system (1) is very sensitive to small change in the initial condition and hence chaotic dynamic is detected.

Discussion and Conclusions

In this paper, a discrete-time prey-predator model with ratio-dependent functional response is proposed and analyzed. It is observed that system (1) has two nonnegative fixed points. The local stability analysis for each of them is investigated analytically. The global stability analysis of system is investigated numerically. It has been shown that, system (1) has a flip bifurcation that occurs at the bifurcation point $a = 4.32$. However, as the growth rate parameter increases further, the system has interesting dynamical behaviors, including cascade of periodic-doubling and chaos. These results show that the discrete time prey predator models have richer dynamics compared with the associated models in the continuous case. Finally, in order to explore the existence of chaotic attractors in system (1), the sensitivity of typical attractors to small varying in the initial conditions is studied.

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