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Mixed Implicit Galerkin – Frank Wolf, Gradient and Gradient Projection Methods for Solving Classical Optimal Control Problem Governed by Variable Coefficients, Linear Hyperbolic, Boundary Value Problem

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Abstract
This paper deals with testing a numerical solution for the discrete classical optimal control problem governed by a linear hyperbolic boundary value problem with variable coefficients. When the discrete classical control is fixed, the proof of the existence and uniqueness theorem for the discrete solution of the discrete weak form is achieved. The existence theorem for the discrete classical optimal control and the necessary conditions for optimality of the problem are proved under suitable assumptions. The discrete classical optimal control problem (DCOCP) is solved by using the mixed Galerkin finite element method to find the solution of the discrete weak form (discrete state). Also, it is used to find the solution for the discrete adjoint weak form (discrete adjoint) with the Gradient Projection method (GPM) , the Gradient method (GM), or the Frank Wolfe method (FWM) to the DCOCP. Within each of these three methods, the Armijo step option (ARSO) or the optimal step option (OPSO) is used to improve (to accelerate the step) the solution of the discrete classical control problem. Finally, some illustrative numerical examples for the considered discrete control problem are provided. The results show that the GPM with ARSO method is better than GM or FWM with ARSO methods. On the other hand, the results show that the GPM and GM with OPSO methods are better than the FWM with the OPSO method.

Keywords: Numerical classical optimal control, Galerkin finite element method, Gradient Projection method, Gradient method, Frank Wolfe method.

مزج طريقة كاليركن – فرانك فولف ,انحدار و مسقط انحدار لحل سيطرة امثلية تقليدية يحكمها مسألة قيم حدودية زائدية خطية ذات معاملات متغيرة

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الخلاصه
يتناول هذا البحث دراسة الحل العددي لمسألة سيطرة امثلية تقليدية من النمط المقسم يحكمها مسألة قيم حدودية زائدية خطية ذات المعاملات المتغيرة. عندما تكون السيطرة التقليدية النمط المقسم ثابتا, تم برهان نظرية الوجود والوحدانية للحل المتقطع للصيغة الضعيفة من النمط المقسم. بوجود فرضيات مناسبة, تم برهان مبرهنة الوجود لسيطرة امثلية تقليدية من النمط المقسم وبرهان مبرهنة الشرط الضروري لمسألة سيطرة امثلية تقليدية من النمط المقسم . تم حل مسألة السيطرة الامثلية التقليدية من النمط المقسم بواسطة مزج طريقة

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كاليركن للعناصر المنتهية لإيجاد الحل لمعادلة الحالة المتقطعة للصيغة الضعيفة من النمط المقسم و الحل المرافق من النمط المقسم للصيغة المرافقة للصيغة الضعيفة للحالة مع أطريقة فرانك فولف او الانحدار او طريقة مسقط الانحدار لإيجاد سيطرة امثلية تقليدية من النمط المقسم. ضمن هذه الطرق الثلاث, تم استخدام خطوة ارميجو او خطوة امثلية لتحسين (لتسريع الخطوة) الحل للسيطرة التقليدية من النمط المقسم. في النهاية, تم إعطاء بعض الأمثلة العددية التوضيحية للمسألة وبينت النتائج ان طريقة مسقط الانحدار مع استخدام خطوة ارميجو افضل من كل من طريقة فرانك فولف او الانحدار مع خطوة ارميجو , ايضا النتائج بينت ان كل من طريقة الانحداراومسقط الانحدار مع استخدام خطوة امثلية افضل من طريقة فرانك فولف مع استخدام خطوة امثلية .

Introduction

Optimal control problems (OCP) have various applications [1, 2]. These problems are usually governed by partial differential equations (PDEs) or ordinary differential equations (ODEs).

Many researchers have been interested to study the numerical solution of optimal control problems described by nonlinear elliptic PDEs [3, 4], by semilinear parabolic PDEs [5-7], or by one dimensional linear hyperbolic PDEs with constant coefficients (LHPDES) [8]. The researchers also include two dimensional linear and nonlinear hyperbolic PDEs with constant coefficients [9-12], or by nonlinear ODEs [13]. These works attracted our attention to focus our interest on studying OCP described by LHPDES but with variable coefficients (LHBVPVC).

This paper investigates the numerical solution of the DCCOCP that is described by the LHBVPVC. Here, the continuous classical optimal control problem (CCOCP) is described, which is discretized by applying the Galerkin finite element method (GFEM). The GFEM is applied for variable space and the implicit finite difference scheme (IFDS) which is employed for the time variable to obtain the discrete CCOCP(DCCOCP). The existence and the uniqueness theorem for the discrete solution (DS) of the discrete weak form (DWF) is stated and proved. The DCCOCP is found numerically by using the mix of the GFEM with the IFDS (GFEM-IFDS) [10] to find the DS of the DWF, while the DCOC is obtained through solving the optimization problem (finding the minimum of the cost function) by separately using one of the following optimization methods: the Gradient method (GM) , the Gradient projection method (GPM), and the Frank Wolfe method (FWM) [14]. Within each one of these three methods the Armijo step option (ARSO) or the optimal step option (OPSO) is used to improve the direction of the optimal search [14]. Some illustrative examples for this considered problem are given to show the accuracy and the efficiency of each of the three methods.

1. The Statement of the CCOCP: Let $K \subset \mathbb{R}^2$ be a bounded open region, ∂K be the boundary of K , and $E = [0, T]$, 0<T<∞ be a time space, $\rho = K \times E$. The CCOCP governed by the LHBVPVC is:

$$\psi_{tt} - \sum_{\tau,\varrho=1}^{2} \frac{\partial}{\partial y_\varrho}\left[\bar{a}_{\varrho\tau}(\vec{x},t)\frac{\partial\psi}{\partial z_\tau}\right] + \bar{\bar{a}}(\vec{x},t)\psi = g(\vec{x},t) + \omega - \omega_d, \text{ in } \rho = K \times E\ , \vec{x} = (y,z)\ , \quad (1)$$

with BC and ICs

$$\psi(\vec{x},t) = 0, \text{ in } \partial\rho = \partial K \times [0,T] \quad (2)$$

$$\psi(\vec{x},0) = \psi^0(\vec{x}), \text{ in }\ , \quad (3)$$

$$\psi_t(\vec{x},0) = \psi^1(\vec{x}), \text{ in } K, \quad (4)$$

where the control is symbolized by $\omega = \omega(\vec{x},t) \in L^2(\rho)$ and its corresponding state is symbolized by $\psi = \psi_\omega(\vec{x},t) \in C^2(\bar{\rho})$, $\bar{a}_{\varrho\tau}(\vec{x},t)$, while $\bar{\bar{a}}(\vec{x},t) \in L^\infty(K), (\forall \varrho, \tau = 1,2)$ are positive arbitrary functions. The desired control is symbolized by $\omega_d = \omega_d(\vec{x},t) \in L^2(\rho)$ and $g(\vec{x},t) \in L^2(\rho)$ is a given function.

The set of the admissible continuous classical controls is represented by W_{ad}, where

$W_{ad} = \{\ \omega \in L^2(\rho) |\ \omega(\vec{x},t) \in U, \text{ a.e. in } \rho\}$,with $U \subset \mathbb{R}^d$ is a convex and compact set .

The cost functional [9] is given by

$$G_0(\omega) = \int_\rho [\tfrac{1}{2}(\psi - \psi_d)^2 + \tfrac{\varpi}{2}(\omega - \omega_d)^2] d\vec{x}dt\ , \quad (5)$$

where, the desired state is symbolized by $\psi_d = \psi_d(\vec{x},t) \in L^2(\rho)$.

The CCOCP is to find $\omega \in W_{ad}$ which minimizes (5).

Now, the weak form (WF) of the problem (1-4) for $\psi \in H_0^1(K)$ is

$$\langle \psi_{tt}, \varphi \rangle + B(t,\psi,\varphi) = (g(\vec{x},t),\varphi)_K + (\omega,\varphi)_K - (\omega_d,\varphi)_K\ , \forall \varphi \in S\ , \quad (6)$$

$$(\psi(0),\varphi) = (\psi^0,\varphi), \text{ in } K, \quad (7)$$

$(\psi_t(0),\varphi)=(\psi^1,\varphi)$, in , (8)

where $\psi^0 \in S$, $\psi^1 \in L^2(K)$ and $B(t,\psi,\varphi)=\sum_{\tau,\varrho=1}^{2}(\bar{a}_{\varrho\tau}(\vec{x},t)\nabla\psi,\nabla\varphi)_K+(\bar{\bar{a}}(\vec{x},t)\psi,\varphi)_K$ is a symmetric bilinear form.

Assumptions A[9]: For each $\varphi,\psi \in S$ and $t \in E$ the following inequality is satisfied

(I)$|B(t,\psi,\varphi)| \le \sigma_2 \parallel \psi \parallel_1 \parallel \varphi \parallel_1$, where $\sigma_1 > 0$

(II) $|B(t,\varphi,\varphi)| \ge \sigma_1 \parallel \varphi \parallel_1^2$, where $\sigma_2 > 0$.

Now, suppose $\psi_t = \zeta$, then (6-8) can be rewritten as

$\langle\zeta_t,\varphi\rangle + B(t,\psi,\varphi) = (g(\vec{x},t),\varphi)_K + (\omega,\psi)_K - (\omega_d,\psi)_K$, $\forall\varphi \in S$, (9)

$\langle\psi_t,\varphi\rangle = \langle\zeta,\varphi\rangle$, (10)

$(\psi_0,\varphi)=(\psi^0,\varphi)$, in , (11)

$(\zeta_0,\varphi)=(\psi^1,\varphi)$, in K (12)

2. The DCCOCP[9]: The CCOCP is discretized by using the GFEM as follows:

First, the region K can be divided into subregions (a polyhedron) for every integer (s), $Z_i^s, i = 1,\dots,n$ be an admissible regular triangulation of $\bar{K}$, i.e. $\bar{K}=\cup_{i=1}^{n} Z_i^s$. Second, let $E_j^s=[t_j^s,t_{j+1}^s]$ be a subdivision of the interval E and for $j=0,1,\dots,m-1$, where each interval has same lengths ($\Delta t=\frac{T}{m}$). Let $S_s \subset S = H_0^1(K)$ be the space of continuous piecewise affine mapping (CPAM) in K. The set of admissible discrete classical controls (DCC) is

$W_{ad}^s=\{\bar{\omega}=\bar{\omega}^s \in W_{ad} \,|\, \bar{\omega}(\vec{x},t)=\bar{\omega}_{ij} \in U^s \; in \; \rho_{ij}\}$, where $\rho_{ij}:=Z_i^s \times E_j^s$.

Now, $\forall\, \varphi \in S_s$, and for $j=0,1,\dots,m-1$, the DWF (9)-(12) can be given by

$\left(\zeta_{j+1}^s-\zeta_j^s,\varphi\right)_K + \Delta t B(\psi_{j+1}^s,\varphi) = \Delta t\left(g(t_j^s),\varphi\right)_K + \Delta t\left(\omega_j^s,\varphi\right)_K - \Delta t(\omega_d(t_j^s),\varphi)_K$, (13)

$(\psi_{j+1}^s-\psi_j^s,\varphi)_K = \Delta t(\zeta_{j+1}^s,\varphi)_K$, (14)

$(\psi_0^s,\varphi)_K=(\psi^0,\varphi)_K$, (15)

$(\zeta_0^s,\varphi)_K=(\psi^1,\varphi)_K$, (16),

where $\psi_j^s=\psi(t_j^s)$, $\zeta_j^s=\zeta(t_j^s) \in S_s$ for $j=0,1,\dots,m$, and $\psi^0 \in S$ and $\psi^1 \in L^2(K)$.

The discrete cost functional (DCF) $G_0^s(\omega^s)$ is defined by

$$G_0^s(\omega^s)=\Delta t\sum_{j=0}^{m-1}\int_K \left[\tfrac{1}{2}(\psi_{j+1}^s-\psi_d(t_{j+1}^s))^2+\tfrac{\varpi}{2}(\omega_j^s-\omega_d(t_j^s))^2\right]d\vec{x} \quad (17)$$

Hence, the DCCOCP is to find a DCOC $\bar{\omega}^s \in W_{ad}^s$, such that

$G_0^s(\bar{\omega}^s)=\min_{\omega^s \in W_{ad}^s} G_0^s(\omega^s)$

2.1 Theorem (existence and uniqueness of the DWF): The DWF (13-16) and for any fixed j, with fixed DCC $\omega^s \in W_{ad}^s$, has a unique solution $\psi_{\omega^s}^s=\psi^s=(\psi_0^s,\psi_1^s,\dots,\psi_m^s)$, for sufficiently small Δt.

Proof: To find the solution $\psi^s=(\psi_0^s,\psi_1^s,\dots,\psi_m^s)$ for any fixed j $(0 \le j \le m-1)$, let $(\varphi_i(\vec{x})$, $(i=1,\dots,n)$ are CPAM in K, with $\varphi_i(\vec{x})=0$ on ∂K) being a span of S_s . Then for any $i=1,\dots,n$ and $\psi_j^s, \zeta_j^s, \psi_{j+1}^s, \zeta_{j+1}^s \in S_s$, equations (13-16) can be formed as:

$\left(\zeta_{j+1}^s-\zeta_j^s,\varphi_i\right)_K + \Delta t\, B(\psi_{j+1}^s,\varphi_i) = \Delta t\left(g(t_j^s),\varphi_i\right)_K + \Delta t\left(\omega_j^s,\varphi_i\right)_K - \Delta t(\omega_d(t_j^s),\varphi_i)_K$, (18)

$(\psi_{j+1}^s-\psi_j^s,\varphi_i)_K=\Delta t(\zeta_{j+1}^s,\varphi_i)_K$, (19)

$(\psi_0^s,\varphi_i)_K=(\psi^0,\varphi_i)_K$, (20)

$(\zeta_0^s,\varphi_i)_K=(\psi^1,\varphi_i)_K$, (21)

Substituting equation (19) in equation (18) yields:

$$\left(\psi_{j+1}^s,\varphi_i\right)_K+(\Delta t)^2 B(\psi_{j+1}^s,\varphi_i)$$
$$=\left(\psi_j^s,\varphi_i\right)_K+\Delta t\left(\zeta_j^s,\varphi_i\right)_K+(\Delta t)^2\left(g(t_j^s),\varphi_i\right)_K+(\Delta t)^2\left(\omega_j^s,\varphi_i\right)_K-(\Delta t)^2\left(\omega_d(t_j^s),\varphi_i\right)_K. \quad (22)$$

Now by using the GFEM, we obtain

$$\psi_0^s=\sum_{k=1}^{n}v_k^0\varphi_k,\ \psi_j^s=\sum_{k=1}^{n}v_k^j\varphi_k,\ \psi_{j+1}^s=\sum_{k=1}^{n}v_k^{j+1}\varphi_k,\ \zeta_0^s=\sum_{k=1}^{n}w_k^0\varphi_k,\ \zeta_j^s=\sum_{k=1}^{n}w_k^j\varphi_k$$

and $\zeta_{j+1}^s=\sum_{k=1}^{n}w_k^{j+1}\varphi_k$,

where $v_k^j=v_k(t_k^j)$ and $w_k^j=w_k(t_k^j)$ are unknown constants, and $\forall j=0,1,\dots,m$ to be determined.

Substituting ψ_0^s , ψ_j^s , ψ_{j+1}^s , ζ_0^s, ζ_j^s and ζ_{j+1}^s into equations ((19)-(22)) yields the following linear algebraic system (LAGS) of 1st ODEs for $(j = 0,1, \dots, m-1)$

$$(D + (\Delta t)^2 F)v^{j+1} = Dv^j + \Delta t\, D\, w^j + (\Delta t)^2 \vec{b}_1(t_j^s) + (\Delta t)^2 \vec{b}_2(t_j^s) , \quad (23)$$

$$w^{j+1} = \frac{v^{j+1} - v^j}{\Delta t} , \quad (24)$$

$$Dv^0 = e^0 , \quad (25)$$

$$Dw^0 = e^1, \quad (26)$$

where $D = (a_{ik})_{n\times n}$, $a_{ik} = (\psi_i, \psi_k)$, $F = (B_{ik})_{n\times n}$,
$B_{ik} = B(t, \varphi_k, \varphi_i)$, $v_{n\times 1}^{j+l} = (v_1^{j+l}, v_2^{j+l}, \dots, v_n^{j+l})^T$, $w_{n\times 1}^{j+l} = (w_1^{j+l}, w_2^{j+l}, \dots, w_n^{j+l})^T$, (for $l = 0,1$),
$e^0 = (e_i^0)_{n\times 1}$, $e^1 = (e_i^1)_{n\times 1}$,$e_i^0 = (\psi^0, \varphi_i)$, $e_i^1 = (\psi^1, \varphi_i)$, $b_{1i} = (g(t_j), \varphi_i)$,$\vec{b}_2 = (b_{2i})_{n\times 1}$,
$b_{2i} = (\omega_j^s - \omega_d(t_j^s), \varphi_i)$, $\forall\, i, k = 1, \dots, n$.

From the Assumption A on the operator $B(.,.)$, the matrices D and F are positive definite (PD). Similarly for $(D + (\Delta t)^2 F)$, equations (23)-(26) has a unique solution.

3. Existence of the DCCOCP: The following assumptions are useful to study the existence of the discrete control.

Assumptions B: The cost functional is of **Carathéodory** type, and satisfies (for each$j = 0,1, \dots, m-1$)

$$\left|\frac{1}{2}\left(\psi_{j+1}^s - \psi_d(t_{j+1}^s)\right)^2 + \frac{\varpi}{2}\left(\omega_j^s - \omega_d(t_j^s)\right)^2\right| \le \gamma_j^s(\vec{x}) + \theta_j(\psi_j^s)^2 ,$$

where $\gamma_j^s(\vec{x}) = \gamma^s(\vec{x}, t_j) \in L^2(\mathrm{K})$ and $\theta_j \ge 0$**.**

3.1 Theorem: The operator $\omega^s \mapsto \psi^s = \psi_{\omega^s}^s$ is continuous on $L^2(\mathrm{K})$.

Proof: Let $\omega^s = (\omega_0^s, \omega_1^s, \dots, \omega_{m-1}^s)$, $\omega^{sr} = (\omega_0^{sr}, \omega_1^{sr}, \dots, \omega_{m-1}^{sr})$, $\psi^s = (\psi_0^s, \psi_1^s, \dots, \psi_{m-1}^s)$,
$\psi^{sr} = (\psi_0^{sr}, \psi_1^{sr}, \dots, \psi_{m-1}^{sr})$, $\zeta^s = (\zeta_0^s, \zeta_1^s, \dots, \zeta_{m-1}^s)$, and $\zeta^{sr} = (\zeta_0^{sr}, \zeta_1^{sr}, \dots, \zeta_{m-1}^{sr})$.

To prove that $\psi_{\omega^{sr}}^s = \psi^{sr} \longrightarrow \psi^s = \psi_{\omega^s}^s$ as $r \longrightarrow \infty$, if $\omega^{sr} \longrightarrow \omega^s$, the mathematical induction is used. Firstly, from the projection theory and equations (20) and (21), one can have
$\psi_0^{sr} \longrightarrow \psi_0^s$, and $\zeta_0^{sr} \longrightarrow \zeta_0^s$, as $r \longrightarrow \infty$.

For any fixed j, suppose $\psi_j^{sr} \longrightarrow \psi_j^s$ and $\zeta_j^{sr} \longrightarrow \zeta_j^s$ as $r \longrightarrow \infty$, we want to prove that $\psi_{j+1}^{sr} \longrightarrow \psi_{j+1}^s$ as $r \longrightarrow \infty$.

Suppose that $\psi_{j+1}^s = L(\psi_j^s , \zeta_j^s, \omega_j^s)$ and $\psi_{j+1}^{sr} = L(\psi_j^{sr} , \zeta_j^{sr}, \omega_j^{sr})$, then

$\| \psi_{j+1}^{sr} - \psi_{j+1}^s \|_{\mathrm{K}} = \| L(\psi_j^{sr} , \zeta_j^{sr}, \omega_j^{rn}) - L(\psi_j^s , \zeta_j^s, \omega_j^s) \|_{\mathrm{K}} = \| \psi_{j+1}^s - \psi_{j+1}^s \|_{\mathrm{K}} = 0$,

therefore $\psi_j^{sr} \longrightarrow \psi_j^s$, for any fixed j, thus the operator $\omega^s \mapsto \psi^s = \psi_{\omega^s}^s$ is continuous.

3.2 Lemma: If the DCCs ω^s ,$\overline{\omega}^s$ are bounded in $L^2(\rho)$, and the corresponding discrete state solutions to the DCCs ω_j^s and $\overline{\omega}_j^s = \omega_j^s + \Delta\, \omega_j^s$ are ψ_j^s , $\overline{\psi}_j^s = \psi_j^s + \Delta\, \psi_j^s$, respectively, then $(\forall\, j = 0,1, \dots, m)$:

$\| \Delta\psi_{\hbar}^s \|_1^2 \le c \| \Delta\, \omega^s \|_\rho^2$ and $\| \Delta\zeta_{\hbar}^s \|_{\mathrm{K}}^2 \le c \| \Delta\, \omega^s \|_\rho^2$

Or $\| \Delta\psi_{\hbar}^s \|_1^2 \le \bar{c}$, and $\| \Delta\zeta_{\hbar}^s \|_{\mathrm{K}}^2 \le \bar{c}$ (27)

Proof: From the DWF of equations (13)-(16), and for $j = 0,1, \dots, m-1$, we have

$$\left(\Delta\, \zeta_{j+1}^s - \Delta\, \zeta_j^s, \varphi\right)_{\mathrm{K}} + \Delta t\, B\left(\Delta\, \psi_{j+1}^s, \varphi\right) = \Delta \mathrm{t}\left(\Delta\, \omega_j^s, \varphi\right)_{\mathrm{K}} , \quad (28)$$

$$(\Delta\, \psi_{j+1}^s - \Delta\, \psi_j^s, \varphi)_K = \Delta t (\Delta\zeta_{j+1}^s, \varphi)_K , \quad (29)$$

$$\Delta\psi_0^s = \Delta\, \zeta_0^s = 0 , \quad (30)$$

By substituting $\varphi = \Delta\, \zeta_{j+1}^s$ into equation (38), we obtain

$$\| \Delta\, \zeta_{j+1}^s \|_{\mathrm{K}}^2 - \| \Delta\, \zeta_j^s \|_{\mathrm{K}}^2 + \| \Delta\, \zeta_{j+1}^s - \Delta\, \zeta_j^s \|_{\mathrm{K}}^2 + 2\Delta t\, B\left(\Delta\, \psi_{j+1}^s, \Delta\, \zeta_{j+1}^s\right)$$
$$\le \Delta t \| \Delta\, \zeta_{j+1}^s \|_{\mathrm{K}}^2 + \Delta t \| \Delta\, \omega_j^s \|_{\mathrm{K}}^2 \quad (31)$$

Since

$B\left(\Delta\, \psi_{j+1}^s - \Delta\, \psi_j^s, \Delta\, \psi_{j+1}^s - \Delta\, \psi_j^s\right) = (\Delta t)^2 B\left(\Delta\, \zeta_{j+1}^s, \Delta\, \zeta_{j+1}^s\right)$, and

and

$B\left(\Delta\, \psi_{j+1}^s, \Delta\, \psi_{j+1}^s\right) - B\left(\Delta\, \psi_j^s, \Delta\, \psi_j^s\right) = -(\Delta t)^2 B\left(\Delta\, \zeta_{j+1}^s, \Delta\, \zeta_{j+1}^s\right) + 2\Delta t\, B\left(\Delta\, \psi_{j+1}^s, \Delta\, \zeta_{j+1}^s\right)$.

then

$$2\Delta t\, B\left(\Delta \psi_{j+1}^{s}, \Delta \zeta_{j+1}^{s}\right) = [B\left(\Delta \psi_{j+1}^{s}, \Delta \psi_{j+1}^{s}\right) - B\left(\Delta \psi_{j}^{s}, \Delta \psi_{j}^{s}\right) + B\left(\Delta \psi_{j+1}^{s} - \Delta \psi_{j}^{s}, \Delta \psi_{j+1}^{s} - \Delta \psi_{j}^{s}\right)]\,. \quad (32)$$

By putting equation (32) into the LHS of equation (31), taking the summation for both sides of the obtained equation from $j = 0$ to $j = \hbar - 1$, then using equation (30), and then applying assumption B-I on $B(.,.)$, we get

$$\| \Delta \zeta_{\hbar}^{s} \|_{\mathrm{K}}^{2} + \sum_{j=0}^{\hbar-1} \| \Delta \zeta_{j+1}^{s} - \Delta \zeta_{j}^{s} \|_{\mathrm{K}}^{2} + \sigma_2 \| \Delta \zeta_{\hbar}^{s} \|_{1}^{2} + \sigma_2 \sum_{j=0}^{\hbar-1} \| \Delta \psi_{j+1}^{s} - \Delta \psi_{j}^{s} \|_{1}^{2} \leq \Delta t \sum_{j=0}^{\hbar-1} \| \Delta \zeta_{j+1}^{s} \|_{\mathrm{K}}^{2} + \Delta t \sum_{j=0}^{\hbar-1} \| \Delta \omega_{j}^{s} \|_{\mathrm{K}}^{2}\,. \quad (33)$$

Since

$$\| \Delta \zeta_{j+1}^{s} \|_{\mathrm{K}}^{2} \leq 2 \| \Delta \zeta_{j+1}^{s} - \Delta \zeta_{j}^{s} \|_{\mathrm{K}}^{2} + 2 \| \Delta \zeta_{j}^{s} \|_{\mathrm{K}}^{2}. \quad (34)$$

Substituting equation (34) into equation (33) gives

$$\| \Delta \zeta_{\hbar}^{s} \|_{\mathrm{K}}^{2} + \left(1 - \bar{\bar{C}}\Delta t\right) \sum_{j=0}^{\hbar-1} \| \Delta \zeta_{j+1}^{s} - \Delta \zeta_{j}^{s} \|_{\mathrm{K}}^{2} + \sigma_2 \| \Delta \psi_{\hbar}^{s} \|_{1}^{2} + \sigma_2 \sum_{j=0}^{\hbar-1} \| \Delta \psi_{j+1}^{s} - \Delta \psi_{j}^{s} \|_{1}^{2}$$
$$\leq \Delta t \sum_{j=0}^{\hbar-1} \| \Delta \omega_{j}^{s} \|_{\mathrm{K}}^{2} + \Delta t \bar{\bar{C}} \sum_{j=0}^{\hbar-1} (\| \Delta \zeta_{j}^{s} \|_{\mathrm{K}}^{2}) \leq \| \Delta \omega^{s} \|_{\rho}^{2} + \Delta t \bar{\bar{C}} \sum_{j=0}^{\hbar-1} (\| \Delta \zeta_{j}^{s} \|_{\mathrm{K}}^{2}). \quad (35)$$

Now, by choosing $\Delta t < \frac{1}{\bar{\bar{C}}}$, the second and the fourth terms in the LHS of equation (35) are positive. Applying $\bar{\vartheta} = \max(1, \sigma_2)$, yields

$$\bar{\vartheta}(\| \Delta \zeta_{\hbar}^{s} \|_{\mathrm{K}}^{2} + \| \Delta \psi_{\hbar}^{s} \|_{1}^{2}) \leq \| \Delta \zeta_{\hbar}^{s} \|_{\mathrm{K}}^{2} + \sigma_2 \| \Delta \psi_{\hbar}^{s} \|_{1}^{2} \leq \| \Delta \omega_{j}^{s} \|_{\rho}^{2} + \Delta t \bar{\bar{C}} \sum_{j=0}^{\hbar-1} (\| \Delta \zeta_{j}^{s} \|_{\mathrm{K}}^{2}). \quad (36)$$

Applying $\check{h} = 1/\bar{\vartheta}$ and $\acute{C} = \bar{\bar{C}}/\bar{\vartheta}$ into equation (36) produces

$$\| \Delta \zeta_{\hbar}^{s} \|_{\mathrm{K}}^{2} + \| \Delta \psi_{\hbar}^{s} \|_{1}^{2} \leq \check{h} \| \Delta \omega^{s} \|_{\rho}^{2} + \acute{C}\Delta t \sum_{j=0}^{\hbar-1} (\| \Delta \zeta_{j}^{s} \|_{\mathrm{K}}^{2} + \| \Delta \psi_{j}^{s} \|_{1}^{2}) \quad (37)$$

Using the discrete Grownwall's inequality (DGI) on equation (37) gives
$\| \Delta \zeta_{\hbar}^{s} \|_{\mathrm{K}}^{2} + \| \Delta \psi_{\hbar}^{s} \|_{1}^{2} \leq \bar{\kappa}\check{h} \| \Delta \omega^{s} \|_{\rho}^{2}$, which gives

$$\| \Delta \zeta_{\hbar}^{s} \|_{\mathrm{K}}^{2} \leq c \| \Delta \omega^{s} \|_{\rho}^{2}, \text{ and } \| \Delta \psi_{\hbar}^{s} \|_{1}^{2} \leq c \| \Delta \omega^{s} \|_{\rho}^{2}\,. \quad (38)$$

Finally, since ω^{s} and $\bar{\varpi}^{s}$ are bounded in $L^2(\rho)$, equation (27) is satisfied from equation (38).

3.2 Theorem [10]: Suppose that $W_{ad}^{s} \neq \emptyset$ is convex and compact, and if $G_0^{s}(\omega^{s})$ is coercive, there exists a DCOC.

Proof: By using the same technique used in (Theorem 4 in [10]).

4. The Necessary conditions for DCCOCP

The following theorem deals with the state and proof for the necessary conditions of the DCCOCP

4.1 Theorem: Assume that DCF of equation (17) is given and the DAWF (for the state equation) $\eta_{\omega^s}^{s} = \eta^{s} = (\eta_0^{s}, \eta_1^{s}, \dots, \eta_{m-1}^{s})$ is given (for $j = m-1, m-2, \dots, 0$) by

$$\left(\phi_{j+1}^{s} - \phi_{j}^{s}, \varphi\right)_{K} + \Delta t\, B\left(\eta_{j}^{s}, \varphi\right) = \Delta \mathrm{t} \left(\psi_{j+1}^{s} - \psi_{d}(t_{j+1}^{s}), \varphi\right)_{\mathrm{K}} \quad (39)$$

$$\eta_{j+1}^{s} - \eta_{j}^{s} = \Delta \mathrm{t}\, \phi_{j}^{s} \quad (40)$$

$$\eta_{m}^{s} = \phi_{m}^{s} = 0 \quad (41)$$

where $\eta_j^{s}, \phi_j^{s} \in \mathrm{S}_s$ ($\forall\, j = 0,1, \dots, m$).

Then the Fréchet derivative (FD) of DCF can be written as

$$(\mathrm{D}G_0^{s}(\omega^{s}), \omega'^{s} - \omega^{s}) = \Delta t \sum_{j=0}^{m-1} \left(H_{\omega_j^{s}}^{s}(t_j^{s}, \psi_{j+1}^{s}, \eta_j^{s}, \omega_j^{s}), \Delta\omega_j^{s}\right)_{\mathrm{K}}$$
$$= \Delta t \sum_{j=0}^{m-1} (\eta_j^{s} + \varpi(\omega_j^{s} - \omega_d(t_j^{s})), \Delta\omega_j^{s})_{\mathrm{K}}\,, \quad (42)$$

where $\omega'^{s}, \omega^{s} \in W_{ad}^{s}$, $\Delta\omega_j^{s} = \omega'^{s} - \omega^{s}$ for $(j = 0,1, \dots, m)$, and H^{s} is called the Hamiltonian.

Proof: By using equation (31) and set $\varphi = \eta_j^{s}$, then summing over j (for $j = 0$ to $j = m-1$), we get

$$\Delta t \sum_{j=0}^{m-1} \frac{\left(\Delta \zeta_{j+1}^{s} - \Delta \zeta_{j}^{s}, \eta_j^{s}\right)_{\mathrm{K}}}{\Delta t} + \Delta t \sum_{j=0}^{m-1} B\left(\Delta \psi_{j+1}^{s}, \eta_j^{s}\right) = \Delta \mathrm{t} \sum_{j=0}^{m-1} \left(\Delta \omega_j^{s}, \eta_j^{s}\right)_{\mathrm{K}}\,. \quad (43)$$

By setting $\varphi = \Delta\,\psi_{j+1}^{s}$ in equation (39), and summing over j (for $j = 0$ to $j = m-1$), we obtain

$$\Delta t \sum_{j=0}^{m-1} \frac{\left(\phi_{j+1}^{s} - \phi_{j}^{s}, \Delta\,\psi_{j+1}^{s}\right)_{\mathrm{K}}}{\Delta t} + \Delta t \sum_{j=0}^{m-1} B\left(\eta_{j}^{s}, \Delta\,\psi_{j+1}^{s}\right) = \Delta t \sum_{j=0}^{m-1} \left(\psi_{j+1}^{s} - \psi_{d}(t_{j+1}^{s}), \Delta\,\psi_{j+1}^{s}\right)_{\mathrm{K}}, \quad (44)$$

then by subtracting equation (43) from equation (44), we get

$$\Delta t \sum_{j=0}^{m-1} \frac{\left(\Delta\,\zeta_{j+1}^{s} - \Delta\,\zeta_{j}^{s}, \eta_{j}^{s}\right)_{\mathrm{K}}}{\Delta t} - \Delta t \sum_{j=0}^{m-1} \frac{\left(\phi_{j+1}^{s} - \phi_{j}^{s}, \Delta\,\psi_{j+1}^{s}\right)_{\mathrm{K}}}{\Delta t}$$
$$= \Delta t \sum_{j=0}^{m-1} \left(\Delta\,\omega_{j}^{s}, \eta_{j}^{s}\right)_{\mathrm{K}} - \Delta t \sum_{j=0}^{m-1} \left(\psi_{j+1}^{s} - \psi_{d}(t_{j+1}^{s}), \Delta\,\psi_{j+1}^{s}\right)_{\mathrm{K}}. \quad (45)$$

Now, for any given values of ψ_{j}^{s}, ($j = 0,1,\ldots,m$) in a vector space, the following functions are defined almost everywhere on E:

$\psi_{-}^{s}(t) := \psi_{j}^{r}$, $t \in E_{j}^{s}$, for each $j = 0,\ldots,m$,

$\psi_{+}^{s}(t) := \psi_{j+1}^{s}$, $t \in E_{j}^{s}$, for each $j = 0,\ldots,m-1$,

$\psi_{\wedge}^{s}\left(t_{j}^{s}\right) := \psi_{j}^{s}$, $\forall\, j = 0,1,\ldots,m$, where each function $\psi_{\wedge}^{s}\left(t_{j}^{s}\right)$ is affine on each E_{j}^{s}.

These notations are used for ψ, ζ, η and ϕ in the LHS of equation (45), to get

$$\Delta t \sum_{j=0}^{m-1} \frac{\left(\Delta\,\zeta_{j+1}^{s} - \Delta\,\zeta_{j}^{s}, \eta_{j}^{s}\right)_{\mathrm{K}}}{\Delta t} = \int_{0}^{T} \left((\Delta\,\zeta_{\wedge}^{s})', \eta_{-}^{s}\right)_{\mathrm{K}} dt, \quad (46a)$$

and

$$\Delta t \sum_{j=0}^{m-1} \frac{\left(\phi_{j+1}^{s} - \phi_{j}^{s}, \Delta\,\psi_{j+1}^{s}\right)_{\mathrm{K}}}{\Delta t} = \int_{0}^{T} \left((\phi_{\wedge}^{s})', \Delta\,\psi_{+}^{s}\right)_{\mathrm{K}} dt. \quad (46b)$$

By using the discrete integration by parts twice to the RHS of equation (46a), then using equations (32), (41), (31) and (43), we get

$$\int_{0}^{T} \left((\Delta\,\zeta_{\wedge}^{s})', \eta_{-}^{s}\right)_{\mathrm{K}} dt = -\int_{0}^{T} \left(\Delta\,\psi_{+}^{s}, (\phi_{\wedge}^{s})'\right)_{\mathrm{K}} dt + (\Delta\,\psi_{m}^{s}, \phi_{m}^{s})_{\mathrm{K}} - (\Delta\,\psi_{0}^{s}, \phi_{0}^{s})_{\mathrm{K}},$$
$$= \int_{0}^{T} \left((\phi_{\wedge}^{s})', \Delta\,\psi_{+}^{s}\right)_{\mathrm{K}} dt. \quad (47)$$

Substituting equation (47) in equation (45) gives

$$\Delta t \sum_{j=0}^{m-1} \left(\psi_{j+1}^{s} - \psi_{d}(t_{j+1}^{s}), \Delta\,\psi_{j+1}^{s}\right)_{\mathrm{K}} = \Delta \mathrm{t} \sum_{j=0}^{m-1} \left(\Delta\,\omega_{j}^{s}, \eta_{j}^{s}\right)_{\mathrm{K}}. \quad (48)$$

On the other hand, since the FD of the DCF exists, then

$$G_{0}^{s}(\omega^{s} + \Delta\omega^{s}) - G_{0}^{s}(\omega^{s})$$
$$= \Delta t \sum_{j=0}^{m-1} \left(\psi_{j+1}^{s} - \psi_{d}(t_{j+1}^{s}), \Delta\,\psi_{j+1}^{s}\right)_{\mathrm{K}} + \Delta \mathrm{t} \sum_{j=0}^{m-1} \left(\varpi(\omega_{j}^{s} - \omega_{d}(t_{j}^{s})), \Delta\,\omega_{j}^{s}\right)_{\mathrm{K}} + \varepsilon\,(\Delta\,\omega^{s})\,\|\,\Delta\,\omega^{s}\,\|_{\rho} \quad (49)$$

where $\varepsilon\,(\Delta\,\omega^{s}) \longrightarrow 0$ and $\|\,\Delta\,\omega^{s}\,\|_{\rho} \longrightarrow 0$ as $\Delta\,\omega^{s} \longrightarrow 0$.

By substituting equation (48) into equation (49), one can have

$$G_{0}^{s}(\omega^{s} + \Delta\omega^{s}) - G_{0}^{s}(\omega^{s})$$
$$= \Delta \mathrm{t} \sum_{j=0}^{m-1} \left(\Delta\,\omega_{j}^{s}, \eta_{j}^{s}\right)_{\mathrm{K}} + \Delta \mathrm{t} \sum_{j=0}^{m-1} \left(\varpi(\omega_{j}^{s} - \omega_{d}(t_{j}^{s})), \Delta\,\omega_{j}^{s}\right)_{\mathrm{K}} + \varepsilon_{3}(\Delta\,\omega^{s})\,\|\,\Delta\,\omega^{s}\,\|_{\rho}, \quad (50)$$

where $\varepsilon\,(\Delta\,\omega^{s}) \longrightarrow 0$ and $\|\,\Delta\,\omega^{s}\,\|_{\rho} \longrightarrow 0$ as $\Delta\,\omega^{s} \longrightarrow 0$.

Finally, the FD of the DCF is

$$\left(DG_{0}^{s}(\omega^{s}), \omega'^{s} - \omega^{s}\right) = \Delta t \sum_{j=0}^{m-1} (\eta_{j}^{s} + \varpi(\omega_{j}^{s} - \omega_{d}(t_{j}^{s})), \Delta\omega_{j}^{s})_{\mathrm{K}}.$$

4.1 Corollary: The inequality

$$\Delta t \sum_{j=0}^{m-1} (\eta_{j}^{s} + \varpi(\omega_{j}^{s} - \omega_{d}(t_{j}^{s})), \Delta\omega_{j}^{s})_{\mathrm{K}} \geq 0, \forall \omega_{j}^{s} \in W_{ad}^{s} \quad (51)$$

is equivalent with the minimum principle blockwise $\forall j = 0,1,\ldots m-1$

$$\left(\eta_{j}^{s} + \varpi(\omega_{j}^{s} - \omega_{d}(t_{j}^{s})), \omega_{j}^{s}\right)_{T_i} = \min_{\omega^{s} \in W_{ad}^{s}} \left(\eta_{j}^{s} + \varpi(\omega_{j}^{s} - \omega_{d}(t_{j}^{s})), \omega_{j}'^{s}\right)_{T_i} \quad (52)$$

Proof: It can be proved by using the same technique used in [9].

5. Optimization methods: The following algorithm shows the numerical calculation for the DCOC by using the mixed GFEM-IFDS with each of the methods of GM, FWM, or GPM (with ARSO and

OPSO). We will use the norm $\|\cdot\|$ with respect to the vector space Q, where the vector space $U \subset \mathrm{K}$ a is a convex subset and the functional $G: \mathrm{K} \subset Q \longrightarrow \mathbb{R}$.

5.1 ALGORITHIM: Let $b, c \in (0,1)$, $\{\delta^s\}$ be a sequence with $\delta^s \in (0,\infty)$, or $\delta^s \in (0,1]$, for each s. $\mu > 0$, and let $\omega^s \in U$ be an initial control.

Step1: Set $s := 0$.

Step 2: Solve the DWF of equations (19-22) (the DAWF of equations (41-43)) by using GFEM-IFDS to get the state ψ^s (the adjoint solution η^s). Then we calculate $G(\omega^s)$ and $DG(\omega^s)$ from equation (17) and equation (44), respectively.

Step 3: Find a new direction (new control) $u^s \in U$ (i.e. a direction $u^s - \omega^s$), by using the following methods (separately):

(a) GM: Find $u^s \in U$, such that:

$$u^s = \omega^s - \frac{1}{\mu} DG(\omega^s)$$

(b) FWM: Find $u^s \in U$, such that

$$(DG(\omega^s), u^s - \omega^s) = \min_{u \in U} (DG(\omega^s), u - \omega^s)$$

(c) GPM: Find $u^s \in U$, such that

$$\xi^s = (DG(\omega^s), u^s - \omega^s) + \frac{\mu}{2} \| u^s - \omega^s \|^2 = \min_{u \in U} (DG(\omega^s), u - \omega^s) + \frac{\mu}{2} \| u - \omega^s \|^2$$

Step 4: Solve the DWF (15-18) to find the state solution ψ^s corresponding to the new control u^s

Step 5: Calculate $\xi^s = (DG(\omega^s), u^s - \omega^s)$, ($\xi^s = -\frac{1}{\mu} \| DG(\omega^s) \|^2$, in the GM).

If $\xi^s = 0$, stop.

Step 6: Choose δ^s by using one the following methods:

ARSO: Assume the initial value $\delta^s \in [0, +\infty)$(or $\delta^s \in [0,1]$). If δ^s satisfies the inequality

$$\mathrm{X}_s(\delta^s) = G\big(\omega^s + \delta^s(u^s - \omega^s)\big) - G(\omega^s) \leq \delta^s b \xi^s,$$

we set $\delta := \delta / c$ and choose the last $\delta \in (0,\infty)$ for δ^s, which satisfies the above inequality. If not satisfied, we set $\delta := \delta c$ and choose for δ^s the first $\delta^s \in (0,\infty)$ (or $\delta^s \in (0,1]$ in GM).

OPSO: Find an $\delta^s \in [0,1]$, such that

$$(DG(\omega^s), u^s - \omega^s) = \min_{\delta^s \in [0,1].} (DG(\omega^s), u - \omega^s)$$

Step 7: Set $\omega^{s+1} = \omega^s + \delta(u^s - \omega^s)$, $s := s + 1$ and we go to step 2.

6. Numerical results for solving the DCOCP

This section contains some illustrative examples to show the activity of the methods which are given in algorithm (5.1). Mat lab software is used to achieve the solution of the methods. The GFEM is used in step (2) to find the DS ψ^s (η^s), with $n = 9$, $m = 20$, and $\Delta t = 0.05$. In the GM, GPM and FWM, the parameters are taken the values of $b = c = 0.5$ and $\mu = 0.5$.

6.1 Application 1**:** Consider the following CCOCP governed by the LHBVPVC:

$$\psi_{tt} - \frac{\partial}{\partial y}\left[(y^2+1)\frac{\partial \psi}{\partial y}\right] - \frac{\partial}{\partial z}\left[(z^2+1)\frac{\partial \psi}{\partial z}\right] + \psi = 2z sin(\pi y) cos(t)(y(z-1) + yz) +$$
$$2yz\, cos(t)\big(sin(\pi y)(z-1) + \pi y\, cos(\pi y)(z-1)\big) +$$
$$\big(2\pi z\, cos(\pi y) cos(t)(z-1) - \pi^2 zy\, sin(\pi y) cos(t)(z-1)\big)(y^2+1) + 2y\, sin(\pi y) cos(t)(z^2 +$$
$$1) + \omega - \omega_d, \quad \text{in } \rho,$$

$\psi(\vec{x}, t) = 0$, in $\partial\rho = \partial \mathrm{K} \times [0, T]$.

$\psi(\vec{x}, 0) = -yz\, sin(\pi y)(z - 1)$, and $\psi_t(\vec{x}, 0) = 0$ in K

where $E = [0,1]$, $\mathrm{K} = [0,1] \times [0,1]$, $\vec{x} = (y, z)$.

The control constraint is $U = [-2,2]$ and the cost function in equation (5) with $\varpi = 1$ is

$\psi_d(\vec{x}, t) = yz(1 - z)\sin(\pi y)\cos(t)$, $\forall (\vec{x}, t) \in \rho$, and

$$\omega_d(\vec{x}, t) = \begin{cases} -1.35 + e^t & , \text{for } 0 \leq t \leq 0.5 \\ 0.35 & , \text{for } 0.5 < t \leq 1 \end{cases}$$

with the initial control $\omega_0(\vec{x}, t) = -0.4 + (t sin(t))$, $\forall (\vec{x}, t) \in \rho$.

First, depending on the above initial control and its corresponding state, the following results are obtained according to the optimization methods with ARSO:

(I) In the GM: the optimal control and corresponding state are obtained after 12 iterations, and the results are: $G_0(\omega^s)$=1.4362e-06, ϱ_s =4.2e-03, and ε_s=2.5438e-04

where ϱ_s and ε_s are the discrete maximum errors for the state and control, respectively.

The optimal control and its corresponding state are shown at $t = 0.5$ by Figure- 1 and Figure-2.

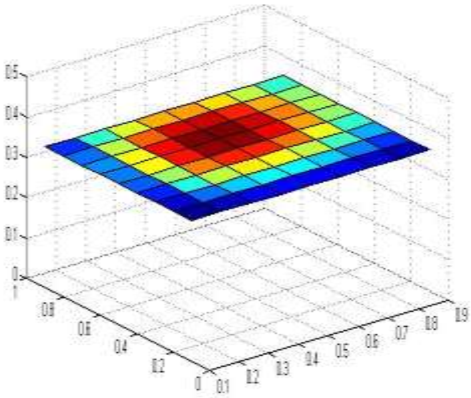

Figure 1-The optimal control at $t = 0.5$

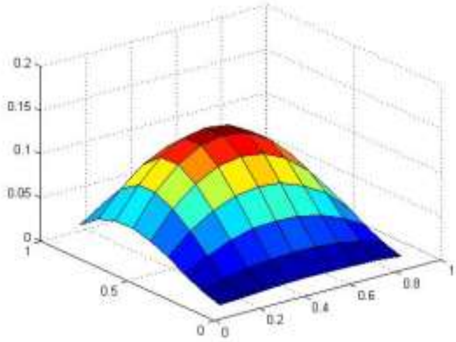

Figure 2-The corresponding state at $t = 0.5$

(II) In the FWM: the optimal control and corresponding state are shown after 121 iterations. The results are: $G_0(\omega^s)$=1.4465e-06, ϱ_s =4.2e-03, and ε_s=7.4935e-04
Figures-(3 and 4) show the optimal control and its corresponding state at $t = 0.5$.

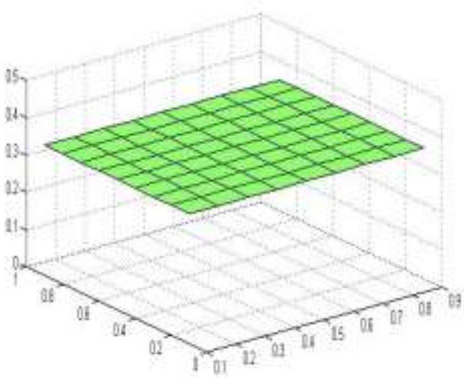

Figure 3-The optimal control at $t = 0.5$

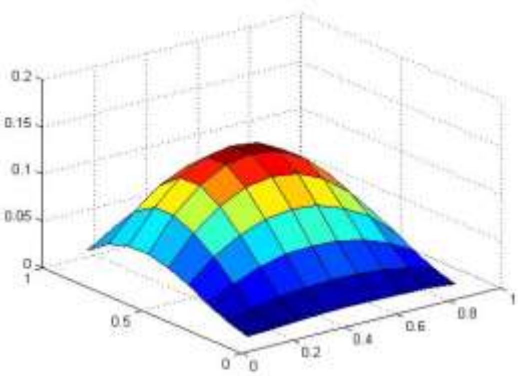

Figure 4- The corresponding state at $t = 0.5$

(III) In the GPM: the optimal control and corresponding state are obtained after *5* iterations, and the results are: $G_0(\omega^s)$=1.4339e-06, ϱ_s =4.2e-03, and ε_s=1.7788e-04.
Figures- 5 and 6 show the optimal control and its corresponding state at $\Delta t = 0.5$:

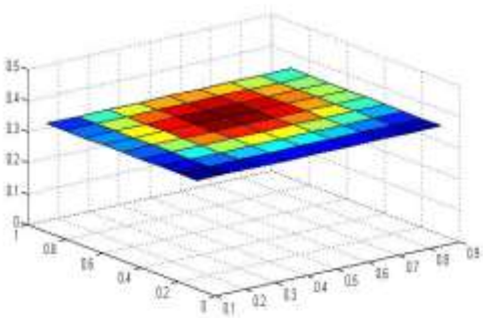

Figure 5- The optimal control at $t = 0.5$

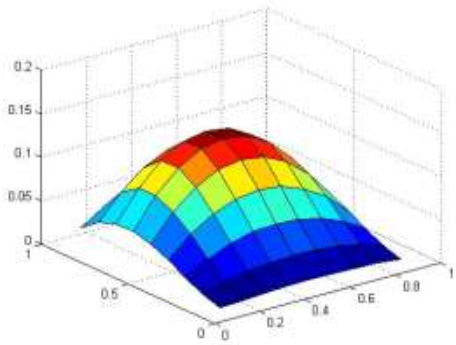

Figure 6-The corresponding state at $t = 0.5$

Second, the following results are obtained by using the optimization methods with OPSO:
(I) In the GM and GPM: the optimal control and corresponding state are given after *2* iterations, and the results in this case are: $G_0(\omega^s)$=1.4351e-06, ϱ_s =4.2e-03, and ε_s=2.1289e-04
The optimal control and its corresponding state are shown at $\Delta t = 0.5$ in figures 7 and 8.

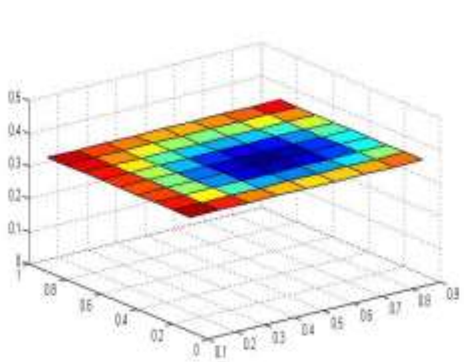

Figure 7-The optimal control at $t = 0.5$

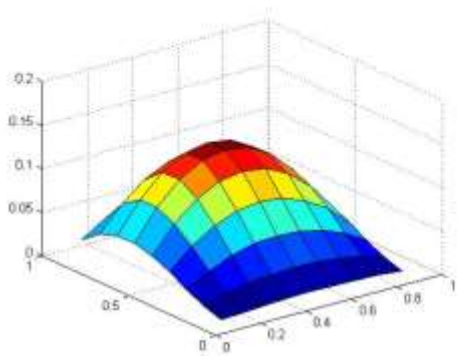

Figure 8-The corresponding state at $t = 0.5$

(II) In the FWM: the optimal control and corresponding state are given after *46* iterations, and the results are: $G_0(\omega^S)$=1.4374e-06, ϱ_s =4.2e-03, and ε_s=4.1159e-04

Figures-(9 and 10) show the optimal control and its corresponding state at $\Delta t = 0.5$:

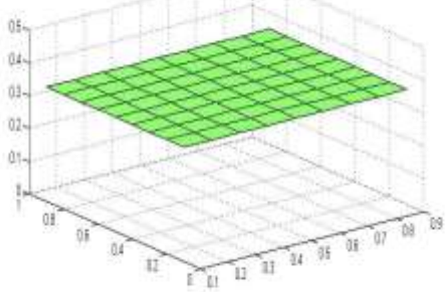

Figure 9-The optimal control at $t = 0.5$

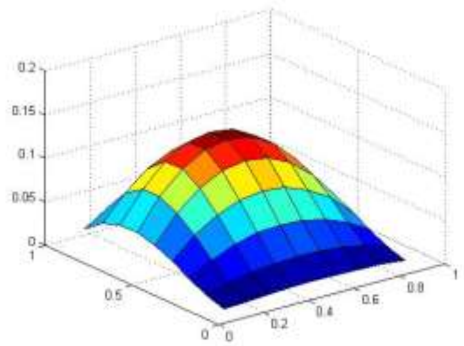

Figure 10-The corresponding state at $t = 0.5$

Application 2**:** Consider the following CCOCP governed by the LHBVPVC:

$\psi_{tt} - \frac{\partial}{\partial y}\left[(1+y+z)\frac{\partial\psi}{\partial y}\right] - \frac{\partial}{\partial z}\left[(1+y-z)\frac{\partial\psi}{\partial z}\right] + e^{-yz}\psi = \frac{1}{2}((y\,sin(\pi y)cos(t)(\tan(z)^2+1)) + (y\,sin(\pi y)cos(t)tan(z)(\tan(z)^2+1)(z-1))/2)(y-z+1) + ((\pi cos(\pi y)cos(t)tan(z)(z-1))/2 - (\pi^2 y\,sin(\pi y)cos(t)tan(z)(z-1))/4)(y+z+1) - (y\,sin(\pi y)cos(t)tan(z))/4 + (sin(\pi y)cos(t)tan(z)(z-1))/4 + (y\,sin(\pi y)cos(t)tan(z)(z-1))/4 - (y\,sin(\pi y)cos(t)(\tan(z)^2+1)(z-1))/4 + (\pi y\,cos(\pi y)cos(t)tan(z)(z-1))/4 - (y\,exp(-yz)sin(\pi y)cos(t)tan(z)(z-1))/4 + \omega - \omega_d$ in ρ, E =[0,1], K = [0,1] × [0,1] , $\vec{x} = (y,z)$

$\psi(\vec{x},t) = 0$, in $\partial\rho = \partial K \times [0,T]$.

$\psi(\vec{x},0) = -(ysin(\pi y)tan(z)(z-1))/4$, and $\psi_t(\vec{x},0) = 0$, in K

The control constraint is $U = [-1,1.5]$ and the cost function (5) with $\varpi = 1$ is

$\psi_d(\vec{x},t) = 0.25y(1-z)\sin(\pi y)\tan(z)\cos(t)$, $\forall(\vec{x},t) \in \rho$, and

$$\omega_d(\vec{x},t) = \begin{cases} 1.5 - 2e^{-t} & ,\text{for } 0 \le t \le 0.6 \\ 0.55 & ,\text{for } 0.6 < t \le 1 \end{cases}$$

with the initial control $\omega_0(\vec{x},t) = -0.8 + 2.5t^2$, $\forall(\vec{x},t) \in \rho$.

First, depending on the above initial control and its corresponding state, the following results are obtained according to the optimization methods with ARSO:

(I) In the GM: the optimal control and corresponding state are obtained after 10 iterations, and the results are: $G_0(\omega^S)$=8.3096e-06, ϱ_s =8.7e-03, and ε_s=3.6968e-04

The optimal control and its corresponding state at $t = 0.5$ are shown by Figures-(11 and 12).

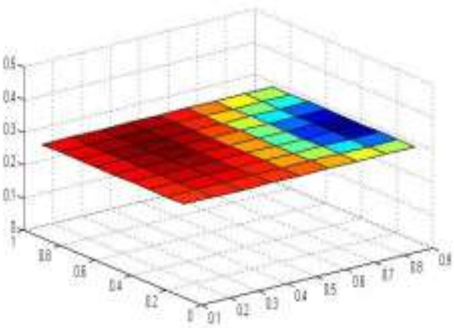

Figure 11- The optimal control at $t = 0.5$

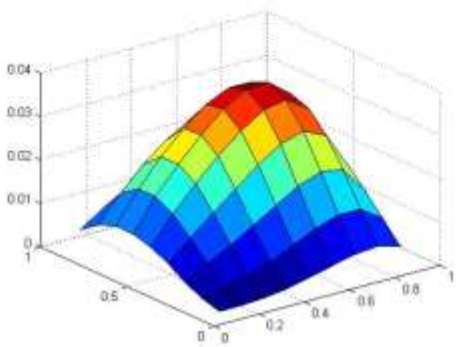

Figure 12 The corresponding state at $t = 0.5$

(II) In the FWM: the optimal control and corresponding state are obtained after 331 iterations, and the results are :$G_0(\omega^s)$=8.3219e-06, ϱ_s =8.7e-03, and ε_s=6.7630e-04
Figures 13 and 14 show the optimal control and its corresponding state.

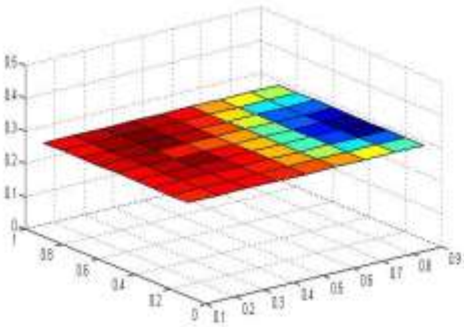

Figure 13- The optimal control at $t = 0.5$

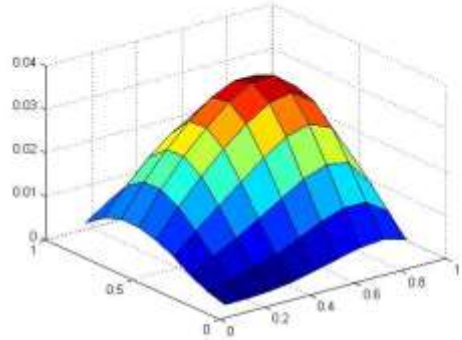

Figure 14- The corresponding state at $t = 0.5$

(III) In the GPM : the optimal control and corresponding state are obtained after 8 iterations, and the results are: $G_0(\omega^s)$=48.3079e-06, ϱ_s =8.7e-03, and ε_s=3.2433e-04
Figures 15 and 16 show the optimal control and its corresponding state.

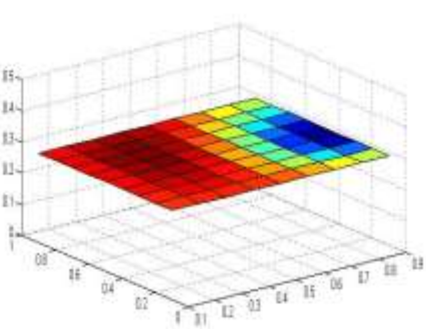

Figure 15-The optimal control at $t = 0.5$

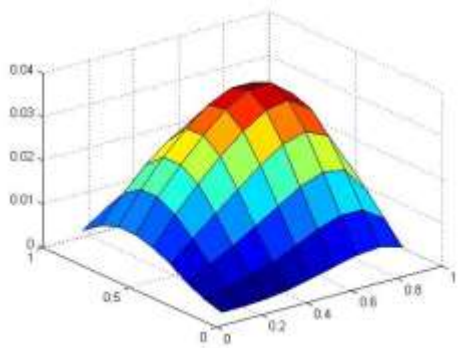

Figure 16-The corresponding state at $t = 0.5$

Second, the following results are obtained by using the optimization methods with OPSO:
(I) In the GM and GPM: the optimal control and corresponding state are given after 2 iterations, and the results in this case are: $G_0(\omega^s)$=8.3044e-06, ϱ_s =8.7e-03, and ε_s=5.2921e-04
The optimal control and its corresponding state are shown by figures 17 and 18.

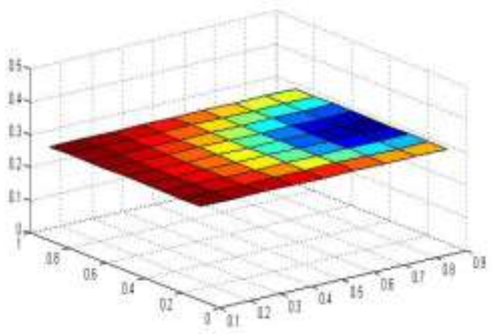

Figure 17-The optimal control at $t = 0.5$

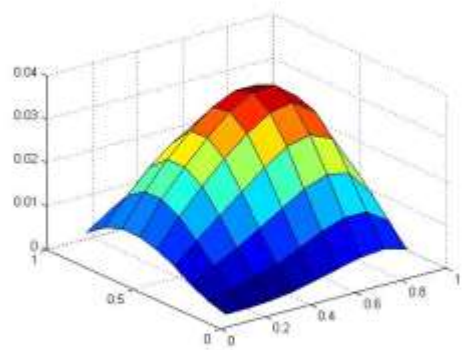

Figure 18- The corresponding state at $t = 0.5$

(II) In the FWM: the optimal control and corresponding state are given after *261* iterations, and the results are: □$_0$(□□)=8.3190e-06, □□ =8.7e-03, and □□=5.7906e-04
Figures-(19 and 20) show the optimal control and its corresponding state at □ $= 0.5$.

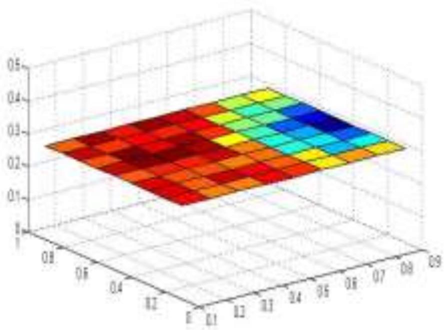

Figure 19-The optimal control at $t = 0.5$

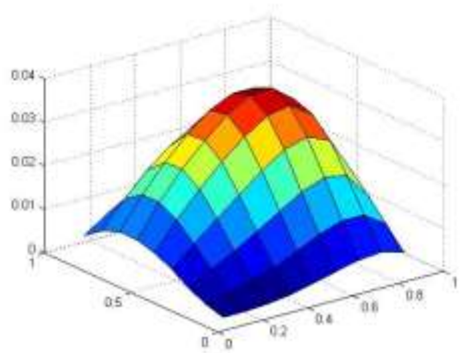

Figure 20- The corresponding state at $t = 0.5$

Conclusions

In this paper, the proof of the existence and uniqueness theorem for the DS of the DWF for the LHBVPVC is achieved. The existence theorem for the DCOC and the necessary conditions for optimality of the problem are proved under suitable assumptions. On the other hand, the DCOCP was solved numerically by using the mixed GFEM-IFDS to find the DS, the DWF and its adjoint of the DAWF, with step length of space variable $h = 0.1$ and step length of time $\Delta t = 0.05$. While the DCOC is obtained by finding the minimum of the cost function by using each one of the optimization methods of GPM, GM and FWM with either ARSO or ORSO step options with parameters ($b = 0.5$, $\mu = 0.5$ and $c = 0.5$). From the numerical solutions we concluded that; the GFEM was a suitable and fast method to solve the DWF and DAWF, beside this we saw from the results obtained using the GPM with ARSO method were better than those obtained using the GM or FWM with ARSO methods, on the other hand the results obtained using the GPM and GM with OPSO methods were better than those obtained using the FWM with the OPSO method. The OPSO method needed less or equal number of iterations than the ARSO method. This comparison happened when we had a quadratic cost function. Finally, when we had a more general function, the OPSM was not easy to be applicable, while the ARSO method can be considered as a general method to improve the direction search.

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References

1. Kunisch, K., Pieper, K. and Rund, A. **2016**. Time optimal control for a reaction diffusion system arising in cardiac electrophysiology - A monolithic approach, *ESAIM Mathematical Modelling and Numerical Analysis*, **50**(2): 381-414.
2. Yuan, W., Xiaochuan, Yang, Y. and Haijuan, C. **2016**. Optimal control of two dimensional parabolic partial differential equations with application to steel billets cooling in continuous casting secondary cooling zone, *Optimal Control Applications and methods,* **37**(6):1314-1328.
3. Chryssoverghi, I. **2007**. Mixed Discretization-Optimization Methods for Nonlinear Elliptic Optimal Control Problems, *LNCE. Springer (Heidelberg),* **4310**: 287-295.
4. Madiha, S. and Muhammad, M. **2019**. Finite Element Approximation of Optimal Control Problem with Weighted Extended B-Splines, *Mathematics*, **7**(452): 1-12
5. Chryssoverghi, I. **2006**. Discretization Methods for Similinear Parabolic Optimal Control Problems, *International Journal of Numerical and Modeling*, **3**(02): 437-457.
6. Chryssoverghi I. and Al- Hawasi, J. **2004**. Discrete Approximation of Semilinear Parabolic Optimal Control Problems, the 1st IC-SCCE international conference from scientific computing to the computational engineering, 8-10 Sept. Athens-Greece.
7. Chryssoverghi, I., Coletsos, J. and Kokkinis, B. **2009**. Classical and Relaxed Optimization methods for Nonlinear Parabolic Optimal Control Problem, the 7th international conference on Large-Scale Scientific Computing, Bulgaria, pp. 247-255 .
8. Toyoğlu, F. **2018**. On the Solution of a Optimal Control Problem for a Hyperbolic System, *Applied and Computational Mathematics,* **7**(3): 75-82.
9. Al-Hawasy, J.A. and Al-Rawdanee E.H. **2019**. Numerical Solution for Classical Optimal Control Problem Governing by Hyperbolic Partial Differential Equation via Galerkin Finite Element with Gradient and Frank Wolfe Methods. *The 2nd Inter. Conf. for Applied and Pure Math., SICAPM2019, Feb 27-28, Baghdad-Iraq.*
10. Al-Hawasy JA, Al-Rawdanee EH. **2019**. Numerical Solution for Classical Optimal Control Problem Governing by Hyperbolic Partial Differential Equation Vai Galerkin Finite Element-Implicit method with Gradient Projection Methods. *Ibn Al-Haitham Journal for Pure and Applied Science*, **32**(2): 71-80.
11. Sandily, R. and Kumar, S. **2016**. Convergence of Discontinuous Finite Volume Discretizations for a Semilinear Hyperbolic Optimal Problem, *International Journal of Numerical Analysis and Modeling*, **13**(6): 926-950.
12. Al-Rawdanee, E.H., Al-Hawasy, J.A. The Mixed of Galerkin Method with Gradient Projection, Gradient and Frank Wolfe Methods for Solving Classical Optimal Control Governing by Nonlinear Hyperbolic Boundary Value Problem, Submitted for publishing in *the Baghdad Science Journal*.
13. Chryssoverghi, I., Coletsos J. and Kokkinis B. **2006**. Discretization methods for optimal control problems with state constraints, *J. Comput. Appl. Math.*, **191**(1): 1–31.
14. Polak, E. **1997**. *Optimization*: *Algorithms and Consistent Approximations*, Springer, 1997.