Modules With Chain Conditions On δ -Small Submodules

Wasan Khalid Hasan, Sahira Mahmood Yaseen*

Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.

Abstract:

Let R be an associative ring with identity and M be unital non zero R-module. A submodule N of a module M is called a δ-small submodule of M (briefly $N <<_{\delta} M$)if N+X=M for any proper submodule X of M with M/X singular, we have X=M .

In this work,we study the modules which satisfies the ascending chain condition (a. c. c.) and descending chain condition (d. c. c.) on this kind of submodules .Then we generalize this conditions into the rings , in the last section we get same results on δ- supplement submodules and we discuss some of these results on this types of submodules.

Keywords: δ-small submodule , δ- supplement submodules, c-singuler submodule.

المقاسات التي تحقق خاصية السلسلة للمقاسات الجزئية δ الصغيرة

وسن خالد حسن و ساهره محمود ياسين*

قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصة :

لتكن R حلقة تجميعية ذات عنصر محايد وليكن M مقاسا احاديا غير صفري ايمن معرفا على R ألمقاس الجزئيN منM يقال بأنه δ صغير اذا كان N+X=M كل مقاس جزئيX منM بحيثM/X منفردا فان X=M في هذا البحث سنقوم بدراسة هذا النوع من المقاسات الجزئية والمقاسات التي تحقق خاصيتي السلسلة على المقاسات الجزئية δ صغيرة . كذالك قمنا بتعميم هذه الشروط على الحلقات وفي الجزء الاخير حصلنا على بعض النتائج عن المقاسات الجزئية δ–المكملة وتوضيح بعض نتائجها.

1.Introduction

Let R be an associative ring with identity and M is a non zero unital right R-module. A submodule of R-module A is called essential ($A \subseteq_e M$)if every non zero submodule of M has non intersection with A. M is called singular module if Z(M)=M where $Z(M)=\{x \in M:ann(x) \subseteq_e R\}$ A submodule N of a module M is called a small submodule of M , denoted by $N << M$, if $N + L \neq M$ for any proper submodule L of M [1].

In [2] Zhou introuced the definition of the concept of δ-small submodule that a submodule N of a module M is called a δ-small submodule of M (briefly $N <<_{\delta} M$)if N+X=M for any proper submodule X of M with M/X singular, we have X=M.

Let M be an R- module, a submodule X of M is called c-singuler ($X \subseteq_{C.S} M$) if is M/X singular module.An ideal I of a ring R is δ-small ideal if I is δ-small R-submodule of R.

*Email: sahira.mahmood@gmail.com

Remark1-1[3]

1-let A be submodule of R-module M if $A \subseteq_e M$ then $A \subseteq_{C.S} M$ **.**

2-Let M and N be R-modules and

$f : M \to N$ be an epimorphism if $A \subseteq_{C.S} M$ then $f(A) \subseteq_{C.S} N$.

3--Let A and B be submodules of R-moduleM if$A \subseteq_{C.S} M$ and $B \subseteq_{C.S} M$**.** then $(A \cap B) \subseteq_{C.S} M$ **.**

4- Every submodule of a singuler module is c-singuler .

Lemma 1.2 [2]: Let M be a module,

1) For submodules N,K , L of M with $K \subseteq N$ then

a) $N \ll_\delta M$ if and only if $K \ll_\delta M$ and $N/K \ll_\delta M/K$

b) $N+L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.

2) If $K \ll \delta M$ and $f : M \to N$ a homo then $f(K) \ll \delta N$.

3) If $K1 \subseteq M1 \subseteq M$, $K2 \subseteq M2 \subseteq M$, and $M=M1 \oplus M2$ then $K1 \oplus K2 \ll \delta M1 \oplus M2$ if and only if $K1 \ll \delta M$ and $K2 \ll \delta M2$.

4) Let A<B<M, If $A \ll_\delta M$ and B is a direct summand then $A \ll_\delta B$.

In [4], If N and L be submodules of a module M.N is called a δ-supplement of L in M if M=N +L and $N \cap L \ll_\delta M$.and if every submodules of M. has a δ-supplement in M.Then M is called a δ-supplement module

An R-module M is said to satisfy the ascending chain condition (a.c.c.) on small submodules. respectively descending chin condition (d.c.c.) on small submodules if every ascending descending chain of small submodules $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_{n \ldots}$ respectively $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq$ Terminates[5].

In this work,we study the modules which satisfies the ascending chain condition (a. c. c.) and descending chain condition (d. c. c.) on δ-small submodules .Then we generalize these conditions into the rings . In the last section we get some results on δ- supplement submodules and we discuss some of these results on this types of submodules.

2.Modules with chain conditions on a δ- small submodules

In this section ,we introduce the definition of module which satisfies the ascending chain condition (a. c. c.) and descending chain condition (d. c. c.) on δ-small submodules as a generalization of chain condition (a. c. c.) and descending chain condition (d. c. c.) on small submodules [5] and we study the relation btween the ring that satisfies (a. c. c.) and descending chain condition (d. c. c.) on δ-small ideals..

Definition (2.1): An R-module M is said to be satisfis the ascending chain condition (a.c.c.) on δ-small submodules. respectively descending chin condition (d.c.c.) on δ-small submodules if every ascending (descending) chain of δ-small submodules $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_{n \ldots}$ respectively $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq \ldots$terminates.

Since every small submodule is δ- small submodule, The following is clear

Remark (2.2): If M satisfy the a.c.c.(d.c.c.) on δ-small submodules then M satisfy the a.c.c.(d.c.c.) on small submodules.

Proposition (2.3): Let M_1 and M_2 be two R-modules and $R=annM_1+annM_2$ .Then $M_1 \oplus M_2$ satisfies a.c.c.(d.c.c.) on δ-small submodules iff M_1 and M_2 satisfies a.c.c.(d.c.c.) on δ-small submodules.

Proof : Since R=annM_1+annM_2 , let $N_1 \oplus K_1 \subseteq N_2 \oplus K_2 \subseteq N_3 \oplus K_3 \subseteq \ldots \subseteq N_n \oplus K_n \ldots$ be ascending chain on δ-small submodules of $M_1 \oplus M_2$ hence, $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_n \ldots$ is ascending chain on δ-small submodules of M_1 and $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_n \ldots$ be ascending chain on δ-small submodules of M_2 . Since M_1 and M_2 satisfies a.c.c. on δ-small submodules then $\exists$ t , r $\in Z^+$ such that $N_t = N_{t+i}$ $= \ldots \forall$ i =1,2,3 ,…and $K_r = K_{r+i}$ $\forall$ i =1,2,3 ,…take s=max{t,r } ,hence $N_s + K_s = N_{s+i} + K_{s+i} = \forall$ i =1,2,3 ,

Conversly let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_n \ldots$ be ascending chain on δ-small submodules of M_1 then $N_1 \oplus \{0\} \subseteq N_2 \oplus \{0\} \subseteq N_3 \oplus \{0\} \subseteq \ldots \subseteq N_n \oplus \{0\} \ldots$ is an ascending chain of δ-small submodules of $M_1 \oplus M_2$, then $\exists$ m $\in Z^+$ such that $N_m \oplus \{0\} = N_{m+i} \oplus \{0\}$ $\forall$ i =1,2,3 then $N_m = N_{m+i}$ $\forall$ i

A similar proof for d.c. c.

Recall that an R-module M is called multiplication if M=MI for some ideal I of R .The following proposition gives a relation beteen δ- small ideals and δ- small submodules of a a finitely generated faithful multiplication modules.

Proposition (2.4): Let M be a finitely generated faithful multiplication R- module, and let N= M I, for some ideal I of R then N is δ-small submodule in M iff I is δ-small ideal in R .

Proof : Assume N is δ-small in M , and N = M I , let I + J = R , for some c-singler ideal J of R . then M I + M J = M R = M . then M=N+MJ,since J is c-singuler ideal in R then J $\subseteq_e$ R[3,P.32] and by [6,prop.1.5] MJ $\subseteq_e$ M then MJ $\subseteq_{C.S}$ M and since N is δ-small in M , then MJ=M=RM then J=R [7].

Conversely, Let N+K=M for some c-singler submodule K of M,Since M multiplication R- module , then K=MJ, for some ideal Jof R [6] Hence N + K = M I + M J =M (I + J) = M But M is a finitely generated faithful multiplication R- module , then I + J = R, Since K=MJ $\subseteq_{C.S}$ M I then M/MJ singular module. Let $\bar{x} \in$ M/MJ , $\bar{x} \neq$ MJ i.e $\bar{x}$ is non zero then $\bar{x}$L = $\bar{0}$ for some L large ideal in R then (x+MJ)L = MJ then xL⊆MJ If xL=0 then L ⊆ annM=0(M is faithful) then L=0 which is a contraduction since L $\subseteq_e$ R then xL≠ 0. and xL⊆ xR≠ 0. , xL⊆MJ xR∩MJ ,hence MJ $\subseteq_e$ M then J $\subseteq_e$ R [6,Prop1.5.] ,thus J $\subseteq_{C.S}$ R[3,.p.32] thenJ=R ,J is δ-small ideal in R . thus MJ=MR=M and then N is δ-small submodule in M.

From the proof of Prop.2.4, we get the following corollary.

Corollary (2.5).Let M be a finitely generated faithful multiplication R- module, and let N= M I, for some ideal I of R then N $\subseteq_{C.S}$ M iff I $\subseteq_{C.S}$ R .

Corollary (2.6) Let M be a finitely generated faithful multiplication R-module, then R satisfies a. c. c. on c-singuler ideal if and only if M satisfies a. c. c. on c-singuler submodules .

Proof: $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots$ be an ascending chain of c-singuler ideals in R then by Corollary 2.5 $M I_1 \subseteq M I_2 \subseteq M I_3 \subseteq \ldots \subseteq M I_k \subseteq \ldots$ is an ascending chain of c-singuler submodules of M . Since M satisfies a. c. c. on c-singuler submodules then then $\exists$ K $\in$ N , , such that $M I_k = M I_{k+1} = \ldots$ But M is a finitely generated faithful module , then $I_k = I_{k+1} = \ldots \forall$ k=1,2,

Conversely , Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots$ be an ascending chain of c-singuler submodule of M . Since M is a multiplication R-module, then $N_i = I_i M$, for some ideal I_i of R for all i . Hence $M I_1 \subseteq M I_2 \subseteq M I_3 \subseteq \ldots \subseteq MI_k \subseteq \ldots$ But M is finitely generated then by Corollary 2.5

$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots$ is an ascending chain of c-singuler ideals in R . Since R satisfies a.c.c on c-singuler ideal , then $\exists$ K $\in$ N , such that $I_k = I_{k+1} = \ldots$, hence M I_k = M I_{k+1} = ……which implies $N_k = N_{k+1} = \ldots$, that is M satisfies a. c. c. on c-singuler submodule of M .

The following results are sequences of this proposition.

Corollary (2.7): Let M be a finitely generated faithful multiplication R-module, then R satisfies a. c. c.(d.c. c.) on δ-small ideal if and only if M satisfies a. c. c.(d.c. c.) on δ-small submodules .

Proof : Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots$ be an ascending chain of δ-small submodule of M . Since M is a multiplication R-module, then $N_i = I_i M$, for some ideal I_i of R for all i .

Hence $M I_1 \subseteq M I_2 \subseteq M I_3 \subseteq \ldots \subseteq M I_k \subseteq \ldots$ But M is finitely generated then by proposition (2.4) $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots$ is an ascending chain of δ-small ideals in R . Since R satisfies a.c.c on δ-small ideal , then $\exists$ K $\in$ N , such that $I_k = I_{k+1} = \ldots$, hence M I_k = M I_{k+1} = ……which implies $N_k = N_{k+1} = \ldots$, that is M satisfies a. c. c. on δ-small submodules. Conversely , let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots$ be an ascending chain of δ-small ideals in R , then by Proposition (2.4) $M I_1 \subseteq M I_2 \subseteq M I_3 \subseteq \ldots \subseteq M I_k \subseteq \ldots$ is an ascending chain of δ-small submodule of M .Since M satisfies a. c. c. on δ-small submodules then then $\exists$ K $\in$ N , , such that M I_k = M I_{k+1} = …. But M is a finitely generated faithful module then

$I_k = I_{k+1} = \ldots$ [7] . Thus R satisfies a. c. c. on δ-small ideals of R .

Proposition (2.8): Let M be an R-module, satisfies a. c. c. on δ-small submodules .and A is δ-small submodule of M then $\frac{M}{A}$ satisfies a. c. c. on δ-small submodules of M

Proof: Let $\frac{A_1}{A} \subseteq \frac{A_2}{A} \subseteq \ldots$be a. c. c. on δ-small submodules of $\frac{M}{A}$ then $A_1 \subseteq A_2 \subseteq \ldots$ But A is δ-small submodule and $\frac{A_i}{A} \ll_\delta \frac{M}{A}$ then $A_i \ll_\delta M$ $\forall$ I [Lemma 1.2] thus $A_1 \subseteq A_2 \subseteq \ldots$ is an ascending chain of δ-small submodule of M . $\exists$ K $\in$ N , , such that $A_n = A_{n+1} = \ldots$thus $\frac{M}{A}$ satisfies a. c. c. on δ-small submodules Similar proof for (d.c.c.).Hence we have the following result :

Theorem(2.9) : Let M be a finitely generated faithful multiplication R-module, then the following are equivalent.

1) M satisfies a.c.c (d.c.c) on δ-small submodules

2) R satisfies a.c.c (d.c.c) on δ-small ideals .

3) S =End $_R$ (M) satisfies a.c.c (d.c.c) on δ-small ideals .

4) M satisfies a.c.c (d.c.c) on δ-small submodules as S- module.

Proof : (1) $\Rightarrow$ (2) By Cor (2.7)

(2) $\Rightarrow$ (3)since M is a finitely generated faithful multiplication R-module, then R $\approx$ S hence R satisfies a.c.c(d.c.c) S =End $_R$ (M) satisfies a.c.c (d.c.c) on δ-small ideals .

(3) $\Rightarrow$ (4) By Cor (2.7)

(4) $\Rightarrow$ (1) By Cor (2.7) R satisfies a.c.c (d.c.c) on δ-small ideals . R $\approx$ S[7] hence R satisfies a.c.c(d.c.c) on δ-small ideals and by cor (2.7) M satisfies a.c.c (d.c.c) on δ-small submodules .

3.Modules with chain conditions on δ- supplement submodule

It is known that Rad(M) is the sum of all small submodules of M . In [2]Zhou introduced the δ(M)as a generalization of Rad(M).

Definition 3.1 [2]: Let ρ be the class of all singular simple modules. For a module M, Let δ(M) $=\cap\{ N \subseteq M , M/N \in \rho\}$be the reject M ofρ.

Lemma 3.2: [2,Lemma 1.5] Let M and N be R- modules

1) $\delta(M) = \Sigma\{ L \subseteq M / L$ is δ - small submodule of M }

2) If $f : M \rightarrow N$ is an R-homomorphism then $f(\delta(N)) \subseteq \delta(N)$. Therefore δ(M) is a fully invariant submodule of M and M. $\delta(R_R) \subseteq \delta(M)$

3) If $M = \oplus_{i\in I} M_i$, then $\delta(M) = \oplus\, \delta(M_i)$

4)If every proper submodule of M is contained in maximal submodule of M then δ(M) is unique largest δ-small submodule of M.

5)Let $m \in M$ then $Rm <<_\delta M$ iff $m \in \delta(M)$.

6)An arbitrary sum of δ-small submodules of M is δ-small submodule of M iff $\delta(M) <<_\delta M$.

Remark (3.3): Let M be a finitly generated R-module . Then for any submodule A of M , A is δ-small iff $A \subseteq \delta(M)$.

Proof : Clear from Lemma 3.2 and [1, Th.2.3.11].

Proposition (3.**4):** Let M be an R-module then the following are equivelent

a)M satisfies a.c.c (d.c.c) on δ-small submodules

b)Every non empty collection of δ-small submodules possesses a maximal (minmal) member.

Proof : Clear.

Proposition (**3.5):** Let M be an R- module then M satisfies a.c.c on δ-small submodules if and only if δ(M) is δ-small and every δ-small submodule is finitly generated .

Proof: Assume M satisfies a.c.c on δ-small submodules Let μ={B:B is a finite sum of δ-small submodules of M } then μ is non empty collection of δ -small submodules by[lemma 1.2] so by Prop.2.4 μ has maximal element say K hence K is δ-small submodule of M then $K \subseteq \delta(M)$. [Lemma 3.2,6] .Suppose that there exists $x \in \delta(M)$. and $x \notin K$ hence Rx is δ-small submodule of M [Lemma 3.2,5] so K+Rx is δ-small submodule thus $K+Rx \in \mu$ and $K \subseteq K+Rx$ this contraduction the maximality of K then K=δ(M) thus δ(M). is δ-small submodule. Consider any δ-small submodule A of M and let G={B:B is finitly generated δ-small submodule of M,$B \subseteq M$}since the zero submodule is contained in G,$G \neq \phi$,by Prop.3.4,G has a maximal element say K,we claim that K=A,Since $K \in G$,K is finitly generated and $K \subseteq A$.

If $K \neq A$ then ther exist $x \in A, x \notin K$,hence K+Rx is member of G ,contining K is contadaction maximality of K then K=A then A is finitly generated

For the converse,consider $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots$ an ascending chain of δ-small submodules ,let $I = \cup_i^\infty I_i$ then $I \subseteq \delta(M)$ since for every i=1,2,3 $I_i \subseteq \delta(M)$ .But δ(M) is δ-small submodule of M so I is δ-small submodule of M thus I is finitly generated $I = Rx_1 + Rx_2 + \ldots + Rx_n$ now each $x_i \in I_i$ for every i so there exisit m such that $x_1, x_2, \ldots, x_m \in I_m$,But this implis that $I = I_m$ so $I_m = I_{m+1} = \ldots$thus M satisfies a.c.c. on δ-small submodules .

From remark (3.3) and similer proof of prop.(3.5) we get the following

Corollary (3.6): Let R be aring then R satisfies a.c.c on δ-small ideal if and only if every δ-small ideal is finitly generated .

Let N and L be submodules of a module M. N is called a supplement of L in M if M=N +L and

$N \cap L << N$.[8]

In [4] If N and L be submodules of a module M . N is called a δ-supplement of L in M if M=N +L and $N \cap L <<_\delta N$.and if every submodules of M. has a δ-supplement in M.Then M is called a δ-supplement module ,R is called a δ-supplement if it is supplement as R- module .It is clear that every supplement submodule is a δ-supplement but the converse is not true [4].

Proposition (3.7):): Let N and L be submodules of a finitly generated faithful multiplication R-module M such that N= M I, and L=NJ for some ideals I,J of R then N is δ- supplement submodule of L in M iff I is δ- supplement ideal of J in R

Proof :If N is δ- supplement submodule of L, then M= N + L and $N\cap L <<_\delta M$,then M I + M J =M , $MI\cap MJ <<_\delta M$ hence M(I+J) =M and $M(I\cap J) <<_\delta M$ then R=I+J , by prop. 2.4 $I\cap J = <<_\delta I$ hence I is δ- supplement ideal of J in R

as the same proof the convers is true.

Corollary (3.8): Let M be a finitely generated faithful multiplication R-module, then R satisfies a. c. c(d.c.c.). on δ- supplement ideal if and only if M satisfies a. c. c(d.c.c.). on δ- supplement submodules .

Proof : Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots$ be an ascending chain of δ- supplement ideals of J_i in R , then $M I_1 \subseteq M I_2 \subseteq M I_3 \subseteq \ldots \subseteq M I_k \subseteq \ldots$ is an ascending chain of δ- supplement submodule of $J_i M$ in M $\forall$ i=1,2,..by prop.3.7 then $\exists$ $K \in N$, , such that $M I_k = M I_{k+1} = \ldots$ But M is a finitely generated faithful module , then $I_k = I_{k+1} = ..$ [7]. Thus R satisfies a. c. c. on δ- supplement ideals of R

Conversely , Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots$ be an ascending chain of δ- supplement submodule of L_i $\forall$ i=1,2,.. Since M is a multiplication R-module, then $N_i = M I_i$ and $L_i = M J_i$ where J_i, I_i ideals of R for all i by prop. 4.1 I_i are δ- supplement ideals of J_i in R , hence $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots$ is an ascending chain of δ- supplement ideals of J_i in R . Since R satisfies a.c.c on δ-supplement ideal , then $\exists$ $K \in N$, such that $I_k = I_{k+1} = \ldots$, hence $M I_k = M I_{k+1} = \ldots$which implies $N_k = N_{k+1} = \ldots$, that is M satisfies a. c. c. on δ- supplement submodules.

The same argument for d.c.c. condition hence omitted.

Refrences

1. Kasch,F. **1982** , *Modules and Rings* , Academic Press Ins, London.
2. Zhou Y., **2000** Generalizations of perfect, semiperfect and semireguler rings, *Algebra Coll.*, 7(3), pp:305-318.
3. Coodearl K.R, **1976** *Ring theory* , pure and Applied Math., No.33 Marid-Dekker.
4. Kosan M. T., **2007** δ-lifting and δ-supplemented Modules, *Algebra coll.*, 14(1) ,pp:53-60.
5. Naoum A.G. and Hadi I.M. **1996** Module with ascending(descending) chain conditions on small submodules , *Iragi Journal of Science* , 37(3), pp:1085-1102 .
6. Naoum A.G. and AL-Aubaidy W.K. **2004** Modules That satisfy a.c.c.(d.c.c.) on large submodules , *Iraqi Journal of Science* , 45(1), pp:172-175 .
7. Naoum A.G. **1984**, on the ring of finitily generated multiplication modules, *Periodica Math. Hungarica* ,29, pp:277-284.
8. Wisbauer R., **1991**. *Foundations of Module and Ring Theory*, Gordon and Breach, Reading .