Some Results on Strongly Fully (m,n)- Stable Modules to Ideal

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Abstract
Let R be a commutative ring with non-zero identity element .For two fixed positive integers m and n , a right R-module M is called strongly fully (m,n)- stable relative to an ideal A of $R^{n\times m}$ if $\theta(N)\subseteq N \cap M^n A$ for each n- generated submodule of M^m and R- homomorphism $\theta: N\to M^m$. In this paper I give some characterizations theorems and properties of strongly fully (m,n) –stable modules relative to an ideal A of $R^{n\times m}$.

Keywords: fully (m,n), stable, modules, ideal

نتائج حول المقاسات تامة الاستقرارية من النمط (m,n) بقوة بالنسبة الى مثالي

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الخلاصة:
لتكن R حلقة أبدالية ذات عنصر محايد .M مقاسا أيمن أحاديا على R. فكرة المقاسات تامة الاستقرارية من النمط (n،m) بقوة بالنسبة الى مثالي A في الحلقة R^{nxm} قدمت في هذا البحث. نقول ان المقاس M تام الاستقرارية من النمط (m, n) بقوة بالنسبة الى مثالي A في R^{nxm} اذا كان $\theta(N)\subseteq N\cap M^n A$ لكل تشاكل مقاسي θ من N الى M^m حيث N مقاس جزئي متولد من النمط n . في هذا البحث تم دراسة علاقة صنف المقاسات تامة الاستقرارية من النمط (m, n) بقوة بالنسبة الى مثالي بأصناف أخرى مثل المقاسات شبه اغمارية من النمط (m, n) بالنسبة الى مثالي

Introduction:
Throughout, R is commutative ring with non –zero identity and all modules are unitary right R-modules. We use the notation $R^{m\times n}$ for the set of all m x n matrices over R. For $G \in R^{m\times n}$,G^T will denote the transpose of G .In general ,for an R- module N ,we write $N^{m\times n}$ for the set of all formal m x n matrices whose entries are elements of N .Let M be a right R- module and N be a left R –module .For $x \in M^{l\times m}$, $s \in R^{m\times n}$ and $y \in N^{n\times k}$ under the usual multiplication of matrices ,x_s (resp.sy) is a well defined element in $M^{l\times m}$ (resp.$N^{n\times K}$).If $X\in M^{l\times m}$, $S \in R^{m\times n}$ and $Y\in N^{n\times k}$, define

$l_M^{l\times m}(S)= \{u\in M^{l\times m}: us =0 , \forall s\in S\}$
$r_N^{n\times k}(S)= \{v \in N^{n\times k}: vs=0 , \forall s\in S \}$
$l_R^{m\times n}(Y)=\{s \in R^{m\times n}: sy=0 ,\forall y\in Y \}$
$r_R^{m\times n}(X)=\{s \in R^{m\times n}: xs=0,\forall x\in X\}$
We will write $N^n = N^{1xn}$, $Nn =N^{nx1}$.

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Strongly fully stable module relative to an ideal A of R have been discussed in [1],an R –modul M is called fully stable relative to an ideal A if θ (N)⊑ N ∩ MA for each submodule N of M and R –homomorphism θ: N→M.

It is an easy matter to see that M is strongly fully stable relative to an ideal A of R .if and only if θ(x R)⊆ x R ∩ MA for each x in M and R-homomorphism. θ: xR →M .In this paper , for two fixed positive integer m and n , we introduce the concepts of strongly fully (m, n) -stable module relative to an ideal and strongly (m, n) - Baer criterion relative to ideal and we prove that an R-module M is strongly fully (m, n)-stable relative to an ideal A of R^{nxm} if and only if any two m-element subsets {$\boldsymbol{\alpha}_1$, ……..,$\boldsymbol{\alpha}_m$} and {$\boldsymbol{\beta}_1$,……..,$\boldsymbol{\beta}_m$} of M^n ,if $\boldsymbol{\beta}$j∈$\sum_{i=1}^{n} \alpha i\, R \cap M^m A$, j = 1 ,……..,m implies r_{Rn} {$\boldsymbol{\alpha}_1$, ……..,$\boldsymbol{\alpha}_m$} ⊈ r_{Rn} {$\boldsymbol{\beta}_1$,……..,$\boldsymbol{\beta}_n$} .

Strongly fully (m, n) –stable module relative to ideal.

Definition 2.1:-An R-module M is called strongly fully (m, n) -stable relative to an ideal A of R n x m if $\boldsymbol{\theta}$ (N) ⊆ N ⋂ Mn A for each n- generated submodule N of M m and R-homomorphism : N → Mm. It is clear that M is strongly fully (1,1)- stable relative to Ideal ,if and only if M is strongly fully stable relative to ideal .

It is an easy matter to see that an R-module .M is strongly fully (m, n) –stable relative to ideal if and only if it is strongly fully (m, q) –stable relative to ideal for all 1≤ q ≤ n ,if and only if it is strongly fully (p,n) – stable relative to ideal for all 1 ≤ p ≤ m ,if and only if fully (p, q) –stable relative to ideal for all 1 ≤ p ≤ m and 1≤ q ≤ n .

In [2] an R-module M is called fully (m,n) –stable if $\boldsymbol{\theta}$ (N) ⊆ N for each n –generated submodule N of Mm and R-homomorphism $\boldsymbol{\theta}$:N → Mm. It is clear that every strongly fully (m, n) –stable module M relative to a non –zero ideal A of R n x m is fully (m, n) – stable .This follows from the fact N ⋂ M^n A ⊆ N for each n- generated submodule N of Mm.

Remark 2.2

1-Let M be anR-module and A be a a non –zero ideal A of R n x m. If M is fully (m,n)-stable and Mm=MnA then M is strongly fully (m ,n) –stable relative to A ,since for each n-generated N of Mm and R- homomorphism.f : N→M^m ,f(N) ⊆ N=N ∩ Mm =N ∩ M^nA.

2-Note that the concepts fully (m, n) – stable R-module and strongly fully (m, n) –stable module relative to an ideal A of Rn x mconsider for the ideal R n x min R^{nxm}, that is an R-module M is fully (m, n)-stable if and only if M is strongly fully (m ,n)-stable module relative to ideal R^{nxm} of R^{nxm}.

The following proposition gives another characterization of strongly fully (m,n) – stable relative to ideal .

Proposition 2.3

An R- module M is strongly fully (m,n) – stable relative to a non –zero ideal A of R^{mxn} if and only if any two m-element subsets {$\boldsymbol{\alpha}_1$, ……..,$\boldsymbol{\alpha}_m$} and {$\boldsymbol{\beta}_1$,……..,$\boldsymbol{\beta}_m$} of M^n ,if $\boldsymbol{\beta}$j∈$\sum_{i=1}^{n} \alpha i\, R \cap M^m A$, j = 1 ,……..,m implies

r_{Rn} {$\boldsymbol{\alpha}_1$, ……..,$\boldsymbol{\alpha}_m$} ⊈ r_{Rn} {$\boldsymbol{\beta}_1$,……..,$\boldsymbol{\beta}_n$} .

Proof: - Assume that M is strongly fully (m, n) –stable relative to an ideal A of R ^{mxn} and the exist two m - element subsets {$\boldsymbol{\alpha}_1$, ……..,$_m$} and {$\boldsymbol{\beta}_1$,……..,$\boldsymbol{\beta}_m$} of M ^{n} such that

$\boldsymbol{\beta}_j \notin \sum_{i=1}^{n} \alpha i\, R \cap M^m A$ for each j = 1,……,m and $r_{R\,n}${$\boldsymbol{\alpha}_1$, ……..,$\boldsymbol{\alpha}_m$} ⊆ $r_{R\,n}${$\boldsymbol{\beta}_1$,……..,$\boldsymbol{\beta}_m$}.

Define f : $\sum_{i=1}^{n} \alpha i\ R \to M^m$ by f ($\sum_{i=1}^{n} \alpha i\ ri$)=$\sum_{i=1}^{n}$ βi ri

Let αi = (a_{i1}, a_{i2} …….a_{in}). If $\sum_{i=1}^{n} \alpha i\ ri$=0 ,then $\sum_{i=1}^{n} \alpha ij\ ri$ =0 , j=1,………m implies that $\boldsymbol{\alpha}_j r^T$= 0 where r = ($r_1$……………..$r_n$)∈ R^n and hence r^T∈ r_{Rn}{$\boldsymbol{\alpha}_1$, ……..,$\boldsymbol{\alpha}_m$} .By assumption $\boldsymbol{\beta}_j$ r^T =0 ,j=1,………..,m ,so $\sum_{i=1}^{n} \beta_i\, r_i = 0$. This shows that f is well define.It is an easy matter to see that f is R- homomorphism strongly fully (m,n) –stable relative to an ideal A of R^{mxn} implies that there exist t∈ R such that

f($\sum_{i=1}^{n} \alpha i\ ri$)=$\sum_{k=1}^{n}(\sum_{i=1}^{n} \alpha i\ ri)$ t_k ,j=1,………..,m

for each $\sum_{i=1}^{n} \alpha i\ ri$∈ $\sum_{i=1}^{n} \alpha i\ R$,let r_i =(0,0,……..,1,0,…….0)∈ R ,where 1 in the ith position and 0 otherwise

$\boldsymbol{\beta}_i$ =f($\sum_{i=1}^{n} \alpha i$) = $\sum_{k=1}^{n} \alpha_i t_k$ ∈ M ^{m} A ,thus $\boldsymbol{\beta}_i$ ∈ $\sum_{i=1}^{n} \alpha i\ R$ ∩ M^m A which is contradiction.

Thus $r_{R\,n}${$\boldsymbol{\alpha}_1$, ……..,$_m$} ⊈ $r_{R\,n}${$\boldsymbol{\beta}_1$,……..,$\boldsymbol{\beta}_m$}.

Conversely assume that there exist n- generated submodule of M^m and R- homomorphism : $\sum_{i=1}^{n} \alpha i\ R$ → M ^{m} such that $\boldsymbol{\theta}$ ($\sum_{i=1}^{n} \alpha i\ R$) ∉ $\sum_{i=1}^{n} \alpha i\ R$ ∩ M^n A .

Then there exists an element **β**= (=$\sum_{i=1}^{n} \alpha i \ \ ri$)∈ $\sum_{i=1}^{n} \alpha i \ \ R$ such that
(**β**) ∉ $\sum_{i=1}^{n} \alpha i \ \ R$ ∩ M^n A ,take β_j = **β** ,j=1,……,m .

Than we have m- element subset { (**β**),……… **θ** (**β**)} such that
θ ( ) ∉ $\sum_{i=1}^{n} \alpha i \ \ R$ ∩ M^m A .

Let **η** =$(t_1,……..t_n)^T$∈ $r_{R\,n}${α_1, ……..,α_m} then α_j **η** =0 i.e $\sum_{i=1}^{n} \alpha ij \ tji = 0$ ∀ j,……..,m .
α_j =(a_{1j},……………..,$a_{n\,j}$) and { **θ**(**β**),……… **θ** (**β**)} **η**
=$\sum_{k=1}^{n}$ θ (β)t_k=$\sum_{k=1}^{n}$ θ ($\sum_{i=1}^{n} \alpha i \ \ ri$)t_k=$\sum_{k=1}^{n}$(θ ($\sum_{i=1}^{n} \alpha i \ \ ri$ t_k)=0
hence $r_{R\,n}${α_1, ……..,$_m$} ⊆ $r_{R\,n}${ **θ**(β_1),……… **θ** (β_n)} which is a contradiction .Thus M is strongly fully (m, n)- stable relative to ideal .

Corollary 2.4

Let M be strongly fully (m, n)-stable module relative to an ideal A of R^{mxn} , then for any two m- element subsets {α_1, ……..,α_m} and {β_1,……..,β_m} of M^n,
$r_{R\,n}${α_1, ……..,$_m$} ⊆ $r_{R\,n}${β_1,……..,$_m$} implies
(α_1R +……….+ α_mR) ∩ M^mA = (β_1 R +……….+ β_m R) ∩ M^mA

Corollary 2.5 [1]

let M be a strongly fully stable module relative to an ideal A of R , then for each x,y in M ,y ∉ (x) , r_R (x)= r_R (y) implies (x) ∩ M A=(y) ∩ M A .

Recall that a submodule N of an R – module M is (m, n)- pure submodule if for all C∈ R^{nxm} . N^m∩ M^nC =N^nC [3].

The following proposition gives a partial answer for the question: - When the submodule of strongly fully (m, n)-stable module relative to ideal.

Proposition 2.6

let M be a strongly fully (m ,n)- stable module to a non –zero ideal A of $R^{nx\,m}$. Then every (m, n)-pure submodule of M is strongly fully (m, n)-stable module relative to A.

Proof:- let N be (m, n)- pure submodule of M.For each n-generated submodule K of N and an R – homomorphism f:K → N^m, put g=i o f:K → M^m(where i is the inclusion mapping of N^m to M^m),then by assumption f(K) = g(K) ⊆ M^n A , and since f(K) ⊆ N^m. Hence f (K) ⊆ K ∩ M^nA ∩ N^m. since N is (m, n) –pure submodule of M then N^m ∩ M^nA =Nn A, for each ideal A of R^{nxm} , therefore f(K) ⊆ K ∩ N^nA .

Thus N is strongly fully (m, n) stable module relative to A.

Corollary 2.7[1]

let M be a strongly fully stable R –module relative to anon -zero ideal A of R . Then every pure submodule of M is strongly fully stable module relative to A.

A.M .Sharky in [4] ,has introduced the concepts of Baer , s Criterion relative to an ideal .

Let M be an R-module , A be an ideal of R and N be a submodule of M . She says that N satisfies Baer ' s Criterion relative to A if for each R-homomorphism f:N → M, there exist r ∈ R such that f(n)-n r ∈ MA , for each n ∈N .M is said to satisfy Baer's Criterion relative to A ,if each submodule of M satisfies Baer' s Criterion relative to A . We introduce the concept of strongly (m,n)Baer 's Criterion relative to A .

Definition 2.8

For affixed positive integers n and m ,we say that an R –module M satisfies strongly (m, n)- Baer' s Criterion relative to an ideal A of R^{nxm} , if for any n- generated submodule N of M^m and any R – homomorphism **θ** :N → M^m there exists t ∈ R^{nxm} such that **θ** (x)=xt ∈ M^nA for each x in N .

It is clear that if M satisfies strongly (m, n) Bear ' s Criterion relative to an ideal A then M satisfies strongly (p ,q)- Bear' s Criterion relative to A ,∀ 1≤ p ≤ m and 1 ≤ q ≤ n.

Proposition 2. 9

Let M be an R- module and A be a non zero ideal of R^{nxm} .Then M satisfies strongly (m, n) Bear , s Criterion relative to an ideal A ,if and only if

$l_M{}^m$ r_{Rm}(α_1R+………..+ α_nR) ⊆ (α_1R+………..+ α_nR) ∩ M^n A for any n- element subset { α_1,………..,α_n) of M^m .

Proof: -First assume that strongly (m, n)) Bear ' s Criterion relative to an ideal A holds for n-generated submodule of) M^m, let α_i=(a_{i1},…………,a_{im}) ,for each i=1,……,n and **β**={β_1,……..,β_n}∈ $l_M{}^m$ r_{Rm}(α_1R+………..+ α_nR)

Define $\boldsymbol{\theta}$: $\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R \to M^m$ by $\boldsymbol{\theta}(\sum_{i=1}^{n} \alpha_i \text{ ri}) = \sum_{i=1}^{n} \beta_i$ ri If $\sum_{i=1}^{n} \alpha_i\, r_i = 0$,then $\sum_{i=1}^{n} a_{ij} r_i = 0$,this implies that $\boldsymbol{\alpha}_i(r^T) = 0$ where $r = (r_1, \ldots\ldots, r_n) \in R^m$,hence $r^T \in r_{Rm}(\boldsymbol{\alpha}_1, \ldots\ldots., \boldsymbol{\alpha}_n)$ .By assumption $\boldsymbol{\beta}_i r^T = 0, \forall i = 1, \ldots.., n$ so$\sum_{i=1}^{n} \beta_i\, r_i = 0$ .This shows that f is well defined . It is an easy matter to se e that $\boldsymbol{\theta}$ is an R- homomorphism . By assumption there exists $t \in R$ such that $\boldsymbol{\theta}(\sum_{i=1}^{n} \alpha_i \text{ ri}) = \sum_{k=1}^{n}(\sum_{i=1}^{n} \alpha_i \text{ ri}) t_k \in M^n A$ $t = (t_1, \ldots\ldots.., t_n) \in R^m$ for each

$\sum_{i=1}^{n} \alpha_i$ ri $\in \sum_{i=1}^{n} \alpha_i$ R .Let ri= (0,…….0,1,0,……,0) $\in R^m$ where 1 in the ith positive and 0 otherwise .

$\boldsymbol{\beta}_i = \sum_{k=1}^{n} \alpha_i t_k$ thus $(\boldsymbol{\alpha}_i) = \sum_{k=1}^{n} \alpha_i t_k \in M^n$ A thus $\boldsymbol{\beta}_i \in \sum_{i=1}^{n} \alpha_i\, R \cap M^n A$ which is contradiction . This implies that $l_M^m\ r_{Rm}(\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R) \subseteq (\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R) \cap M^n A$.

Conversely, assume that $l_M^m\ r_{Rm}(\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R) \subseteq (\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R) \cap M^n A$, for each $\{ \boldsymbol{\alpha}_1, \ldots\ldots\ldots., \boldsymbol{\alpha}_n\}$ of M^m. Then for each R –homomorphism f : $\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R \to M^m$ and s $= (s_1, \ldots\ldots.., s_n) \in r_{Rm}(\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R)$, $\sum_{k=1}^{n}(\sum_{i=1}^{n} \alpha_i \text{ ri}) s_k = 0$ for each $\sum_{i=1}^{n} \alpha_i$ ri $\in \sum_{i=1}^{n} \alpha_i$ R hence $\sum_{k=1}^{n} f(\sum_{i=1}^{n} \alpha_i \text{ ri}) s_k = \sum_{k=1}^{n} f(\sum_{i=1}^{n} \alpha_i\, r_i\, s_k) = 0$,

thus $f(\sum_{i=1}^{n} \alpha_i\, r_i) \in l_M^m\ r_{Rm}(\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R) = (\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R) \cap M^n A$, then $f(\sum_{i=1}^{n} \alpha_i \text{ ri}) = f(\alpha_i r^T) = f(\alpha_i) r^T \in (\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R) \cap M^n A$. Then M satisfies strongly (m, n) - stable relative to A .

Corollary 2.10

An R-module is strongly fully (m ,n) stable relative to an ideal A of R ^{nxm} if and only if $l_M^m\ r_{Rm}(\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R) \subseteq (\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R) \cap M^n A$ for any n-element subset $\{ \boldsymbol{\alpha}_1, \ldots\ldots., \boldsymbol{\alpha}_n \}$ of M^m.

Proposition 2.11

Let A be an ideal of R^{nxm} and M be an R –module such that $r_R(N \cap K) = r_R(N) + r_R(K)$ for each two n-generated submodule of M^m.If M satisfies strongly (m,1) – Bear ' s criterion relative to A .Then M satisfies strongly (m, n) - Bear ' s criterion relative to A for each $n \geq 1$.

Proof:-Let $L = x_1 R + \ldots\ldots\ldots + x_n R$ be n-generated submodule of M^m and $f: l \to M^m$ an R –homomorphism .We use induction on n . It is clear that M satisfies strongly (m, n) –Bear, s criterion, if n=1 .Suppose that M satisfies strongly (m, n) – Bear, s criterion for all k-generated submodule of M^m,for $k \leq n-1$.

Write $N = x_1 R$, $k = x_2 R+\ldots\ldots\ldots.+ x_n R$, then for each $w_1 \in N$ and $w_2 \in k$,$f/_N(w_1) = w_1 r = f/_K(w_2) = w_2 s$ for some r ,s $\in$R . I t is clear $r - s \in r_R(N \cap K) = r_R(N) + r_R(K)$,suppose that r –s =u + v with $u \in r_R(N)$, $v \in r_R(K)$ and let r –u = s+v . then for any $w = c + w_2 \in L$ with $w_1 \in N$ and $w_2 \in K$,f(w)=wt

$f(w_1) + f(w_2) = (w_1 + w_2)t$

$f(w_1) - w_1 t = w_2 t - f(w_2)$

$f(w_1) - w_1(r - u) = w_2(s + v) - f(w_2)$

$f(w_1) - w_1 r + w_1 u = w_2 s + w_2 v - f(w_2) \in M^n A$.

Corollary 2.12[1]

Let M be an R – module and A be a non –zero ideal of R .Then a strongly Bear ' s criterion relative to A holds for each cyclic submodule of M if and only if $l_M(r_R(x)) = R_x \cap MA$ for each $x \in M$.

Corollary 2.13

Let A be a non –zero ideal of R and M be an R –module such that $r_R(N \cap K) = r_R(N) + r_R(K)$ for every finitely generated submodule N and K of M .Then M is strongly fully stable relative to A if and only if M satisfies strongly Bear ' s criterion relative to A for finitely generated submodules.

Corollary 2.14

An R-module M is strongly fully -(m, n) stable relative to A of R^{nxm} , if and only if $l_M\ r_R((\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R) \subseteq (\boldsymbol{\alpha}_1 R+\ldots\ldots\ldots.+ \boldsymbol{\alpha}_n R) \cap M^n A$ for any n-element subset $\{\boldsymbol{\alpha}_1, \ldots.. \boldsymbol{\alpha}_n\}$ of M .

Recall that an R- module M is (m, n) –quasi injective if for each R –homomorphism from an n-generated submodule of M^n to M extends to one from M^m to M [5].Now, we introduce the concept of strongly (m, n) –quasi –injective relative to ideal.

Definition 2.15

An R –module M is said to be strongly (m, n)- quasi –injective relative to a non –zero ideal A of R^{nxm} if for each n- generated submodule N of M and R –homomorphism f: N → M there exists an R-homomorphism g: M^m →M such that
f(x) =g(x) ∈ M^nA ∀x ∈ N.

It is clear an R-module M is strongly principally quasi injective relative to A if and only if strongly (1, 1)-quasi –injective relative to A.

Proposition 2.16

Let M be an R- module and A a non zero ideal of R^{nxm} . If M is a strongly fully (m, n)-stable relative to A , then M is strongly (m, n)-quasi –injective relative to A .

Proof:-Let N =$\boldsymbol{\alpha}_1$R +…….+$\boldsymbol{\alpha}_n$R for each {$\boldsymbol{\alpha}_1$,…….,$\boldsymbol{\alpha}_n$} ∈ M^m (N is n- generated submodule of M^m) and f:N →M be an R- homomorphism then f(N) ⊆ N ∩ M^n A, thus there exist t∈ R such that f($\sum_{i=1}^{n} \alpha_i$ri) = $\sum_{k=1}^{n}(\sum_{i=1}^{n} \alpha_i$ ri)t_k , t = (t_1,……,t_n) ∈ Rm.

Define g : M^m →M by g($\boldsymbol{\alpha}_i$) = $\boldsymbol{\alpha}_i$ t_i , i = 1,……., n ∀ $\boldsymbol{\alpha}$ i∈ M^m . It is clear that g is well defined R-homomorphism and f($\sum_{i=1}^{n} \alpha_i r_i$) =g ($\sum_{i=1}^{n} \alpha_i r_i$) = $\sum_{k=1}^{n}(\sum_{i=1}^{n} \alpha_i r_i) t_k$. Therefore M is strongly (m, n)-quasi injective relative to A .

Corollary 2.17[1]

Let M be an R-module M and A a non –zero ideal of R . If M is a strongly fully -stable relative to A, then M is strongly principally quasi –injective relative to A.

In [6] , a submodule N of an R-module M is said to be fully invariant if **θ**(N) ⊆N for each R-endomorphism **θ** of M . In case that each submodule of M is fully invariant, then M is called duo module.

Theorem 2.18

Let M be an R- module and A be a non zero ideal of R^{nxm}. Then M is a strongly fully (m, n) -stable relative to A and duo module .

Proof:-→ by proposition (2.16),M is strongly (m, n) - quasi –injective module relative to A and it is clear that M is duo module .

Conversely, let N be an n-generated submodule of M^m and f:N→ M be an R homomorphism since M is strongly (m, n) – quasi injective relative to A ,then there exists an R – homomorphism
g:M^m → M such that f(n)=g(n) ∈ M^nA for each n∈ N . Now, since M is duo module ,then
g(N) ⊆ N , hence g(N) ⊆ N ∩ M^n A then f(n) ∈ N ∩ M^n A , for each n ∈ N . Therefore
f(N) ⊆ N ∩ M^n A .

Corollary 2.19

Let M be an R –module, and A be anon –zero ideal of R. M is a strongly fully –stable module relative to A if and only if M is strongly principally quasi –injective relative to A and duo module.

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