Influence of MHD on Steady State Newtonian Fluid Flow in A vertical Channel With Porous Wall Using HAM

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Abstract

The aim of this paper is to analyses steady state three-dimensional magnetohydrodynamic(MHD) flow of fluid injected uniformly into the vertical channel with porous wall through one side of the channel. The equations which were used to describe the flow are the momentum and energy equations, these equations were written to get their non-dimensional form. It is found that these equations are controlled by many dimensionless parameter, such as Hartmann number M, Reynolds number Re and Peclet number Pe. The homotopy analysis method(HAM) is employed to obtain a analytical solutions for velocity and heat transfer fields. The effect of each dimensionless parameters upon the normal and tangential velocity, pressure and temperature distributions are analyzed and shown about (15) graphs by using the Mathematica package.

Keywords: Velocity and temperature distributions, porous wall channel, Homotopy analysis method.

تأثير الحقل المغناطيسي على الجريان المستقر لمائع نيوتيني في قناة عموديه ذات جدار مثقبه باستخدام طريقه HAM

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الخلاصة:

الهدف من هذا البحث هو دراسة تأثير الحقل المغناطيسي على الجريان المستقر و المنتظم في اتجاه واحد لمائع نيوتيني في قناة عموديه ثلاثية الابعاد ذات جدار مثقبه. المعادلات التي استخدمت لوصف هذا الجريان هي معادلات الحركه ومعادله الطاقه وقد تم كتابه هذه المعادلات من اجل الحصول على الصيغ اللابعديه لكل منهما. لقد تبين أن معادلتي الحركة و الطاقه تحكمها بعض المعلمات اللابعدية, مثل عدد هارتمان M وعدد رينولدز Re وعدد بكلت Pe . لقد حصلنا على تعبير السرعه و انتقال الحرارة بأستخدام طريقة هوموتوبي التحليلية (HAM). قمنا بدراسه تأثير كل من الاعداد الفيزيائيه المذكوره اعلاه على السرعه العموديه والسرعه المماسيه و الضغط ودرجه الحراره تحت تأثير المجال المغناطيسي. هذه الدراسة قد تمت من خلال رسم حوالي (15) بيان بأستخدام البرنامج الجاهز Mathematica.

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Introduction

Fluid is that state of matter, which is capable of changing shape and is capable of flowing. Fluids may be classified as real "viscous" and ideal "perfect" according to whether the fluid is capable of exerting shearing stress or not. Real fluid is called Newtonian if the relation between stress and rate of strain is linear, otherwise is called non-Newtonian fluid. Within the past fifty years, many problems dealing with the flow of Newtonian and non-Newtonian fluids through porous channels have been studied by engineers and mathematicians. The analysis of such flows finds important applications in different scientific fields. Examples of these applications are the boundary layer control, transpiration cooling, gaseous diffusion, prevent corrosion, reactants addition and drag reduction. From a technological point of view, flows of this type are always important, especially in transpiration cooling, which is very effective process to protect certain structural elements in turbo jet and rocket engines, like combustion chamber walls, or gas turbine blades from influence of hot gases. Examples, of such flow of fluid in vertical porous channel, is found in [1] as the simplest subclass for which one can hope to gain an analytic solution.

The magnetohydrodynamic (MHD) phenomenon is characterized by an interaction between the hydrodynamic and the electromagnetic field. The study of MHD flow in a channel with porous wall also has applications in many devices like MHD power generators, MHD pumps, accelerators, etc. Some recent contributions in the field may be mentioned in [2 - 6]. The flow of Newtonian and non- Newtonian fluids through porous channel has been investigated by numerous authors.

The case of a two-dimensional, incompressible, steady, laminar suction flow of a Newtonian fluid in a porous channel was studied by Berman[7]. He has solved the Navier-Stokes equations by using a perturbation method for very low cross-flow Reynolds number. After his pioneering work, this problem has been studied by many researchers considering various variations in the problem [8,9]. Wang and Skalak[10] were the first persons who present the solution for a three-dimensional problem of fluid injection through one side of a long vertical channel for Newtonian fluid. They have obtained a series solution for small value of Reynolds number and numerical solution for both small and large Reynolds number. Huang[11] re-examined Wang and Skalak problem using a method based upon quasilinearization. Authors like Ascher[12], Baris[13], Sharma and Chaudhary[14] and many others have extended Berman’s series solution and solve it by different techniques.

In the recent year Baris continued the last mentioned research by substituting thermodynamically compaitible fluid instead of Newtonian fluid. the used analytical method by Baris was traditional perturbation solution which was one of the old analytical methods. These scientific problems are modeled by ordinary or partial differential equations and should be solved using special techniques, because in most cases, analytical solutions can't be applied to these problems. In recent years, much attention has been devoted to the newly developed methods to construct an analytical solution of these equations. One of these techniques is Homotopy Analysis Method(HAM),which was introduced by Liao[15-17] and has been successfully applied to solve many types of nonlinear problems[1,18-21].

Basic ideas of HAM:-

This method is proposed by Liao [15-17]. Below the outline of the HAM will be presented.
Consider a non-linear equation governed by

$$A(u)+f(r)=0 \tag{1}$$

where A is a non-linear operator, $f(r)$ is a known function and u is an unknown function. By means of homotopy analysis method, one first construct a family of equations

$$(1-p)\ell[v(r,p)-u_0(r)]=ph\{A[v(r,p)]-f(r)\}, \tag{2}$$

where ℓ is an auxiliary linear operator, $u_0(r)$ is an initial guess, h is an auxiliary parameter, $p\in[0,1]$ is an embedding parameter, $v(r,p)$ is an unknown function of r and p. Liao [18,19] expanded $v(r,p)$ in Taylor series about the embedding parameter

$$\mathrm{v(r,p)}=\mathrm{u}_0(r)+\sum_{m=1}^{\infty}u_m(r)\,p^m, \tag{3}$$

where

$$u_m(r) = \frac{1}{m!} \frac{\partial^m v(r,p)}{\partial p^m}\bigg|_{p=0}. \tag{4}$$

The convergence of the series (3) depends upon the auxiliary parameter h. If it is convergent at $p=1$, one has

$$u(r) = u_0(r) + \sum_{m=1}^{\infty} u_m(r) \tag{5}$$

Differentiating the zeroth order deformation equation (2) m-time with respect to p and then dividing them by $m!$ and finally setting $p=0$ we obtain the following m-th order deformation problem:

$$\ell[u_m(r) - \chi_m u_{m-1}(r)] = hR_m(r), \tag{6}$$

in which

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases} \tag{7}$$

$$R_m(r) = \frac{1}{(m-1)!}\left\{\frac{d^{m-1}}{dp^{m-1}} A\left[u_0(r) + \sum_{m=1}^{\infty} u_m(r)\, p^m\right]\right\}\bigg|_{p=0} \tag{8}$$

There are many different ways to get the higher order deformation equation. However, according to the fundamental theorem in calculus, the term $u_m(r)$ in the series (3) is unique. Note that the HAM contains an auxiliary parameter h, which provides us with a simple way to control and adjust the series solution (5).

Governing equation:-

The steady three –dimensional flow of Newtonian, laminar, and incompressible fluid in a vertical channel with porous wall is considered and the body force per unit mass is taken to be equal to the gravitational acceleration. The channel is assumed to be infinite and uniform and figures-1 and 2 shows the physical model and coordinate system. Through which the fluid is injected uniformly in to the channel through one side. The fluid is injected through a vertical porous plate at y=D with uniform velocity U. The fluid strikes another vertical impermeable plate at y=0. The fluid flows out through the opening of the plates, due to the action of gravity along the z-axis. The distance between the walls is assumed D, is small compared to the dimensions of the plates, i.e. L >> B >> D. Due to this assumption the edge effect can be ignored and the isobars are parallel to the z-axis. Then, the basic equations of our problem can be given by:

Continuity Equation $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{9}$$

N-S Equation $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon \nabla^2 u - \frac{\sigma B^2 u}{\rho} \tag{10}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \upsilon \nabla^2 v \tag{11}$$

where ρ is the density, μ is the viscosity and p is the pressure with the dimensionless quantities :

$$u = \frac{Ux}{D} f'(\eta)\ ,\ v = -Uf(\eta)\ ,\ w = \frac{D^2 g\rho}{\mu} h(\eta) \tag{12}$$

where $\eta = \dfrac{y}{D}$, and with the boundary conditions:

$$f(0)=0\ ,\ f(1)=0\ ,\ f'(0)=0\ ,\ f'(1)=0 \tag{13}$$

Energy Equation $$\rho c_p \left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=k\Delta T \qquad (14)$$

where ρc_p is the heat capacity, k is the thermal conductivity and T is the temperature field is introduced as below

$$T=T_\circ+(T_1-T_\circ)\theta(\eta) \qquad (15)$$

where $T_\circ$ and T_1 are temperatures of the impermeable and porous plates respectively and with constant value and with the boundary conditions

$$\theta(0)=0\ ,\ \theta(1)=1 \qquad (16)$$

Substituting the Eq.(12) into Eqs (10) , (11) and (14) , we obtain following ordinary differential equations with boundary conditions (5):

$$f''''+\mathrm{Re}[ff'''-f'f'']-Mf''=0 \qquad (17)$$

$$p(x,\eta)=-\frac{1}{2}\rho U^2 f^2-\frac{\mu U}{D}f'+\frac{1}{2}\frac{Ux^2\mu}{D^3}f'''(0)+c_\circ \qquad (18)$$

where $c_\circ$ is the constant of integration, $\mathrm{Re}=\dfrac{UD\rho}{\mu}$ is Reynolds number and $M=\dfrac{\sigma D^2B^2}{\mu}$ is called Hartmann number.

Energy equation with corresponding boundary conditions (8) reduces to:

$$\theta''+Pef\theta'=0 \qquad (19)$$

where $Pe=\dfrac{\rho UDc_p}{k}$ is called Peclet number.

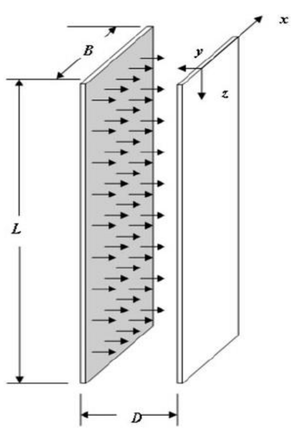


Figure.1 -Sketch of the problem under discussion.

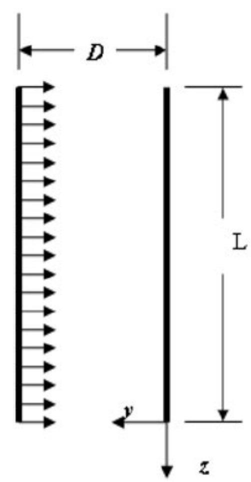


Figure.2- Fluid flow in a vertical channel with porous wall.

HAM solution:-

In this section, we attempt to obtain analytical solutions for the imposed problem. The HAM proposed by Liao[15-17] is employed to solve the problem. Many types of nonlinear problems were solved with HAM in the literatures [1] and [18-21] which verify the validity of the method.

Basic procedure

For the HAM solving procedure, we first select initial guess solutions as follows:

$$f_\circ(\eta)=3\eta^2-2\eta^3 \qquad (20)$$

$$\theta_{\circ}(\eta)=\eta \tag{21}$$

Then we define the linear operators

$$L_1(f)=f'''' \ , \ L_2(\theta)=\theta'' \tag{22}$$

Which have the property

$$L_1(c_1+c_2\eta+c_3\eta^2+c_4\eta^3)=0 \ , \ L_2(c_5+c_6\eta)=0 \tag{23}$$

where $c_i(i=1-6)$ are constants

Further more, The nonlinear operators can be defined as

$$N_1[f(\eta;p)=f''''+\mathrm{Re}[ff'''-f'f'']-Mf''=0 \tag{24}$$

$$N_2[\theta(\eta;p)=\theta''+Pe[f\theta'']=0 \tag{25}$$

where $p\in[0,1]$ is an embedding parameter, as p increases from 0 to 1, $f(\eta;p)$ and $\theta(\eta;p)$ vary from the initial guess $f_0(\eta)$ and $\theta_0(\eta)$ to the exact solution $f(\eta)$ and $\theta(\eta)$, respectively.

We develop the so called zeroth-order deformation equations and corresponding boundary conditions:

$$(1-p)L_1[f(\eta;p)-f_0(\eta)]=ph_1N_1[f(\eta;p)] \tag{26}$$

$$(1-p)L_2[\theta(\eta;p)-\theta_0(\eta)]=ph_2N_2[\theta(\eta;p)] \tag{27}$$

$$f(0;p)=0 \ , \ f'(0;p)=0 \ , \ f(1;p)=1 \ , \ f'(1;p)=1 \tag{28}$$

$$\theta(0;p)=0 \ , \ \theta(1;p)=1 \tag{29}$$

Differentiating the zeroth-order deformation Eqs. (26) and (27) m-times with respect to *p* and then dividing them by m!, finally setting $p=0$, we obtain the following mth-order deformation equations as

$$L_1[f_m(\eta)-\chi_m f_{m-1}(\eta)]=h_1R_m^f(\eta) \tag{30}$$

$$L_2[\theta_m(\eta)-\chi_m\theta_{m-1}(\eta)]=h_2R_m^{\theta}(\eta) \tag{31}$$

$$f_m(0)=f_m'(0)=f_m(1)=f_m'(1)=0 \tag{32}$$

$$\theta_m(0)=\theta_m(1)=0 \tag{33}$$

for both boundary conations In which h_1 and h_2 are an auxiliary parameters.

$$R_m^f(\eta)=f_{m-1}''''+\mathrm{Re}\sum_{i=0}^{m-1}[f_{m-1-i}f_i'''-f_{m-1-i}'f_i'']-Mf_{m-1}'' \tag{34}$$

$$R_m^{\theta}(\eta)=\theta_{m-1}''+Pe\sum_{i=0}^{m-1}f_{m-1-i}\theta_i' \tag{35}$$

and

$$\chi_m=\begin{cases}0 & when \ \ m\le 1\\ 1 & when \ m>1\end{cases}$$

We use the symbolic calculation software MATHEMATICA and solve the set of linear differential Eqs.(30) and (31) with boundary conditions (34) and (35) up to first few order of approximation. It is found that $f(\eta)$ and $\theta(\eta)$ can be written as

$$f(\eta)=\sum_{m=0}^{k}f_m(\eta) \quad , \quad \theta(\eta)=\sum_{m=0}^{k}\theta_m(\eta)$$

And

$$f_m=\sum_{i=0}^{4m+3}\lambda_{m,i}a_{m,i}\eta^i \tag{36}$$

where $a_{m,i}$ is a coefficient of $f_m(\eta)$ for $m \geq 1$, and
$$\lambda_{m,i} = \begin{cases} 0 & m \geq 0, i \leq 1 \\ 0 & m \geq 0, i > 4m+3 \\ 1 & otherwise \end{cases} \quad (37)$$

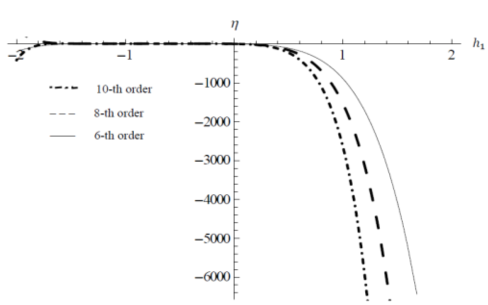
and
$$\theta_m = \sum_{i=0}^{4m+1} \delta_{m,i} b_{m.i} \eta^i \quad (38)$$

where $b_{m,i}$ is a coefficient of $\theta_m(\eta)$ for $m \geq 1$, and
$$\delta_{m.i} = \begin{cases} 0 & m \geq 0, i = 0 \\ 0 & m \geq 0, i > 4m+1 \\ 0 & m \geq 0, 2 \leq i \leq 3 \\ 1 & otherwise \end{cases} \quad (39)$$

To obtain the general solution for N-S equations we will differentiate Eq.(36) w.r.t η four times and substitute the result into Eqs.(30) &(34) and by using Eq.(32) and doing simple calculation the general solution has the form
$$f = \sum_{m=0}^{\infty} f_m = \lim_{D \to \infty} [\sum_{k=1}^{4D+3} (\sum_{m=k-1}^{4D} a_{m,k} \eta^k)] \quad (40)$$
By the same way, to get the general solution of energy equation we will differentiate Eq.(38) w.r.t η twice and substitute the result into Eqs.(31) &(35) and by using Eq.(33) and doing simple calculation the general solution has the form
$$\theta = \sum_{m=0}^{\infty} \theta_m = \lim_{D \to \infty} [\sum_{k=1}^{4D+1} (\sum_{m=k-1}^{4D} b_{m,k} \eta^k)] \quad (41)$$
where $a_{m,k}$ and $b_{m,k}$ are the coefficients of the functions f_m and θ_m respectively and we can obtain the values of it by equating the equal powers of η.

Convergence of the solutions

The analytical expressions given by $f(\eta)$ and $\theta(\eta)$ contain the auxiliary parameter h_1, h_2 respectively which controls the convergence region and rate of approximation for the HAM. It is clear from figure-3 that 10th-order approximation is admissible for the N–S equations. While, figure-4 disclosed that for energy equation, 10th-order approximation is proper. For $h_1 = -0.6$ and $h_2 = -0.6$ they are apparent from the calculations that the series given in $f(\eta)$ and $\theta(\eta)$ converge in the whole region of η. To show the convergence of our solution explicitly, we have made table (1) and (2). It can be seen that the solution converge as order of approximant increase and 10th-order approximation is enough to get a reasonable result.



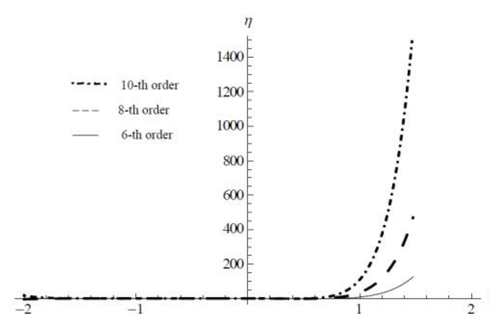


Figure.3- 10th-order of approximation h_1 curve for $f''(0)$

Figure.4 10th-order of approximation h_2 curve for $\theta'(0)$.

Table 1- The value of h_1

THE VALUE OF h_1	10^{TH}-ODER	8^{TH}-ORDER	6^{TH}-ORDER
-0.2	3.40515	**3.46737**	**3.54601**
-0.4	**3.29169**	**3.29817**	**3.33276**
-0.6	**3.28957**	**3.29202**	**3.28866**
-0.8	**3.29208**	**3.28598**	**3.29778**
-1.0	**3.26973**	**3.32564**	**3.22263**
-1.2	**3.46692**	**3.17748**	**3.16221**
-1.4	**4.2035**	**1.64254**	**3.54518**
-1.6	**-2.49731**	**-3.63785**	**5.05828**
-1.8	**-66.5057**	**-15.3632**	**8.40992**

Table 2- The value of h_2

THE VALUE OF h_2	10^{TH}-ORDER	8^{TH}-ORDER	6^{TH}-ORDER
-0.2	**1.15437**	**1.14196**	**1.12323**
-0.4	**1.17571**	**1.1735**	**1.16717**
-0.6	**1.17696**	**1.17681**	**1.17604**
-0.8	**1.17695**	**1.17694**	**1.17709**
-1.0	**1.17705**	**1.17662**	**1.17839**
-1.2	**1.17277**	**1.18547**	**1.17524**
-1.4	**1.21732**	**1.22609**	**1.6533**
-1.6	**1.79421**	**1.33302**	**1.17242**
-1.8	**4.99284**	**1.59514**	**1.27876**

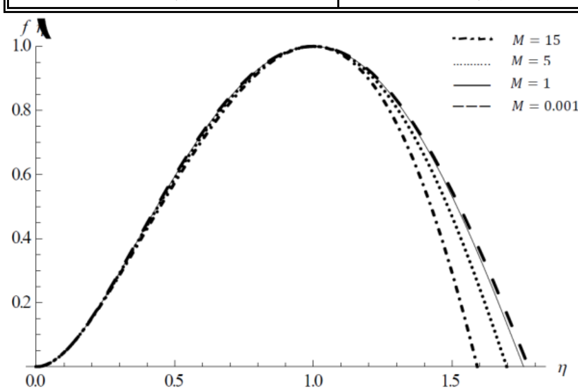


Figure.5 - Normal velocity distribution with $\mathrm{Re}=10, h_1=-0.6, M=0.001,1,5,15$.

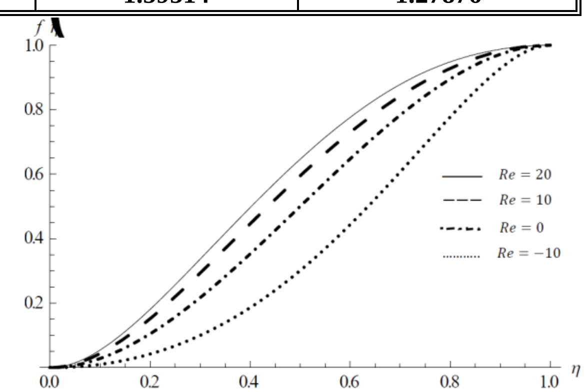


Figure.6 - Normal velocity distribution with $M=1, h_1=-0.6, \mathrm{Re}=-10,0,10,20$

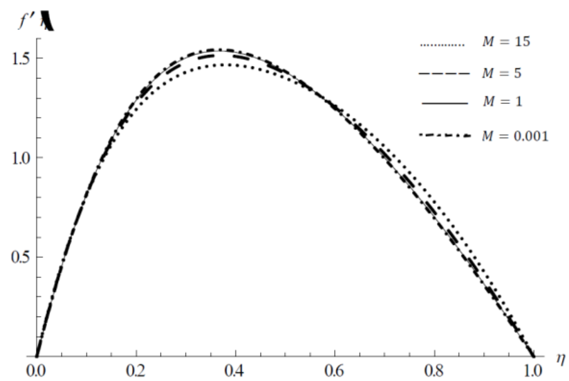


Figure.7 - Tangential velocity distribution with $\mathrm{Re}=10, h_1=-0.6, M=0.001,1,5,15$

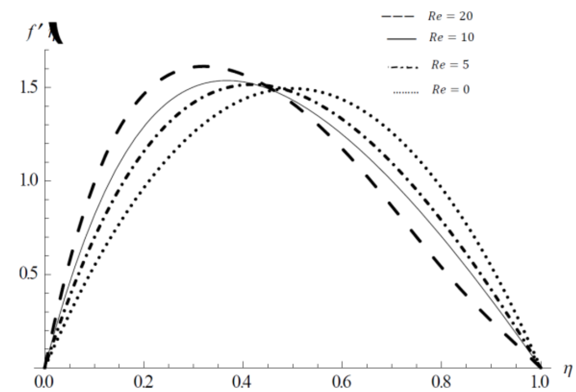


Figure.8 - Tangential velocity distribution with $\mathrm{Re}=20, h_1=-0.6, M=0.001,1,5,15$.

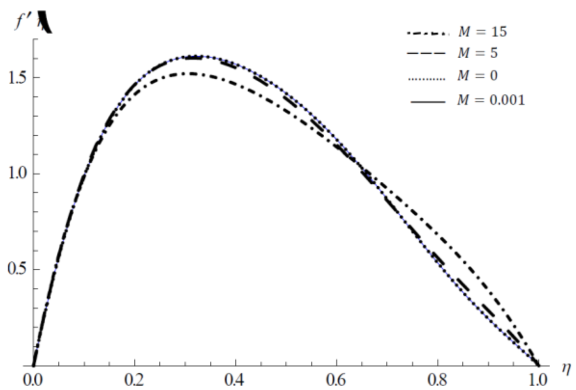


Figure.9 - Tangential velocity distribution with $M=1, h_1=-0.6, \mathrm{Re}=0,5,10,20$.

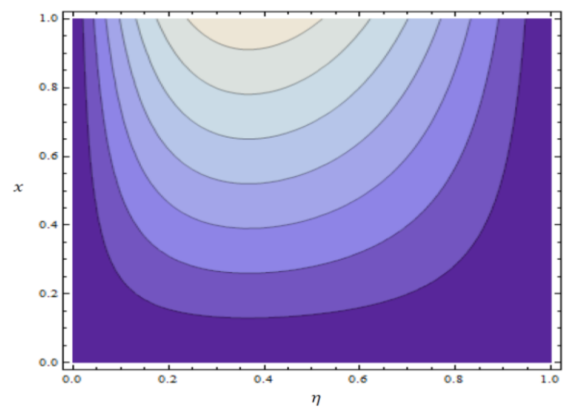


Figure.10 - Tangential velocity contours for $\mathrm{Re}=10$.

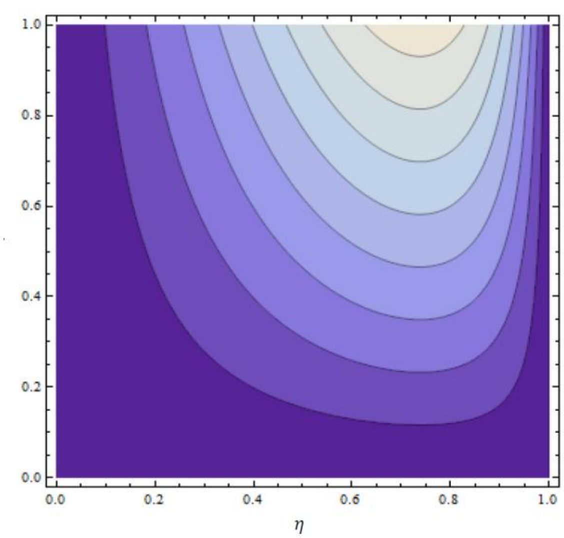


Figure.11 - Tangential velocity contours for $\mathrm{Re}=-10$.

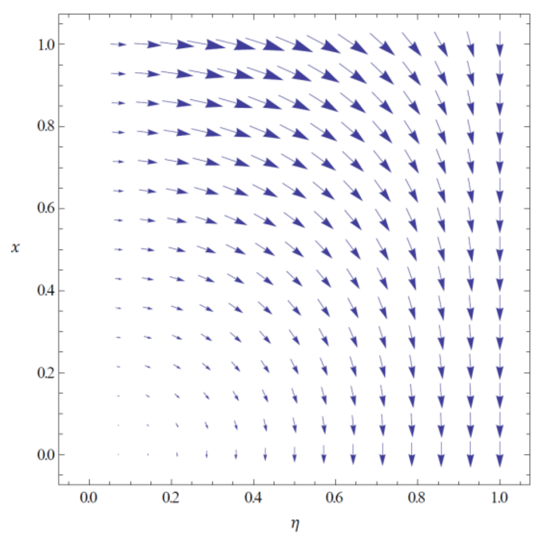


Figure.12 - Tangential velocity vectors for $\mathrm{Re}=10$.

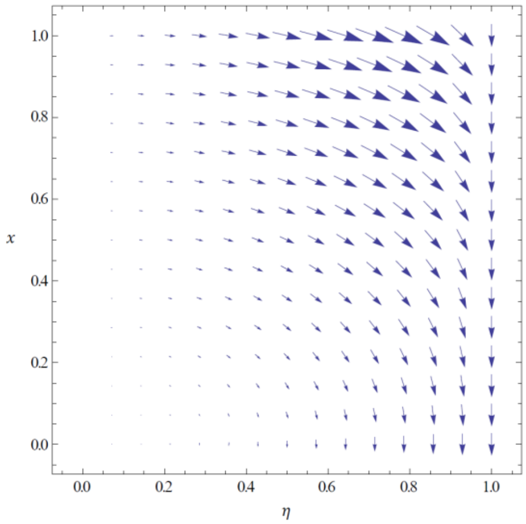


Figure.13 - Tangential velocity contours for $\mathrm{Re} = -10$.

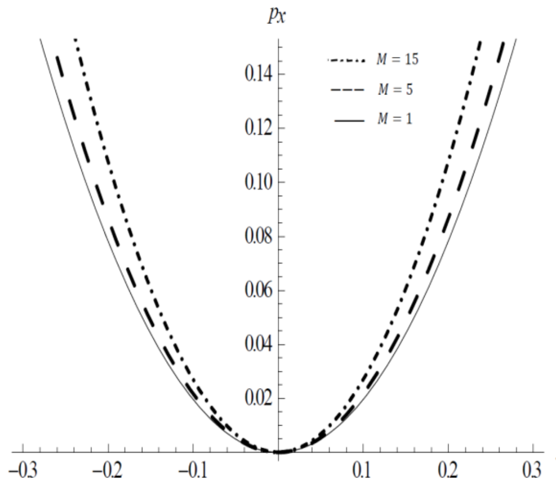


Figure.14 - Pressure distribution with $\mathrm{Re} = 10, h_1 = -0.6$, $M = 1,5,15$.

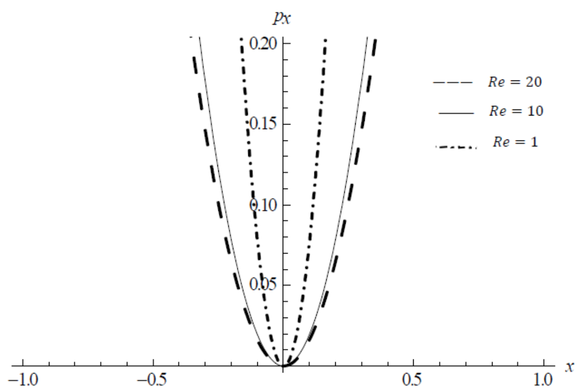


Figure.15 - Pressure distribution with $M = 1, h_1 = -0.6$, $\mathrm{Re} = 1,10,20$.

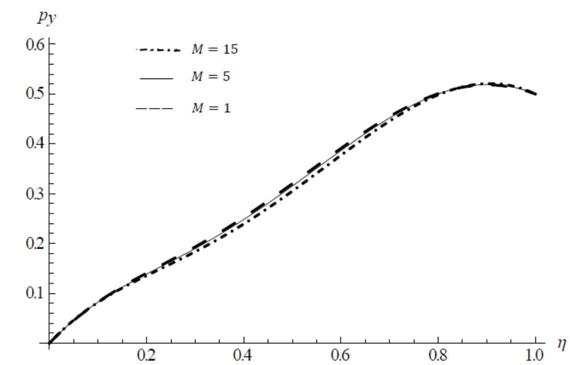


Figure.16 - Pressure distribution with $\mathrm{Re} = 10, h_1 = -0.6$, $M = 1,5,15$.

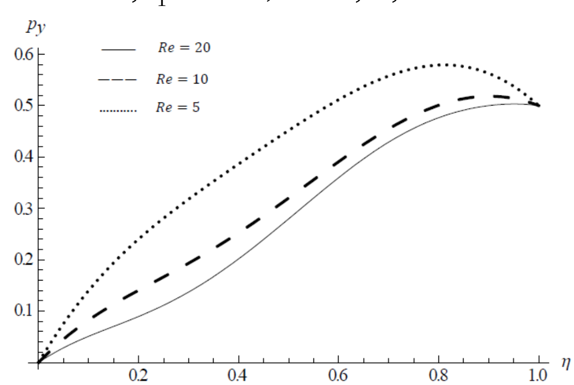


Figure.17 - Pressure distribution with $M = 1, h_1 = -0.6$, $\mathrm{Re} = 1,10,20$

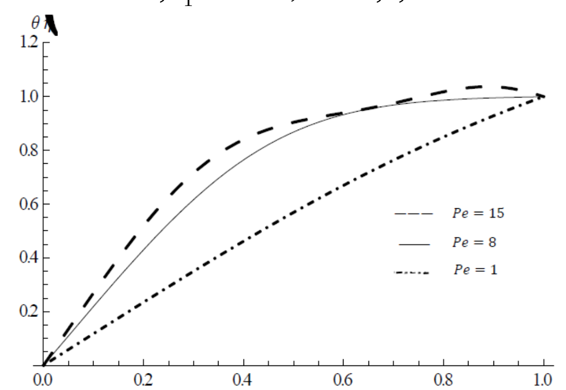


Figure.18 - Temperature distribution with $\mathrm{Re} = 10, h_2 = -0.6$, $M = 1$, $Pe = 1,8,15$.

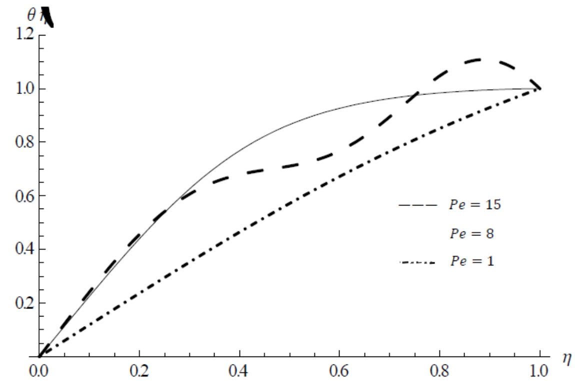


Figure.19 - Temperature distribution with
$\mathrm{Re}=20, h_2=-0.6, M=1, Pe=1,8,15$.

Results and discussion

Utilizing the analytical solutions, calculations are performed to investigate the effect of MHD parameter "M", Reynolds number "Re" and Peclet number "Pe" on fluid flow in a channel with porous wall. And we get the following results:

- As Hartmann number increase, there is small decreasing in the normal velocity range see figure-5, and have different behavior among $\eta=0.4$ to $\eta=1$ enhances the minimum of the tangential velocity and moves away from porous wall, see figure-7 and 8. And pressure is increase in x-direction see figure-14 and decrease in y-direction see figure-16.
- As Reynolds number increase, there is increasing in the normal velocity see figure-6, and tangential velocity have different behavior i.e. enhances the maximum of tangential velocity and moves it away from porous wall, see figure-9. Regarding to the figures-10 ,11 ,12 and 13 the position of maximum tangential velocity gets closer to porous surface when Reynolds number has a negative value. And if Reynolds number increase the pressure is decreasing see figures-15 and 17.
- As Peclet number increase, there is increasing in temperature range, see figures-18 and 19.

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