Iraqi Journal of Science, 2025, Vol. xx, No. x, pp: xx
DOI: 10.24996/ijs.2025.66.2.15

ISSN: 0067-2904

Oscillation and Convergence of Solutions of Second Order Neutral Differential Equations with Periodic Coefficients

Adyar Kamel Neghmish , Hussain Ali Mohamad*
Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq

Received: 21/11/2023   Accepted: 17/2/2024   Published: xx

Abstract

This research aims to study the behavior of solutions of second-order neutral differential equations with periodic coefficients. Some necessary and sufficient conditions have been obtained that classify all solutions of these equations into three categories: either oscillatory, non-oscillatory, and convergent to zero, or non-oscillatory and divergent. The extent to which periodic coefficients influence the occurrence of oscillation, convergence, or divergence for each solution has been explained. The equation under consideration contained a variable delay and a constant delays in which the coefficients are periodic. Not much previous research has discussed the oscillation of solutions of second-order neutral equations with periodic coefficients. In each case, some illustrative examples have been provided that illustrate the ease of achieving the conditions for the obtained results.

Keywords: Oscillation, Neutral differential equations, Periodic coefficients, Second Order.

تذبذب و تقارب حلول المعادلات التفاضلية المحايدة من الرتبة الثانية ذات المعاملات الدورية

ديار كامل نغميش , حسين علي محمد*
قسم الرياضيات، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق

الخلاصة

يهدف هذا البحث إلى دراسة سلوك حلول المعادلات التفاضلية المحايدة من الرتبة الثانية ذات المعاملات الدورية. لقد تم الحصول على بعض الشروط الضرورية والكافية التي تصنف جميع حلول هذه المعادلات إلى ثلاث فئات: إما متذبذبة او غير متذبذبة ومتقاربة إلى الصفر، أو غير متذبذبة ومتباعدة. وتم توضيح مدى تأثير المعاملات الدورية على حدوث التذبذب أو التقارب أو التباعد لكل حل. تحتوي المعادلة قيد النظر على تباطؤ متغير وتباطؤ ثابت تكون فيه المعاملات دورية. لم تناقش الكثير من الأبحاث السابقة تذبذب حلول المعادلات المحايدة من الرتبة الثانية ذات المعاملات الدورية. وفي كل حالة تم تقديم بعض الأمثلة التوضيحية التي توضح سهولة تحقيق شروط النتائج المتحصل عليها.

1. Introduction

Consider the neutral equation with periodic coefficients (NEPC):

*Email: hussainam_math@csw.uobaghdad.edu.iq

$$[\varphi(t)-p(t)\varphi(\tau(t))]'' +\sum_{i=1}^{n} Q_i(t)\,\varphi(t-\sigma_i)=0. \qquad (1.1)$$

$p\in C[[t_0,\infty);\ R^+],\ \ \tau, Q_i\in C[[t_0,\infty); R\ ], \tau(t)$ is increasing function and $\lim_{t\to\infty}\tau(t)=\infty$, Q_i are periodic of period $\sigma,\ \sigma_i=m_i\,\sigma,\ m_i\in N,\ i=1,2,\cdots,n$.
$\sigma_{min}=\min\{\sigma_i,\ i=1,2,\cdots,n\}>0,\ \sigma_{max}=\max\{\sigma_i,\ i=1,2,\cdots,n\},\ Q(t)\not\equiv 0$ for $t\geq 0$. Let

$$\psi(t)=\varphi(t)-p(t)\varphi\big(\tau(t)\big),\ \ t\geq t_0. \qquad (1.2)$$

A function $\varphi(t)$ is said to be a solution of eq.(1.1) if $\varphi(t)-p(t)\varphi(\tau(t))\in \mathrm{C}^2[[\tau(t_0),\infty);R]$ and $\varphi(t)$ satisfies eq.(1.1). A solution $\varphi(t)$ is said to be oscillate, if there exists a sequence $\{t_m\}, t_m\to\infty$ as $m\to\infty$ such that $\varphi(t_m)=0$ [3]. As we indicated in the abstract there is a paucity of published research on the topic of oscillation of solutions of neutral equations with periodic coefficients. However, some papers that discussed this topic or similar to it in somewhat will be presented. Barnes [1] obtained conditions on the coefficients that ensure that all solutions of the differential equation are bounded, based on the well-known result, which states that all solutions of the Hill equation $\ddot{y}+p(t)y=0$ are bounded, if $p(t+T)=p(t)>0$, and if $\int_0^T p(t)dt<\frac{4}{T}$. Cheng et al. [2] discussed a type of neutral second-order differential equation with delay and variable coefficient: $(x(t)-a(\theta)\varphi(\theta-\rho(\theta)))''+b(\theta)\varphi(\theta)=f(\theta,\varphi(\theta-\gamma(\theta)))$. They used Krasnoselsky's fixed point theorem and the properties of the neutral factor $(A\varphi)(\theta):=\varphi(\theta)-a(t)\varphi(\theta-\rho(\theta))$ and some sufficient conditions for the existence of periodic solutions were reached. Ladas et al.[3] studied (NEPC) and prove that the equation

$$[\varphi(t)-p\varphi(t-\tau)]' +Q(t)\varphi(t-\sigma)=0, \qquad (E1)$$

oscillates, if and only if $[\omega(t)-p\omega(t-\tau)]'+\omega(t-\sigma)=0$ oscillates. Liu et al. [4] investigated the existence of oscillatory solutions of forced second-order nonlinear differential equations where $\Phi(v)=v^{\eta}$ is an increasing function η is the ratio of two positive odd integers. Some sufficient conditions for the global existence of the oscillatory solution are obtained by the Schauder–Tychonoff theorem. Mohamad et al.[5] studied the oscillation property of the second order half linear dynamic equation, where some sufficient conditions were obtained to ensure the oscillation of all solutions of that equation. Mohamad et al.[6] discussed the oscillation criteria for nonlinear systems of neutral differential equations. Sufficient conditions were obtained to ensure that all bounded solutions to this system either oscillate or converge to zero as $t\to\infty$. Neghmish et al. [7] discussed the oscillation of solutions of neutral equations of the first order with periodic coefficients of period σ_i. Sufficient conditions have been obtained to ensure that all solutions of equations with periodic coefficients oscillate. Qaraada et al. [8] standards for oscillation were obtained by studying the oscillatory behavior of solutions of third-order equations of the form: $\big(f(t)\big(y''(t)\big)^{\gamma}\big)'+\int_a^b p(t,v)x^{\gamma}\big(s(t,v)\big)\,dv=0$, where $y(t)=x(t)+p(t)x\big(\tau(t)\big)$. Tunç et al. [9] established some specific assumptions that guarantee the asymptotic stability of a trivial solution of a neutral linear differential equation with periodic coefficients, also they estimated the decay rate of the solutions of the equation under consideration to reach the desired results. Li et al. [10] studied the oscillatory behavior of a class of second-order neutral differential equations under the assumptions that allow applications to differential equations with both delayed and advanced arguments. Yoshida [11] showed that the characteristic criteria can be expressed precisely for a type of second-order linear ordinary differential equation with periodic coefficients (Hill's equation) which appear as covariant equations for some periodic solutions of dynamical systems. The main purpose of this paper is to obtain sufficient conditions that guarantee the oscillation or convergence of all solutions of second-order neutral differential equations with periodic coefficients and to demonstrate the effect of the periodicity of these coefficients on oscillation.

2. Main results

In this section, some results were presented that guarantee either the oscillation of all solutions of equation (1.1) or the convergence of non-oscillation solutions to zero or divergence.

Theorem 1. *Assume that* $Q_i(t) \geq 0$, $0 \leq p(t) \leq p_0 < 1$, $\tau(t) \leq t$ *and*

$$n\hat{\sigma} > 1, \quad \hat{\sigma} = \min_i \int_0^{\sigma} Q_i(t)\,dt\,, \quad i = 1,2,\dots,n. \tag{2.1}$$

Then every solution $\varphi(t)$ of eq. (1.1) either oscillates or tends to zero or $|\varphi(t)| \to \infty$ as $t \to \infty$.

Proof. Suppose that $\varphi(t)$ be a nonoscillatory solution of eq.(1.1), and let $\varphi(t) > 0$, $\varphi(\tau(t)) > 0$, $\varphi(t-\sigma_i) > 0$, $t \geq t_0$ then

$$\psi''(t) = -\sum_{i=1}^{n} Q_i(t)\,\varphi(t-\sigma_i) \leq 0, \ \ t \geq t_0. \tag{2.2}$$

Hence $\psi'(t)$ is nonincreasing function, so either $\psi'(t) < 0$ or $\psi'(t) > 0$ for $t \geq t_1 \geq t_0$.
Case 1. If $\psi'(t) < 0$, $t \geq t_1$, then there is $t_2 \geq t_1$ such that $\psi(t) < 0$ and $\lim_{t\to\infty} \psi(t) = -\infty$, then from (1.2) yields
$\varphi(t) < p(t)\varphi(\tau(t)) \leq \varphi(\tau(t))$, $t \geq t_1$ that is $\varphi(t)$ is positive decreasing, so it is bounded, on the other side (1.2) leads to $\psi(t) \geq -p(t)\varphi\big(\tau(t)\big) > -\varphi(\tau(t))$ that is

$$\varphi\big(\tau(t)\big) > -\,\psi(t)$$

Hence $\lim_{t\to\infty} \varphi(t) = \infty$, this leads to a contradiction, so case 1, cannot be occurred.
Case 2. Let $\psi'(t) > 0$, $t \geq t_1$, so either $\psi(t) < 0$ or $\psi(t) > 0, t \geq t_2 \geq t_1$, thus there are two possible subcases to consider:
Case 2.1. $\psi(t) < 0$, $t \geq t_2 \geq t_1$, let $\lim_{t\to\infty} \psi(t) = L \leq 0$, we claim that $L = 0$, otherwise, $\psi(t) \leq L < 0$, then there is $t_3 \geq t_2$ such that (1.2) leads to
$\psi(t) \geq -p(t)\varphi(\tau(t)) \geq -\varphi(\tau(t))$, that is $-\varphi(t) \leq \psi\big(\tau^{-1}(t)\big)$,

$$-\,\varphi(t-\sigma_i) \leq \psi(\tau^{-1}(t-\sigma_i))\,, \ \ t \geq t_3, \ \ i = 1,2,\dots,n. \tag{2.3}$$

Therefore eq. (1.1) is reduced to

$$\psi''(t) = -\sum_{i=1}^{n} Q_i(t)\,\varphi(t-\sigma_i) \leq \sum_{i=1}^{n} Q_i(t)\psi\big(\tau^{-1}(t-\sigma_i)\big), \tag{2.4}$$

$$\psi''(t) \leq L\sum_{i=1}^{n} Q_i(t)\,, \ \ t \geq t_3. \tag{2.5}$$

Integrating inequality (2.5) from t to $t+\sigma$ yields

$$\psi'(t+\sigma) - \psi'(t) \leq L\int_t^{t+\sigma} \sum_{i=1}^{n} Q_i(s)\,ds,$$

$$\psi'(t) \geq -Ln\hat{\sigma}, \ \ \text{where} \ \ \hat{\sigma} = \min_i \int_0^{\sigma} Q_i(t)\,dt\,, i = 1,2,\dots,n.$$

Integrating the last inequality from t_3 to t we get

$$\psi(t) - \psi(t_3) \geq -Ln\hat{\sigma}(t-t_3), \tag{2.6}$$

Letting $t \to \infty$, one can conclude that (2.6) implies $\lim_{t\to\infty} \psi(t) = \infty$, we get a contradiction. Hence $L = 0$, since $\psi(t) < 0$ it follows that $\varphi(t) \leq p(t)\varphi\big(\tau(t)\big) < \varphi\big(\tau(t)\big)$, so $\varphi(t)$ is decreasing function and positive, let $\lim_{t\to\infty} \varphi(t) = l \geq 0$, $\psi(t) = \varphi(t) - p(t)\varphi\big(\tau(t)\big) \geq \varphi(t) - p_0\varphi\big(\tau(t)\big)$,
letting $t \to \infty$, we obtain $0 \geq (1-p_0)l$, which is possible only when $l = 0$.

Case 2.2. $\psi(t) > 0, \psi'(t) > 0,\ \ t \geq t_1 \geq t_0$, in this case there is $k > 0$ such that $\psi(t) \geq k$, from (1.2)

$$\psi(t) \leq \varphi(t),\ \ t \geq t_2 \geq t_1, \tag{2.7}$$

then eq. (1.1) is reduced to

$$\psi''(t) \leq -\sum_{i=1}^{n} Q_i(t)\psi(t-\sigma_i) \leq -k\sum_{i=1}^{n} Q_i(t). \tag{2.8}$$

Integrating (2.8) from t to $t+\sigma$ to get

$$\psi'(t+\sigma) - \psi'(t) \leq -k\int_t^{t+\sigma}\sum_{i=1}^{n} Q_i(s)\, ds,$$

$$\psi'(t) \geq kn\hat{\sigma}$$

Integrating from t_2 to t to get

$$\psi(t) - \psi(t_2) \geq kn\hat{\sigma}(t-t_2)$$

As $t \to \infty$ one can get that

$\lim_{t\to\infty}\psi(t) = \infty$, hence (2.7) implies $\lim_{t\to\infty}\varphi(t) = \infty$. □

Theorem 2. Suppose that $Q_i(t) \leq 0, 1 < p_0 \leq p(t) \leq k,\ \tau(t) \geq t$ and

$$n\breve{\sigma} > 1, \quad \breve{\sigma} = \min_i \int_0^{\sigma} |Q_i(t)|\, dt\,,\ \ i = 1, \dots, n. \tag{2.9}$$

Then every solution $\varphi(t)$ of eq. (1.1) either oscillates or tends to zero or $|\varphi(t)| \to \infty$ as $t \to \infty$.

Proof. Suppose that eq.(1.1) has non-oscillatory solution $\varphi(t)$, let $\varphi(t) > 0,\ \varphi(\tau(t)) > 0$, $\varphi(t-\sigma_i) > 0,\ i = 1, \dots, n\,, t \geq t_0$. Then

$$\psi''(t) = \sum_{i=1}^{n} |Q_i(t)|\,\varphi(t-\sigma_i) \geq 0, \tag{2.10}$$

Hence $\psi'(t)$ is nondecreasing function, so either $\psi'(t) > 0$ or $\psi'(t) < 0$ for $t \geq t_1 \geq t_0$.

Case 1. If $\psi'(t) > 0,\ t \geq t_1$, then there is $t_2 \geq t_1$ such that $\psi(t) > 0$ and $\lim_{t\to\infty}\psi(t) = \infty$. From (1.2) one can obtain $\varphi(t) \geq p(t)\varphi(\tau(t)) \geq \varphi(\tau(t))$, so $\varphi(t)$ is nonincreasing, but on the other side $\psi(t) \leq \varphi(t)$ which implies $\lim_{t\to\infty}\varphi(t) = \infty$, we get a contradiction. Therefore, case 1 cannot occur.

Case 2. Let $\psi'(t) < 0,\ t \geq t_1$, so either $\psi(t) < 0$ or $\psi(t) > 0, t \geq t_2 \geq t_1$, thus there are two possible subcases to consider:

Case 2.1. $\psi(t) > 0,\ t \geq t_2 \geq t_1$, let $\lim_{t\to\infty}\psi(t) = L \geq 0$, we claim that $L = 0$, otherwise $\psi(t) \geq L > 0$, then there is $t_3 \geq t_2$ such that (1.2) yields

$\psi(t) \leq \varphi(t)$, that is $\psi(t-\sigma_i) \leq \varphi(t-\sigma_i)$.

From equation (2.10) the following can be concluded

$$\psi''(t) = \sum_{i=1}^{n} |Q_i(t)|\,\varphi(t-\sigma_i) \geq \sum_{i=1}^{n} |Q_i(t)|\,\psi(t-\sigma_i), \tag{2.11}$$

$$\psi''(t) \geq L\sum_{i=1}^{n} |Q_i(t)|\,,\ \ t \geq t_3. \tag{2.12}$$

Integrating inequality (2.12) from t to $t+\sigma$

$$\psi'(t+\sigma) - \psi'(t) \geq L\int_t^{t+\sigma}\sum_{i=1}^{n} |Q_i(s)|\, ds,$$

$$-\psi'(t) \geq Ln\breve{\sigma} \quad \text{where} \quad \breve{\sigma} = \min_i \int_0^{\sigma} |Q_i(t)|\, dt\,,\ \ i = 1, \dots, n.$$

$$\psi'(t) \leq -Ln\breve{\sigma},\quad t \geq t_3. \tag{2.13}$$

Integrating inequality (2.13) from t to $t+\breve{\sigma}$ we get

$$\psi(t+\breve{\sigma}) - \psi(t) \leq -Ln\breve{\sigma}^2$$

$$\psi(t) \geq Ln\breve{\sigma}^2, \;\; t \geq t_4 \geq t_3. \tag{2.14}$$

Substituting (2.14) in (2.11) to obtain

$$\psi''(t) \geq Ln\breve{\sigma}^2 \sum_{i=1}^{n} Q_i(t), \;\; t \geq t_3 \geq t_2.$$

Repeating this procedure m times to get

$$\psi(t) \geq Ln^m\breve{\sigma}^{m+2}. \tag{2.15}$$

Letting $m \to \infty$, inequality (2.15) leads to $\lim_{t\to\infty} \psi(t) = \infty$ a contradiction since $\psi(t)$ is decreasing. Hence $L = 0$, since $\psi(t) > 0$ it follows that $\varphi(t) \geq p(t)\varphi(\tau(t)) > \varphi(\tau(t))$, so $\varphi(t)$ is decreasing function, let $\lim_{t\to\infty} \varphi(t) = l \geq 0$,

$$\psi(t) = \varphi(t) - p(t)\varphi(\tau(t)) \leq \varphi(t) - p_0\varphi(\tau(t)),$$

Letting $t \to \infty$, we obtain $0 \leq (1 - p_0)l$, which is possible only when $l = 0$.

Case 2.2. $\psi(t) < 0$, $\psi'(t) < 0$, $t \geq t_2 \geq t_1$, there exist $b > 0$, such that $\psi(t) \leq -b$, from (1.2)

$\psi(t) \geq -p(t)\varphi(\tau(t))$, $t \geq t_2 \geq t_1$, that is $\varphi(\tau(t)) \geq -\frac{\psi(t)}{p(t)}$, $t \geq t_2 \geq t_1$, or

$$\varphi(t) \geq -\frac{\psi(\tau^{-1}(t))}{p(\tau^{-1}(t))}, \;\; t \geq t_2 \geq t_1. \tag{2.16}$$

then eq. (2.10) is reduced to

$$\psi''(t) \geq -\sum_{i=1}^{n} |Q_i(t)| \frac{\psi(\tau^{-1}(t-\sigma_i))}{p(\tau^{-1}(t-\sigma_i))} \geq \frac{b}{k}\sum_{i=1}^{n} |Q_i(t)|. \tag{2.17}$$

Integrating (2.17) from t to $t + \sigma$ to get

$$\psi'(t+\sigma) - \psi'(t) \geq \frac{b}{k}\int_t^{t+\sigma} \sum_{i=1}^{n} |Q_i(s)|\, ds,$$

$$\psi'(t) \leq -\frac{b}{k}n\breve{\sigma}$$

Integrating from t_2 to t to get

$$\psi(t) - \psi(t_2) \leq -\frac{b}{k}n\breve{\sigma}(t - t_2)$$

As $t \to \infty$ one can get $\lim_{t\to\infty} \psi(t) = -\infty$. From (2.16) can be obtained

$\varphi(t) \geq -\frac{\psi(\tau^{-1}(t))}{k}$, $t \geq t_2$, which implies $\lim_{t\to\infty} \varphi(t) = \infty$. □

Theorem 3. *Assume that* $Q_i(t) \geq 0$, $1 < p_0 \leq p(t) \leq k$, $\tau(t) \leq t$ *and*

$$\prod_{i=0}^{\infty} p(\tau^{-i}(T)) < \infty, \;\; T \geq t_0. \tag{2.18}$$

Then every solution $\varphi(t)$ *of eq.(1.1) either oscillates or tends to zero or* $\lim_{t\to\infty} \varphi(t) = \infty$.

Proof. Suppose that $\varphi(t)$ be a non-oscillatory solution of eq.(1.1), and let $\varphi(t) > 0$, $\varphi(\tau(t)) > 0$, $\varphi(t - \sigma_i) > 0$ then

$$\psi''(t) = -\sum_{i=1}^{n} Q_i(t)\,\varphi(t - \sigma_i) \leq 0, \qquad t \geq t_0.$$

Hence $\psi'(t)$ is nonincreasing function, so either $\psi'(t) < 0$ or $\psi'(t) > 0$ for $t \geq t_1 \geq t_0$.

Case 1. If $\psi'(t) < 0$, $t \geq t_1$, hence $\psi'(t) < 0$, $\psi(t) < 0$ and $\lim_{t\to\infty} \psi(t) = -\infty$, then there is $t_2 \geq t_1$ and $\delta > 0$ such that $\psi(t) \leq -\delta$, $t \geq t_2$. From (1.2) it follows

$\varphi(t_2) = \psi(t_2) + p(t_2)\varphi(\tau(t_2)) \leq -\delta + p(t_2)\varphi(\tau(t_2)$

$\varphi(\tau^{-1}(t_2)) \leq -\delta + p(\tau^{-1}(t_2))\varphi(t_2) \leq -\delta + p(\tau^{-1}(t_2))[-\delta + p(t_2)\varphi(\tau(t_2))]$

$\varphi(\tau^{-1}(t_2)) \leq -\delta(1 + p(\tau^{-1}(t_2)) + p\,(\tau^{-1}(t_2))p(t_2)\varphi(\tau(t_2)),$

$$\varphi\big(\tau^{-2}(t_2)\big) \le -\delta[1 + p\big(\tau^{-2}(t_2)\big) + p\big(\tau^{-2}(t_2)\big)p\big(\tau^{-1}(t_2)\big)] + p\big(\tau^{-2}(t_2)\big)p\big(\tau^{-1}(t_2)\big)p(t_2)\varphi\big(\tau(t_2)\big)$$

$$\varphi\big(\tau^{-m}(t_2)\big) \le -\delta(1 + \sum_{i=0}^{m-1}\prod_{j=0}^{m-i-1} p\big(\tau^{j-m}(t_2)\big) + \varphi\big(\tau(t_2)\big)\prod_{i=0}^{m} p\big(\tau^{-i}(t_2)\big) \quad (2.19)$$

As $m \to \infty$ then it follows from (2.19), $\lim_{m\to\infty} \varphi\big(\tau^{-m}(t_2)\big) = -\infty$, which is a contradiction.

Case 2. Let $\psi'(t) > 0,\ t \ge t_1,$ so either $\psi(t) > 0$ or $\psi(t) < 0, t \ge t_2 \ge t_1$.

Case 2.1 $\psi(t) > 0, t \ge t_2,$ so there is a constant $b > 0$ such that $\psi(t) \ge b$, from (1.2) it follows that

$$\varphi(t) \ge p(t)\varphi\big(\tau(t)\big) > \varphi(\tau(t)),\ \ t \ge t_2.$$

Hence $\varphi(t)$ is increasing for $t \ge t_2,$ also from (1.2) we obtain
$\varphi(t) \ge \psi(t),$ then eq. (1.1) is reduced to

$$\psi''(t) = -\sum_{i=1}^{n} Q_i(t)\,\varphi(t-\sigma_i) \le -\sum_{i=1}^{n} Q_i(t)\,\psi(t-\sigma_i). \quad (2.20)$$

$$\psi''(t) \le -b\sum_{i=1}^{n} Q_i(t)\,, t \ge t_3 \ge t_2\,. \quad (2.21)$$

Integrating (2.21) from t to $t+\sigma$ to get

$$\psi'(t+\sigma) - \psi'(t) \le -b\int_t^{t+\sigma}\sum_{i=1}^{n} Q_i(s)ds$$

$$\psi'(t) \ge b\sum_{i=1}^{n}\int_0^{\sigma} Q_i(s)ds \ge nb\hat{\sigma}. \quad (2.22)$$

Where $\hat{\sigma} = \min_i \int_0^{\sigma} Q_i(s)\,ds, i = 1, \dots, n.$ Integrating (2.22) from t_3 to t to obtain

$$\psi(t) - \psi(t_3) \ge nb\hat{\sigma}(t - t_3)$$

As $t \to \infty$ one can find that $\psi(t) \to \infty$, since $\varphi(t) \ge \psi(t),$ hence $\varphi(t) \to \infty$.

Case 2.2. $\psi(t) < 0,\ t \ge t_1,$ let $\lim_{t\to\infty}\psi(t) = L \le 0,$ we claim that $L = 0,$ otherwise $\psi(t) \le L < 0,$ then there is $t_2 \ge t_1$ such that from (1.2) yields

$\psi(t) \ge -p\,(t)\varphi(\tau(t) \ge -k\varphi(\tau(t)),$ that is $-\varphi(t) \le \frac{\psi\left(\tau^{-1}(t)\right)}{k},$

$$-\varphi(t-\sigma_i) \le \frac{\psi\big(\tau^{-1}(t-\sigma_i)\big)}{k},\ \ t \ge t_2,\ \ i = 1,2,\dots,n.$$

Then eq.(1.1) leads to

$$\psi''(t) = -\sum_{i=1}^{n} Q_i(t)\,\varphi(t-\sigma_i) \le \sum_{i=1}^{n} Q_i(t)\frac{\psi\big(\tau^{-1}(t-\sigma_i)\big)}{k}$$

$$\psi''(t) \le \frac{L}{k}\sum_{i=1}^{n} Q_i(t)\,, t \ge t_2. \quad (2.23)$$

Integrating (2.23) from t to $t+\sigma$ to get

$$\psi'(t+\sigma) - \psi'(t) \le \frac{L}{k}\int_t^{t+\sigma}\sum_{i=1}^{n} Q_i(s)ds$$

$$\psi'(t) \ge -\frac{L}{k}\sum_{i=1}^{n}\int_0^{\sigma} Q_i(s)ds \ge -\frac{nL\hat{\sigma}}{k}. \quad (2.24)$$

Integrating (2.24) from t_2 to t to get

$$\psi(t) - \psi(t_2) \ge -\frac{nL\hat{\sigma}}{k}(t - t_2)$$

As $t \to \infty$ one can get that $\lim_{t\to\infty}\psi(t) = \infty$, a contradiction, hence $L = 0$.

$\varphi(t) = \psi(t) + p(t)\varphi(\tau(t)) \ge \psi(t) + p_0\varphi(\tau(t)),$

$$\varphi(\tau^{-1}(t)) \ge \psi\big(\tau^{-1}(t)\big) + p_0\varphi(t). \quad (2.25)$$

Let $\limsup_{t\to\infty} \varphi(t) = l \geq 0$, so there exists a sequence $\{t_n\}, t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \varphi(t_n) = l$, hence (2.25) leads to

$$\varphi(\tau^{-1}(t_n)) \geq \psi\big(\tau^{-1}(t_n)\big) + p_0\varphi(t_n).$$

As $n \to \infty$ it follows that $l \geq p_0 l$, that is $(1-p_0)l \geq 0$, which is possible only when $l = 0$. Thus $\lim_{t\to\infty} \varphi(t) = 0$. □

Theorem 4. *Assume that* $Q_i(t) \leq 0$, $0 < p(t) \leq p_0 < 1$, $\tau(t) \geq t$ *and*

$$\limsup_{m\to\infty} \left(\prod_{i=0}^{m} \frac{1}{p\big(\tau^{i-1}(T)\big)} - (m+1)\delta \right) < \infty,\ \ T \geq t_0. \qquad (2.26)$$

Then every solution $\varphi(t)$ *of eq.(1.1) either oscillates or tends to zero or* $\lim_{t\to\infty} |\varphi(t)| = \infty$.

Proof. Suppose that $\varphi(t)$ be a nonoscillatory solution of eq.(1.1), and let $\varphi(t) > 0$, $\varphi(\tau(t) > 0$, $\varphi(t-\sigma_i) > 0, t \geq t_0$ then eq.(1.1) reduce to

$$\psi''(t) = \sum_{i=1}^{n} |Q_i(t)|\, \varphi(t-\sigma_i) \geq 0,\ \ t \geq t_0.$$

Hence $\psi'(t)$ is nondecreasing function, so either $\psi'(t) > 0$ or $\psi'(t) < 0$ for $t \geq t_1 \geq t_0$.

Case 1. If $\psi'(t) > 0$, $t \geq t_1$, hence $\psi(t) > 0$ and $\lim_{t\to\infty} \psi(t) = \infty$, then there is $t_2 \geq t_1$ and $\delta > 0$, such that $\psi(t) \geq \delta$, $t \geq t_2$. From (5.2) it follows $p(t_2)\,\varphi\big(\tau(t_2)\big) = \varphi(t_2) - \psi(t_2)$

$$\varphi\big(\tau(t_2)\big) = \frac{1}{p(t_2)}[\varphi\,(t_2) - \psi(t_2) = \frac{1}{p(t_2)}\,\varphi(t_2) - \frac{1}{p(t_2)}\,\psi(t_2) \leq \frac{1}{p(t_2)}\,\varphi(t_2) - \delta,$$

$$\varphi(t_2) \leq \frac{1}{p\big(\tau^{-1}(t_2)\big)}\,\varphi\big(\tau^{-1}(t_2)\big) - \delta.$$

$$\varphi\big(\tau(t_2)\big) \leq \frac{1}{p(t_2)}\,\varphi(t_2) - \delta,$$

$$\leq \frac{1}{p(t_2)}\left[\frac{1}{p\big(\tau^{-1}(t_2)\big)}\,\varphi\big(\tau^{-1}(t_2)\big) - \delta\ \right] - \delta,$$

$$\leq \frac{1}{p(t_2)p\big(\tau^{-1}(t_2)\big)}\,\varphi\big(\tau^{-1}(t_2)\big) - 2\delta,$$

$$\varphi\Big(\tau\big(\tau(t_2)\big)\Big) \leq \frac{1}{p(t_2)p\big(\tau(t_2)\big)}\,\varphi(t_2) - 2\delta,$$

$$\varphi\big(\tau^2(t_2)\big) \leq \frac{1}{p(t_2)p\big(\tau(t_2)\big)p\big(\tau^{-1}(t_2)\big)}\,\varphi\big(\tau^{-1}(t_2)\big) - 3\delta.$$

Repeating this procedure m times to get

$$\varphi\big(\tau^m(t_2)\big) \leq \varphi\big(\tau^{-1}(t_2)\big) \prod_{i=0}^{m} \frac{1}{p\big(\tau^{i-1}(t_2)\big)} - (m+1)\delta. \quad (2.27)$$

As $m \to \infty$, taking into account condition (2.26), it can be concluded that (5.27) implies $\limsup_{t\to\infty} \varphi(t) < \infty$. On the other side, $\lim_{t\to\infty} \psi(t) = \infty$ implies that $\lim_{t\to\infty} \varphi(t) = \infty$, we get a contradiction.

Case 2 If $\psi'(t) < 0, t \geq t_1$, so either $\psi(t) < 0$ or $\psi(t) > 0, t \geq t_2 \geq t_1$, thus there are two possible subcases to consider.

Case 2.1 $\psi(t) < 0, t \geq t_2$, in this case, in similar way as pervious cases one can get $\lim_{t\to\infty} \psi(t) = -\infty$, which implies $\lim_{t\to\infty} \varphi(t) = \infty$.

Case 2.2 $\psi(t) > 0$, $t \geq t_2$, this case is similar to case 2.1 in theorem 2, so it can be obtained $\lim_{t\to\infty} \psi(t) = 0$, which implies $\lim_{t\to\infty} \varphi(t) = 0$.

3. Examples

In this section, some examples were given to illustrate the obtained results

Example 1. Consider the equation with periodic coefficient

$$\left[\varphi(t) - \left(1 - \tfrac{1}{2}\sin^2 t\right)\varphi(t-2\pi)\right]'' + 10\cos^2 t\,\varphi(t-\pi) + \tfrac{5}{2}\sin^2 t\,\varphi(t-2\pi) = 0. \quad (3.1)$$

$P(t) = 1 - \frac{1}{2}\sin^2 t\,, Q_1(t) = 10\ \cos^2 t\,, Q_2(t) = \frac{5}{2}\sin^2 t\,, \tau(t) = t - 2\pi, \sigma_1 = \pi$

$\sigma_2 = 2\pi,\ \ \sigma = \pi,\ \ 0 \le p(t) \le \frac{1}{2},\ \ Q_1, Q_2\ \ge 0, n = 2,$

$$\int_0^{\sigma} Q_1(t)\,dt = 10\int_0^{\pi}\cos^2 t\,dt = 5\pi,\quad \int_0^{\sigma} Q_2(t)\,dt = \frac{5}{2}\int_0^{\sigma}\sin^2 t\,dt = \frac{5\pi}{4}.$$

Hence $\hat{\sigma} = \frac{5\pi}{4},\ \ n\hat{\sigma} = \frac{5\pi}{2} > 1$

So all conditions of theorem 1 holds, hence according to theorem 1, every solution of eq.(3.1) oscillates, for instance $\varphi(t) = 4\sin^3 t$ is such an oscillatory solution.

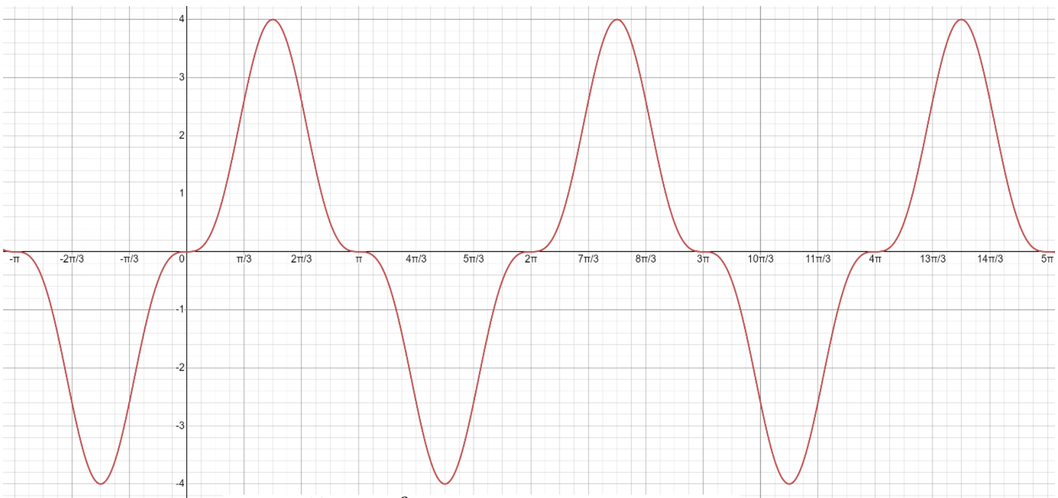

Fig. 1: $\varphi(t) = 4\sin^3 t$ oscillates and periodic of period 2π

Example 2. Consider the second order neutral equation with periodic coefficients:

$$[\varphi(t) - (1 + a\cos^2 2t)\varphi(t+\pi)]'' - 80a\sin^2 2t\varphi\left(t - \frac{\pi}{2}\right) - 20a\cos^2 2t\,\varphi(t-\pi) = 0, (3.2)$$

$p(t) = 1 + a\cos^2 2t\,,\ \ a > 0,\ \ Q_1(t) = -80\,a\sin^2 2t, Q_2(t) = -20a\cos^2 2t,$

$\tau(t) = t + \pi,\ \ \sigma_1 = \frac{\pi}{2},\ \ \sigma_2 = \pi,\ \ \sigma = \frac{\pi}{2},\ \ a > 0,\ \ n = 2,\ \ 1 \le p(t) \le 1 + a$

$$\int_0^{\frac{\pi}{2}} |Q_1(t)|\,dt = 80a\int_0^{\frac{\pi}{2}}\sin^2 2t\,dt = 20a\pi,$$

$$\int_0^{\frac{\pi}{2}} |Q_2(t)|\,dt = 20a\int_0^{\frac{\pi}{2}}\cos^2 2t\,dt = 5a\pi.$$

Thus $\breve{\sigma} = 5a\pi,\ n\breve{\sigma} = 10a\pi > 1$ if $a \ge 0.032$. Hence all conditions of theorem 2 hold. According to theorem2, each solution of (3.2) oscillates, for instance $\varphi(t) = 3\cos^3 2t$ is such an oscillatory solution.

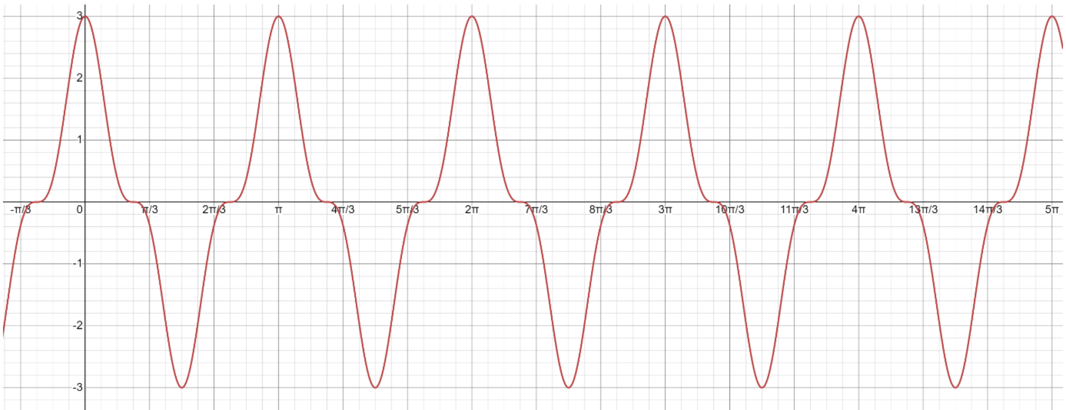

Fig.2: $\varphi(t) = 3\cos^3 2t$, oscillates and periodic of period π on $[t_0, \infty)$.

4. Conclusion

In this research, neutral second-order differential equations with periodic coefficients were studied, and from this study some sufficient conditions were obtained to ensure the oscillation of each solution of these equations or the convergence of non-oscillatory solutions to zero. Through these conditions, it is shown the extent to which periodic coefficients affect oscillation or convergence is revealed. The extracted conditions are easily applicable as shown in the examples presented above.

References

[1] E. R. Barnes, "Stability conditions for second order ordinary differential equations with periodic coefficients", *IEEE Conference on Fuzzy Set Theory and Applications, New Orleans, LA, USA*,1977, pp. 1256-1261, 1977. doi: 10.1109/CDC.1977.271762.

[2] Z. Cheng and F. Li, "Positive Periodic Solutions for a Kind of Second-Order Neutral Differential Equations with Variable Coefficient and Delay. Mediterr. J. Math. (2018) 15:134 https://doi.org/10.1007/s00009-018-1184-y

[3] G. Ladas, CH. G. Philos and Y. G. Sficas, "Oscillation in neutral equations with periodic coefficients", 113(1), 123-134, 1991. https://doi.org/10.2307/2048448.

[4] Y.J. Liu, J.W. Zhang and J. Yan, "Existence of oscillatory solutions of second order delay differential equations", J. Comput. Appl. Math. 277, pp.17–22, 2015. https://doi.org/10.1016/j.cam.2014.08.025

[5] H. A. Mohamad and F. A. Ahmed, "Oscillation and Asymptotic Behavior of Second Order Half Linear Neutral Dynamic Equations". *Iraqi Journal of Science, Vol. 63, No. 12, pp: 5413-5424. 2022. DOI: 10.24996/ijs.2022.63.12.27.*

[6] H. A. Mohamad and N. A. Abdulkareem,"Almost Oscillatory Solutions of Couple System of Differential Equations of Neutral Type". Ibn Al-Haitham Jour. for Pure & Appl. Sci. 34 (1) 2021. http://jih.uobaghdad.edu.iq/index.php/j/index

[7] A. K. Neghmish and H. A. Mohamad, "Oscillation of First Order Neutral Differential Equations with Periodic Coefficients". *Turkish Online Journal of Distance Education*. pp.1-8, *2023.*

[8] B. Qaraad, O. Bazighif, T. A. Nofal and A. H. Ali, "Neutral differential equations with distribution deviating arguments", *Journal of Ocean Engineering and Science*, article in press. Accepted 15 June 2022.

[9] C. Tunc and Y. Altun, "On the Nature of Solutions of Neutral Differential Equations with Periodic Coefficients". Appl. Math. & Inf. Sci. vol. **11**, No. 2, 393-399, 2017. http://dx.doi.org/10.18576/amis/110206.

[10] T. Li and Y. V. Rogovchenko, "Oscillation of second-order neutral differential equations". Wiley online library, Math. Nachr. 1–13 (2015) / doi/ 10.1002/mana.201300029.

[11] H. Yoshida, "A type of second order linear ordinary differential equations with periodic coefficients for which the characteristic exponents have exact expressions". *Celestial Mechanics* 32, 73–86, 1984. https://doi.org/10.1007/BF01358404.