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Approximate Estimation Methods for Exponential Rayleigh Distribution

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Abstract

This paper introduces approximate estimators for the scale parameters, reliability, and hazard rate functions of the exponential Rayleigh distribution using Modified Maximum Likelihood and Bayesian method. The Modified Maximum Likelihood estimator requires iterative techniques such as the modified Newton–Raphson method due to the unavailability of closed-form expressions. Bayesian estimators are derived using both symmetric and asymmetric loss functions, and Lindley's approximation method is employed for integrals that lack closed-form solutions. Finally, the estimators obtained are compared through Monte Carlo simulation study.

Keywords: Exponential Rayleigh Distribution; Modified Maximum Likelihood; Bayes Estimation; Lindley’s Approximation; scale invariant squared Error Loss Function; Squared Error Loss Function; linear-exponential (Linex) Loss Function; precautionary Loss Function.

طرق التقدير التقريبية لتوزيع رايلي الأسي

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الخلاصة

يقدم هذا البحث مقدرات تقريبية لمعلمات المقياس ودوال المعولية ومعدل الخطورة لتوزيع رايلي الأسي باستخدام طرق الامكان الاعظم المعدلة وتقديرالنظرية البايزية . يتطلب مقدر الامكان الاعظم المعدل تقنيات تكرارية مثل طريقة نيوتن–رافسون المعدلة بسبب عدم توفر التعبيرات ذات الشكل المغلق. يتم اشتقاق المقدرات البايزية باستخدام دوال الخسارة المتماثلة وغير المتماثلة، ويتم استخدام طريقة ليندلي التقريبية للتكاملات التي تفتقر إلى الحلول ذات الشكل المغلق. وأخيراً تم مقارنة المقدرات التي تم الحصول عليها من خلال دراسة محاكاة مونت كارلو.

1. Introduction

The hazard rate has been widely used to analysis survival data. The reliability function, which refers to the probability of survival is now widely employed not just by engineers and

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practitioners, but also by retailers who use it to explain product pricing and performance. They emphasize the importance of quality, dependability, security, and customer service in the event until failures occur, etc [1].The Bayes' method adopts a different approach to the non - Bayesian methods, as this method attentions on the use of initial information about the unknown parameter to be estimated, as it is assumed that relying on the current sample information alone is not sufficient in statistical analysis because this parameter represents a random variable that has a probability distribution and is not a fixed value. This method was adopted when the estimators obtained in the non-Bayesian school became sometimes inconsistent with the available initial information about some of the parameters. Here, the estimation will depend on initial information about the parameter to be assessed as well as the sample observations information. This method of estimation was introduced in (1702-1761), by Thomas Bayes. The word Bayesian, however, originated into use only around 1950, and it is not clear that Bayes would have recognized the very broad understanding of probability that is associated with his name [2] . There has been a great deal of concern in the use of many kinds of parametric models lately for the analysis of what is variously denoted to as lifetime, survival time or failure time data that have as end point the time until the failure occurs [3].

In this paper using non – Bayesian inference such as modified maximum likelihood estimation and Bayesian inference to estimate the two-parameters, reliability and hazard rate functions of Exponential Rayleigh ER distribution.
The probability density function of Exponential Rayleigh (ER) distribution can be obtained by [4]:

$$f(x;\gamma,\beta)_{ER} = (\gamma+\beta x)\, e^{-\left(\gamma x+\frac{\beta}{2}x^2\right)} \quad ; x \geq 0; \gamma,\beta > 0. \tag{1}$$

And zero otherwise, where γ and β are scale parameters.

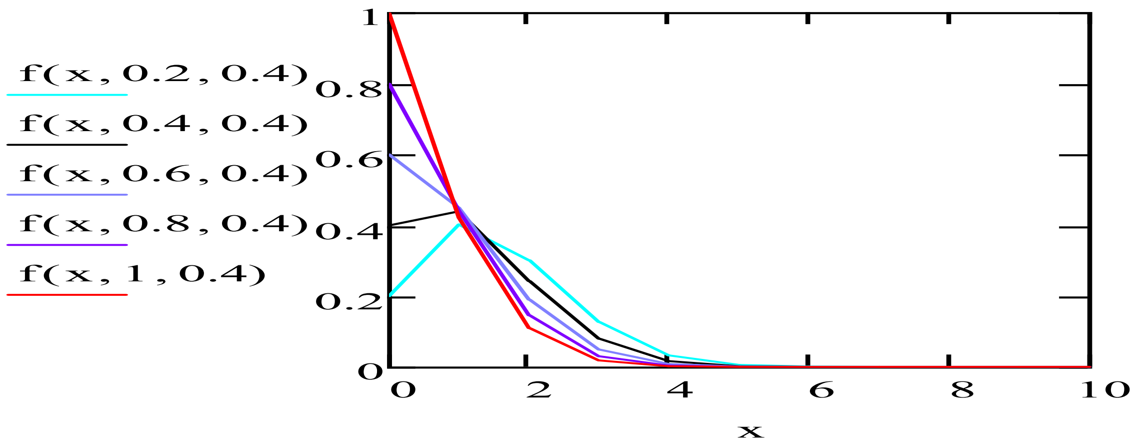

Figure 1: Plot of the probability density function of ER distribution for $\beta = 0.4$ and different values of
($\gamma = 0.2, 0.4, 0.6, 0.8, 1$) [Mathcad 15].

The cumulative distribution function of ER distribution is given by:

$$F(x;\gamma,\beta)_{ER} = 1 - e^{-\left(\gamma x+\frac{\beta}{2}x^2\right)} \quad ; x \geq 0; \gamma,\beta > 0. \tag{2}$$

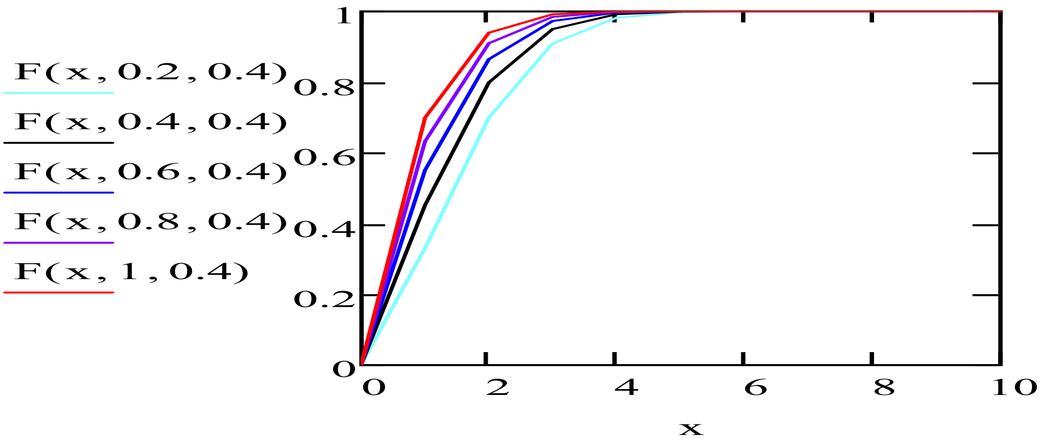

Figure 2: Plot of the cumulative distribution function of ER distribution for $\beta = 0.4$ and different values of ($\gamma = 0.2, 0.4, 0.6, 0.8, 1$) [Mathcad 15].

The reliability and the hazard rate functions at task time (t) are given by [4]:

$$\epsilon_1(t;\gamma,\beta)_{ER} = e^{-(\gamma t + \frac{\beta}{2}t^2)} \quad ; t \geq 0; \gamma, \beta > 0. \qquad (3)$$

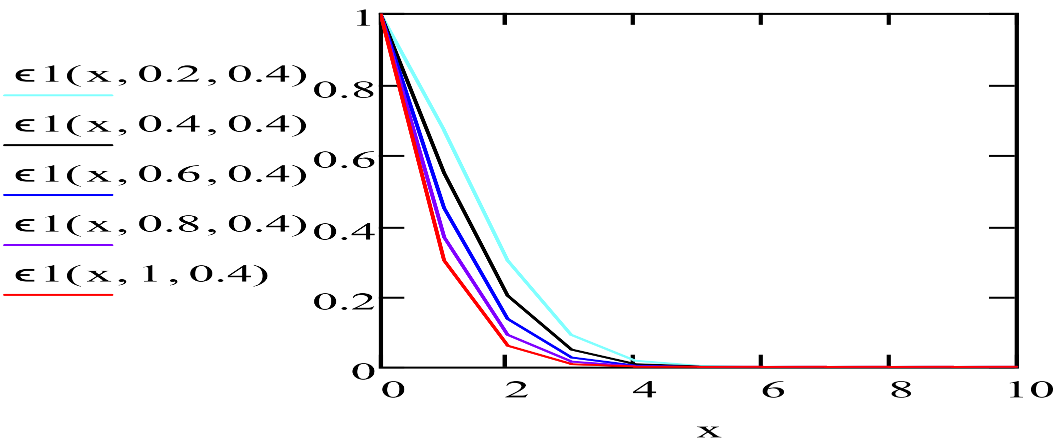

Figure 3: Plot of the Reliability function of ER distribution for $\beta = 0.4$ and different values of ($\gamma = 0.2, 0.4, 0.6, 0.8, 1$) [Mathcad 15].

$$\epsilon_2(t;\gamma,\beta)_{ER} = \gamma + \beta t \quad ; t \geq 0; \gamma, \beta > 0. \qquad (4)$$

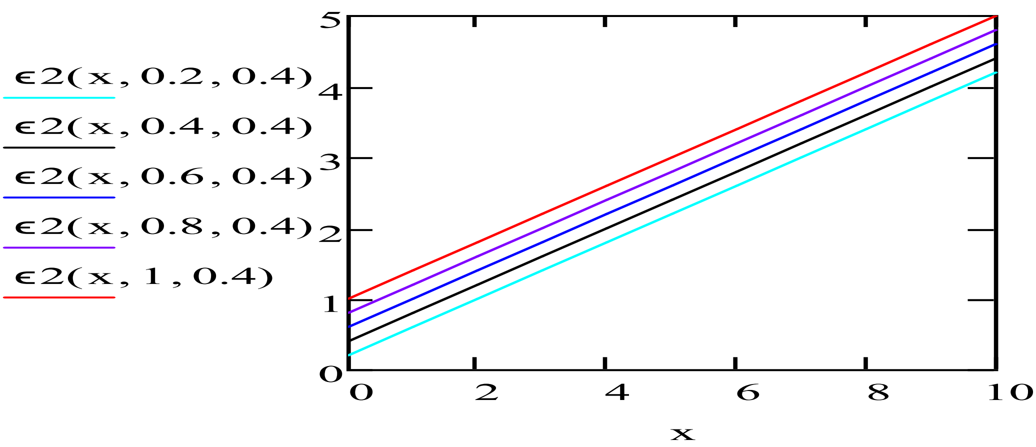

Figure 4: Plot of the hazard rate function of ER distribution for $\beta = 0.4$ and different values of ($\gamma = 0.2, 0.4, 0.6, 0.8, 1$) [Mathcad 15].

2. Non-Bayesian estimation method

- **Modified maximum likelihood estimation**

Let $\underline{x} = (x_1, x_2, \dots, x_n)$ be a random sample of size (n) drawn from ER distribution with p.d.f. given by Equation (1). The complete data likelihood function $LF(\gamma, \beta | \underline{x})_{ER}^{MLE}$ for a given random sample can be given by:

$$LF(\gamma, \beta | \underline{x})_{ER}^{MLE} = \prod_{i=1}^{n} f(x_i; \gamma, \beta)_{ER} = \prod_{i=1}^{n} \left[(\gamma + \beta x_i) e^{-(\gamma x_i + \frac{\beta}{2} x_i^2)} \right]. \quad (5)$$

So, the natural log – likelihood function is obtained by:

$$\mathcal{L}_{ER}^{MLE} = \ln LF(\gamma, \beta | \underline{x})_{ER}^{MLE} = \sum_{i=1}^{n} \ln(\gamma + \beta x_i) - \sum_{i=1}^{n} \left(\gamma x_i + \frac{\beta}{2} x_i^2 \right). \quad (6)$$

Now,

$$\hat{\mathcal{L}}_{\gamma} = \frac{\partial \mathcal{L}_{ER}^{MLE}}{\partial \gamma} = \sum_{i=1}^{n} (\gamma + \beta x_i)^{-1} - \sum_{i=1}^{n} x_i = 0. \quad (7)$$

$$\hat{\mathcal{L}}_{\beta} = \frac{\partial \mathcal{L}_{ER}^{MLE}}{\partial \beta} = \sum_{i=1}^{n} x_i (\gamma + \beta x_i)^{-1} - \frac{1}{2} \sum_{i=1}^{n} x_i^2 = 0. \quad (8)$$

The maximum likelihood estimators denoted by $\hat{\gamma}_{ER}^{MLE}$ and $\hat{\beta}_{ER}^{MLE}$ are the value of γ and β that maximizes $LF(\gamma, \beta | \underline{x})_{ER}^{MLE}$ can be obtained by the solution of Equations (7) and (8). Since the above equations which are nonlinear equations then we need apply the numerical approaches to solve it. One of these approaches is Modified Newton – Raphson method (M-NR).

In Modified Newton – Raphson method, the solution of the likelihood equation at iteration $(h + 1)$ is extracted, through the following iterative process,

$$\begin{bmatrix} \hat{\gamma}_{ER}^{MLE(h+1)} \\ \hat{\beta}_{ER}^{MLE(h+1)} \end{bmatrix} = \begin{bmatrix} \hat{\gamma}_{ER}^{MLE(h)} \\ \hat{\beta}_{ER}^{MLE(h)} \end{bmatrix} - \sigma J_{(h)}^{-1} \begin{bmatrix} \dfrac{\partial \mathcal{L}_{ER}^{MLE}}{\partial \gamma} \\ \dfrac{\partial \mathcal{L}_{ER}^{MLE}}{\partial \beta} \end{bmatrix}_{\substack{\gamma = \hat{\gamma}_{ER}^{MLE(h)} \\ \beta = \hat{\beta}_{ER}^{MLE(h)}}} \quad ; h = 0,1,2, \dots ; \; 0 < \sigma < 1$$

where

$$J_{(h)} = \begin{bmatrix} \dfrac{\partial^2 \mathcal{L}_{ER}^{MLE}}{\partial \gamma^2} & \dfrac{\partial^2 \mathcal{L}_{ER}^{MLE}}{\partial \gamma \partial \beta} \\ \dfrac{\partial^2 \mathcal{L}_{ER}^{MLE}}{\partial \beta \partial \gamma} & \dfrac{\partial^2 \mathcal{L}_{ER}^{MLE}}{\partial \beta^2} \end{bmatrix}_{\substack{\gamma = \hat{\gamma}_{ER}^{MLE(h)} \\ \beta = \hat{\beta}_{ER}^{MLE(h)}}}$$

as where the first partial derivatives as in Equations (7), (8) and the second partial derivatives are given by,

$$\hat{\mathcal{L}}_{\gamma\gamma} = \frac{\partial^2 \mathcal{L}_{ER}^{MLE}}{\partial \gamma^2} \Big|_{\substack{\gamma = \hat{\gamma} \\ \beta = \hat{\beta}}} = -\sum_{i=1}^{n} \left(\hat{\gamma} + \hat{\beta} x_i \right)^{-2}. \quad (9)$$

$$\hat{\mathcal{L}}_{\beta\beta} = \frac{\partial^2 \mathcal{L}_{ER}^{MLE}}{\partial \beta^2} \Big|_{\substack{\gamma = \hat{\gamma} \\ \beta = \hat{\beta}}} = -\sum_{i=1}^{n} x_i^{\,2} \left(\hat{\gamma} + \hat{\beta} x_i \right)^{-2}. \quad (10)$$

$$\hat{\mathcal{L}}_{\gamma\beta} = \frac{\partial^2 \mathcal{L}_{ER}^{MLE}}{\partial \gamma \partial \beta} \Big|_{\substack{\gamma = \hat{\gamma} \\ \beta = \hat{\beta}}} = \hat{\mathcal{L}}_{\beta\gamma} = \frac{\partial^2 \mathcal{L}_{ER}^{MLE}}{\partial \beta \partial \gamma} \Big|_{\substack{\gamma = \hat{\gamma} \\ \beta = \hat{\beta}}} = -\sum_{i=1}^{n} x_i \left(\hat{\gamma} + \hat{\beta} x_i \right)^{-2}. \quad (11)$$

Now, based on an invariant property, the reliability and hazard rate functions at task time (t) of the ER distribution are as follows:

$$\hat{\epsilon_1}(t;\gamma,\beta)_{ER}^{MLE} = e^{-(\hat{\gamma}_{ER}^{MLE}\, t + \frac{\hat{\beta}_{ER}^{MLE}}{2} t^2)}. \quad (12)$$

$$\hat{\epsilon_2}(t;\gamma,\beta\,)_{ER} = \hat{\gamma}_{ER}^{MLE} + \hat{\beta}_{ER}^{MLE} t. \quad (13)$$

3. Bayesian estimation

In this section we considered Bayes estimation theory to estimate the scale parameters along with the reliability and hazard rate functions of ER distribution. In order to obtain Bayes Estimators, with respect to the squared error and scale invariant squared error as symmetric loss functions and the linear-exponential (Linex) and precautionary as asymmetric loss functions respectively. Let the prior distributions of γ and β of ER distribution are taken to be independent exponential (c) and Gamma (a,b),respectively with probability distribution functions as:

$$\pi(\gamma) = ce^{-c\gamma} \quad ; c > 0, \gamma > 0. \quad (14)$$

$$\pi(\beta) = \frac{b^a}{\Gamma(a)}\, \beta^{a-1}\, e^{-\beta b} \quad ; \beta > 0, a, b > \quad (15)$$

The joint prior distribution of unknown parameters can be written as:

$$\pi(\gamma,\beta) = \pi(\gamma)\, \pi(\beta) = \frac{b^a c}{\Gamma(a)}\, \beta^{a-1}\, e^{-(c\gamma+\beta b)}. \quad (16)$$

The joint posterior density function of γ and β denoted by $G(\gamma,\beta|\underline{x})$ can be obtained by merging Equations (5) and (16), as follows:

$$G(\gamma,\beta|\underline{x}) = \frac{LF(\gamma,\beta|\underline{x})\,\pi(\gamma,\beta)}{\int_\beta \int_\gamma LF(\gamma,\beta|\underline{x})\,\pi(\gamma,\beta) d\gamma\, d\beta}. \quad (17)$$

The formula of squared error loss function (SELF) and scale invariant squared error loss function (SISELF) for the parameter ω are respectively given by [5] [6]:

$$L(\hat{\omega}_S, \omega) = (\hat{\omega}_S - \omega)^2$$

where $\hat{\omega}_S$ is an estimate of ω under SELF.

$$L(\hat{\omega}_{SI}, \omega) = \left(\frac{\omega - \hat{\omega}_{SI}}{\omega}\right)^2,$$

where $\hat{\omega}_{SI}$ is an estimate of ω under SISELF.

The Bayes estimators of ω under SELF and under SISELF are given respectively by:

$$\hat{\omega}_{BS} = E(\omega|\underline{x}). \quad (18)$$

$$\hat{\omega}_{BSI} = \frac{E(\omega^{-1}|\underline{x})}{E(\omega^{-2}|\underline{x})}. \quad (19)$$

The formula of Linex loss function (LLF) and precautionary loss function (PLF) for ω are respectively [7]and [1]:

$$L(\hat{\omega}_L, \omega) = d\left[e^{z(\hat{\omega}_L - \omega)} - z(\hat{\omega}_L - \omega) - 1\right] \quad ; z \neq 0, d > 0,$$

where $\hat{\omega}_L$ is an estimate of ω under LLF, z is shape parameter and d is scale parameter.

$$L(\hat{\omega}_P, \omega) = \frac{(\hat{\omega}_P - \omega)^2}{\hat{\omega}_P},$$

where $\hat{\omega}_P$ is an estimate of ω under PLF.

So, Bayes estimator of ω under LLF when d=1 and under PLF are given, respectively, by:

$$\hat{\omega}_{BL} = -\frac{1}{z} ln[E(e^{-z\omega}|\underline{x})]. \quad (20)$$

$$\hat{\omega}_{BP} = [E(\omega^2|\underline{x})]^{\frac{1}{2}} = \sqrt{E(\omega^2|\underline{x})}\,. \quad (21)$$

Depending on the Equations (18), (19), (20) and (21), Bayes estimation of any function of the parameter say $k(\gamma,\beta)$, based on squared error, scale invariant squared error, linear-exponential (Linex) and precautionary loss functions can be written respectively as:

$$\hat{k}_{BS}(\gamma,\beta)=E\left(k(\gamma,\beta)|\underline{x}\right)$$

$$=\frac{\int_\beta \int_\gamma k(\gamma,\beta)\, LF(\gamma,\beta|\underline{x})\, \pi(\gamma,\beta)d\gamma\, d\beta}{\int_\beta \int_\gamma LF(\gamma,\beta|\underline{x})\, \pi(\gamma,\beta)d\gamma\, d\beta}. \quad (22)$$

$$\hat{k}_{BSI}(\gamma,\beta)=\frac{E\left(\left(k(\gamma,\beta)\right)^{-1}|\underline{x}\right)}{E\left(\left(k(\gamma,\beta)\right)^{-2}|\underline{x}\right)}=\frac{E_I}{E_J}. \quad (23)$$

Where,

$$E_I=\frac{\int_\beta \int_\gamma \left(k(\gamma,\beta)\right)^{-1} LF(\gamma,\beta|\underline{x})\, \pi(\gamma,\beta)d\gamma\, d\beta}{\int_\beta \int_\gamma LF(\gamma,\beta|\underline{x})\, \pi(\gamma,\beta)d\gamma\, d\beta}. \quad (24)$$

And

$$E_J=\frac{\int_\beta \int_\gamma \left(k(\gamma,\beta)\right)^{-2} LF(\gamma,\beta|\underline{x})\, \pi(\gamma,\beta)d\gamma\, d\beta}{\int_\beta \int_\gamma LF(\gamma,\beta|\underline{x})\, \pi(\gamma,\beta)d\gamma\, d\beta}. \quad (25)$$

$$\hat{k}_{BL}(\gamma,\beta)=-\frac{1}{z}ln[E\left(e^{-zk(\gamma,\beta)}|\underline{x}\right)]$$

$$=-\frac{1}{z}\ln\left[\frac{\int_\beta \int_\gamma e^{-zk(\gamma,\beta)}\, LF(\gamma,\beta|\underline{x})\, \pi(\gamma,\beta)d\gamma\, d\beta}{\int_\beta \int_\gamma LF(\gamma,\beta|\underline{x})\, \pi(\gamma,\beta)d\gamma\, d\beta}\right]. \quad (26)$$

$$\hat{k}_{BP}(\gamma,\beta)=[E\left(\left(k(\gamma,\beta)\right)^2|\underline{x}\right)]^{\frac{1}{2}}=\sqrt{E\left(\left(k(\gamma,\beta)\right)^2|\underline{x}\right)}$$

$$=\left[\frac{\int_\beta \int_\gamma \left(k(\gamma,\beta)\right)^2 LF(\gamma,\beta|\underline{x})\, \pi(\gamma,\beta)\, d\gamma\, d\beta}{\int_\beta \int_\gamma LF(\gamma,\beta|\underline{x})\, \pi(\gamma,\beta)\, d\gamma\, d\beta}\right]^{\frac{1}{2}}$$

$$=\sqrt{\frac{\int_\beta \int_\gamma \left(k(\gamma,\beta)\right)^2 LF(\gamma,\beta|\underline{x})\, \pi(\gamma,\beta)d\gamma\, d\beta}{\int_\beta \int_\gamma LF(\gamma,\beta|\underline{x})\, \pi(\gamma,\beta)d\gamma\, d\beta}}. \quad (27)$$

It is clear that, the above ratio of two integrals cannot be simplified into a closed form. So, we applied an approximation forms to get Bayes estimators of the $\gamma,\beta,\epsilon_1(t)$ and $\epsilon_2(t)$ of ER distribution.

To evaluate the ratio of two integrals, Lindley (1980) introduced an approximation as follows [8]:

$$\frac{\int_\lambda \emptyset(\lambda)\, e^{\mathcal{L}(\lambda)}\, d\lambda}{\int_\lambda g(\lambda)\, e^{\mathcal{L}(\lambda)}\, d\lambda}. \quad (28)$$

Where,

$\lambda=(\lambda_1,\lambda_2,\lambda_3,\dots,\lambda_m)$ are the vector of parameters to be estimated, $\mathcal{L}(\lambda)$ denote the natural log – likelihood function and $\emptyset(\lambda)$ or $g(\lambda)$ denote any function of λ.

Now, let $g(\lambda)$ denote the prior density function of λ and $\emptyset(\lambda)=k(\lambda)g(\lambda)$, such that $k(\lambda)$ denote any function. So, Equation (28) yields:

$$E\left(k(\lambda)\right)=\frac{\int_\lambda k(\lambda)\, e^{\mathcal{L}(\lambda)+\psi(\lambda)}\, d\lambda}{\int_\lambda e^{\mathcal{L}(\lambda)+\psi(\lambda)}\, d\lambda}. \quad (29)$$

Thus, depending on Lindley the expected value can be asymptotically approached as,

$$E\left(k(\lambda)\right)=[k+\frac{1}{2}\sum_i\sum_j\left(k_{ij}+2k_i\psi_j\right)\eta_{ij}+\frac{1}{2}\sum_i\sum_j\sum_f\sum_l \mathcal{L}_{ijf}\,\eta_{ij}\,\eta_{fl}\,k_l]_{\hat{\lambda}}. \quad (30)$$

Such that

$i,j,f,l = 1,2,3,\dots,m$, $k \equiv k(\lambda)$; $k_i = \frac{\partial k}{\partial \lambda_i}$; $k_{ij} = \frac{\partial^2 k}{\partial \lambda_i \partial \lambda_j}$; $\mathcal{L}_{ijf} = \frac{\partial^3 \mathcal{L}}{\partial \lambda_i \partial \lambda_j \partial \lambda_f}$,

$\psi \equiv \psi(\lambda) = \ln g(\lambda)$; $\psi_j = \frac{\partial \psi}{\partial \lambda_j}$ and η_{ij} is the $(i,j)^{th}$ elements of matrix $[\frac{-\partial^2 \mathcal{L}}{\partial \lambda_i \partial \lambda_j}]^{-1}$.

Now, for $m = 2$; $\lambda = (\lambda_1, \lambda_2) = (\gamma, \beta)$, Equation (29) becomes,

$$E\big(k(\gamma,\beta|\,\underline{x}\,)\big) = \frac{\int_\beta \int_\gamma \; k(\gamma,\beta)\, e^{\mathcal{L}(\gamma,\beta|\,\underline{x}\,) + \psi(\gamma,\beta)}\, d\gamma\, d\beta}{\int_\beta \int_\gamma \; e^{\mathcal{L}(\gamma,\beta|\,\underline{x}\,) + \psi(\gamma,\beta)}\, d\gamma\, d\beta}. \tag{31}$$

Where,

$k(\gamma,\beta)$ denote function of γ and β only, $\mathcal{L}(\gamma,\beta|\,\underline{x}\,)$ denote the natural log – likelihood function defined by Equation (6) and $\psi(\gamma,\beta)$ denote the natural log – joint prior density function.

So, for sufficiently large sample size, the ratio of two integrals $I(\underline{x}) = E\big(k(\gamma,\beta|\,\underline{x}\,)\big)$ can be expressed it as follows [9]:

$$I(\underline{x}) = \hat{k} + \tfrac{1}{2}\Big[\big(\hat{k}_{\gamma\gamma} + 2\hat{k}_\gamma \hat{\psi}_\gamma\big)\,\hat{\eta}_{\gamma\gamma} + \big(\hat{k}_{\gamma\beta} + 2\hat{k}_\gamma \hat{\psi}_\beta\big)\,\hat{\eta}_{\gamma\beta} + \big(\hat{k}_{\beta\gamma} + 2\hat{k}_\beta \hat{\psi}_\gamma\big)\,\hat{\eta}_{\beta\gamma} + \big(\hat{k}_{\beta\beta} + 2\hat{k}_\beta \hat{\psi}_\beta\big)\,\hat{\eta}_{\beta\beta}\big] + \tfrac{1}{2}[(\hat{k}_\beta\, \hat{\eta}_{\gamma\beta} + \hat{k}_\gamma\, \hat{\eta}_{\gamma\gamma})(\,\hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma}) + (\hat{k}_\beta\, \hat{\eta}_{\beta\beta} + \hat{k}_\gamma\, \hat{\eta}_{\beta\gamma})(\hat{\mathcal{L}}_{\beta\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\beta}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})\Big]. \tag{32}$$

Where, $\hat{\gamma}$ and $\hat{\beta}$ are the MLE's of γ and β, respectively

η_{ij} is the $(i,j)^{th}$ elements of matrix $\left[\frac{-\partial^2 \mathcal{L}(\gamma,\beta|\,\underline{x}\,)}{\partial\gamma\partial\beta}\right]^{-1}$; where sub-scripts (i,j) denote of γ and β, respectively.

Now,

$$\psi(\gamma,\beta) = \ln \pi(\gamma,\beta)$$
$$= a \ln b + (a-1)\ln\beta - c\gamma - \beta b + \ln c - \ln\Gamma(a). \tag{33}$$

$$\hat{\psi}_\gamma = \frac{\partial\, ln\, \pi(\gamma,\beta)}{\partial\gamma}\Big|_{\substack{\gamma=\hat{\gamma}\\ \beta=\hat{\beta}}} = -c\,. \tag{34}$$

$$\hat{\psi}_\beta = \frac{\partial\, ln\, \pi(\gamma,\beta)}{\partial\beta}\Big|_{\substack{\gamma=\hat{\gamma}\\ \beta=\hat{\beta}}} = \frac{a-1}{\hat{\beta}} - \text{b}. \tag{35}$$

And,

$$\hat{\eta}_{\gamma\gamma} = -\frac{1}{\hat{\mathcal{L}}_{\gamma\gamma}}\;, \; \hat{\eta}_{\beta\beta} = -\frac{1}{\hat{\mathcal{L}}_{\beta\beta}},\; \hat{\eta}_{\gamma\beta} = \hat{\eta}_{\beta\gamma} = -\frac{1}{\hat{\mathcal{L}}_{\gamma\beta}} = -\frac{1}{\hat{\mathcal{L}}_{\beta\gamma}}. \tag{36}$$

And

$$\hat{\mathcal{L}}_{\gamma\gamma\gamma} = \frac{\partial^3 \mathcal{L}_{ER}^{MLE}}{\partial\gamma^3}\Big|_{\substack{\gamma=\hat{\gamma}\\ \beta=\hat{\beta}}} = 2\sum_{i=1}^{n}\big(\hat{\gamma} + \hat{\beta}x_i\big)^{-3}. \tag{37}$$

$$\hat{\mathcal{L}}_{\beta\beta\beta} = \frac{\partial^3 \mathcal{L}_{ER}^{MLE}}{\partial\beta^3}\Big|_{\substack{\gamma=\hat{\gamma}\\ \beta=\hat{\beta}}} = 2\sum_{i=1}^{n} x_i^3\,\big(\hat{\gamma} + \hat{\beta}x_i\big)^{-3}. \tag{38}$$

$$\hat{\mathcal{L}}_{\gamma\gamma\beta} = \frac{\partial^3 \mathcal{L}_{ER}^{MLE})}{\partial\gamma\,\partial\gamma\,\partial\beta}\Big|_{\substack{\gamma=\hat{\gamma}\\ \beta=\hat{\beta}}} = \hat{\mathcal{L}}_{\gamma\beta\gamma} = \frac{\partial^3 \mathcal{L}_{ER}^{MLE}}{\partial\gamma\,\partial\beta\,\partial\gamma}\Big|_{\substack{\gamma=\hat{\gamma}\\ \beta=\hat{\beta}}} = \hat{\mathcal{L}}_{\beta\gamma\gamma} = \frac{\partial^3 \mathcal{L}_{ER}^{MLE}}{\partial\beta\,\partial\gamma\,\partial\gamma}\Big|_{\substack{\gamma=\hat{\gamma}\\ \beta=\hat{\beta}}} = 2\sum_{i=1}^{n} x_i\,\big(\hat{\gamma} + \hat{\beta}x_i\big)^{-3}\;. \tag{39}$$

$$\hat{\mathcal{L}}_{\beta\gamma\beta} = \frac{\partial^3 \mathcal{L}_{ER}^{MLE}}{\partial\beta\,\partial\gamma\,\partial\beta}\Big|_{\substack{\gamma=\hat{\gamma}\\ \beta=\hat{\beta}}} = \hat{\mathcal{L}}_{\gamma\beta\beta} = \frac{\partial^3 \mathcal{L}_{ER}^{MLE}}{\partial\gamma\,\partial\beta\,\partial\beta}\Big|_{\substack{\gamma=\hat{\gamma}\\ \beta=\hat{\beta}}} = 2\sum_{i=1}^{n} x_i^2\;\big(\hat{\gamma} + \hat{\beta}x_i\big)^{-3}. \tag{40}$$

So, based on above defined expressions, the values of approximate Bayes estimators of the parameters $\gamma, \beta,$ reliability and hazard rate functions of ER distribution under SELF, SISELF, LLF and PLF can be found in the next subsection:

3.1 Approximate Bayes estimate under SELF

- Suppose that $k(\gamma,\beta)=\gamma$, then

$k_\gamma = 1$ and $k_{\gamma\gamma}=k_\beta=k_{\beta\beta}=k_{\gamma\beta}=k_{\beta\gamma}=0$.

So, Bayes estimate of γ under SELF based on Lindley's approximation, is given by:

$$\hat{\gamma}^{L}_{BS}=E\big(\gamma|\underline{x}\big)$$
$$=\hat{\gamma}+\hat{\psi}_\gamma\,\hat{\eta}_{\gamma\gamma}+\hat{\psi}_\beta\,\hat{\eta}_{\gamma\beta}+\frac{1}{2}[\hat{\eta}_{\gamma\gamma}(\,\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma}\;+\hat{\mathcal{L}}_{\gamma\lambda\gamma}\,\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma})$$
$$+\hat{\eta}_{\beta\gamma}(\hat{\mathcal{L}}_{\lambda\lambda\lambda}\,\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\beta}\,\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})]. \tag{41}$$

- Suppose that $k(\gamma,\beta)=\beta$, then

$k_\beta = 1$ and $k_{\beta\beta}=k_\gamma=k_{\gamma\gamma}=k_{\gamma\beta}=k_{\beta\gamma}=0$.

So, Bayes estimate of β under SELF based on Lindley's approximation, is given by:

$$\hat{\beta}^{L}_{BS}(t)=E\big(\beta|\underline{x}\big)$$
$$=\hat{\beta}+\hat{\psi}_\gamma\,\hat{\eta}_{\beta\gamma}+\hat{\psi}_\beta\,\hat{\eta}_{\beta\beta}+\frac{1}{2}[\hat{\eta}_{\gamma\beta}(\,\hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma})$$
$$+\hat{\eta}_{\beta\beta}(\hat{\mathcal{L}}_{\beta\beta\beta}\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\beta}\,\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta}+\mathcal{L}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})]. \tag{42}$$

- Suppose that $k(\gamma,\beta)=\epsilon_1(t)=\;e^{-\,(\gamma t+\frac{\beta}{2}t^2)}$, then

$k_\gamma=-t\;\;e^{-\,(\gamma t+\frac{\beta}{2}t^2)}$; $k_\beta=-\frac{t^2}{2}\;\;e^{-\,(\gamma t+\frac{\beta}{2}t^2)}$; $k_{\gamma\gamma}=t^2\;\;e^{-\,(\gamma t+\frac{\beta}{2}t^2)}$

$k_{\beta\beta}=\frac{t^4}{4}\;\;e^{-\,(\gamma t+\frac{\beta}{2}t^2)}$ and $k_{\gamma\beta}=k_{\beta\gamma}=\frac{t^3}{2}\;\;e^{-\,(\gamma t+\frac{\beta}{2}t^2)}$.

So, Bayes estimate of $\epsilon_1(t)$ under SELF based on Lindley's approximation, is given by:

$$\hat{\epsilon}^{\,L}_{1\,BS}(t)=E\big(\epsilon_1(t)|\underline{x}\big)$$
$$=\;e^{-\,(\hat{\gamma}t+\frac{\hat{\beta}}{2}t^2)}+\frac{1}{2}\,[\big(\hat{k}_{\gamma\gamma}+2\hat{k}_\gamma\,\hat{\psi}_\gamma\big)\,\hat{\eta}_{\gamma\gamma}+\big(\hat{k}_{\gamma\beta}+2\hat{k}_\gamma\,\hat{\psi}_\lambda\big)\,\hat{\eta}_{\gamma\beta}+\big(\hat{k}_{\beta\gamma}+2\hat{k}_\beta\,\hat{\psi}_\gamma\big)\,\hat{\eta}_{\beta\gamma}+\big(\hat{k}_{\beta\beta}+2\hat{k}_\beta\,\hat{\psi}_\beta\big)\,\hat{\eta}_{\beta\beta}]+\frac{1}{2}[\big(\hat{k}_\beta\,\hat{\eta}_{\gamma\beta}+\hat{k}_\gamma\,\hat{\eta}_{\gamma\gamma}\big)(\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma})+\big(\hat{k}_\beta\,\hat{\eta}_{\beta\beta}+\hat{k}_\gamma\,\hat{\eta}_{\beta\gamma}\big)(\hat{\mathcal{L}}_{\beta\beta\beta}\,\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\beta}\hat{\eta}_{\beta\gamma}\;+\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})]. \tag{43}$$

Suppose that $k(\gamma,\beta)=\epsilon_2(t)=\gamma+\beta t$, then

$k_\gamma=1$, $k_\beta=t$ and $k_{\beta\beta}=k_{\gamma\gamma}=k_{\gamma\beta}=k_{\beta\gamma}=0$.

So, Bayes estimate of $\boldsymbol{\epsilon_2}(t)$ under SELF based on Lindley's approximation, is given by:

$$\hat{\epsilon}^{\,L}_{2\,BS}(t)==E\big(\boldsymbol{\epsilon_2}(t)|\underline{x}\big)$$
$$=\hat{\gamma}+\hat{\beta}t+\big[\,\hat{\psi}_\gamma\hat{\eta}_{\gamma\gamma}+\;\hat{\psi}_\beta\,\hat{\eta}_{\gamma\beta}+t\,\hat{\psi}_\gamma\,\hat{\eta}_{\beta\gamma}+t\,\hat{\psi}_\beta\,\hat{\eta}_{\beta\beta}\big]+\;\frac{1}{2}\big[\big(t\,\hat{\eta}_{\gamma\beta}+\hat{\eta}_{\gamma\gamma}\big)\big(\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\gamma}\hat{\eta}_{\gamma\gamma}\big)+\big(t\,\hat{\eta}_{\beta\beta}+\hat{\eta}_{\beta\gamma}\big)\big(\hat{\mathcal{L}}_{\beta\beta\beta}\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\beta}\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma}\big)\big]. \tag{44}$$

3.2 Approximate Bayes estimate under SISELF

- Suppose that $k(\gamma,\beta)=\gamma$

To compute E_I suppose that $(k(\gamma,\beta))^{-1}\;=\gamma^{-1}$, then

$k_\gamma=-\gamma^{-2}$; $k_{\gamma\gamma}=2\gamma^{-3}$ and $k_\beta=k_{\beta\beta}=k_{\gamma\beta}=k_{\beta\gamma}=0$.

$$E_I=E\big(\gamma^{-1}|\underline{x}\big)$$
$$=\gamma^{-1}+\big[\big(\hat{\gamma}^{-3}-\hat{\gamma}^{-2}\hat{\psi}_\gamma\big)\hat{\eta}_{\gamma\gamma}-\hat{\gamma}^{-2}\hat{\psi}_\beta\,\hat{\eta}_{\gamma\beta}\big]-\frac{1}{2}(\hat{\gamma})^{-2}[\,\hat{\eta}_{\gamma\gamma}(\,\hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma}$$
$$+\hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta}+\;\hat{\mathcal{L}}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma})+\;\hat{\eta}_{\beta\gamma}(\hat{\mathcal{L}}_{\beta\beta\beta}\,\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\beta}\,\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})]. \tag{45}$$

Now, to compute E_J suppose that $(k(\gamma,\beta))^{-2}=\gamma^{-2}$, then

$k_\gamma=-2\gamma^{-3}$; $k_{\gamma\gamma}=6\gamma^{-4}$ and $k_\beta=k_{\beta\beta}=k_{\gamma\beta}=k_{\beta\gamma}=0$

$$E_J = E(\gamma^{-2}|\underline{x})$$
$$= \hat{\gamma}^{-2} + \left[(3\hat{\gamma}^{-4} - 2\hat{\gamma}^{-3}\hat{\psi}_\gamma)\hat{\eta}_{\gamma\gamma} - 2\hat{\gamma}^{-3}\hat{\psi}_\beta\,\hat{\eta}_{\gamma\beta}\right] - \hat{\gamma}^{-3}[\,\hat{\eta}_{\gamma\gamma}(\,\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma}) + \hat{\eta}_{\beta\gamma}(\hat{\mathcal{L}}_{\beta\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\beta}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})]. \quad (46)$$

So, Bayes estimate of γ under SISELF based on Lindley's approximation can be obtained by:

$$\hat{\gamma}^L_{BSI} = \frac{E_I}{E_J} = \frac{E(\gamma^{-1}|\underline{x})}{E(\gamma^{-2}|\underline{x})}. \quad (47)$$

- Suppose that $k(\gamma,\beta) = \beta$.

To compute E_I suppose that $\frac{1}{k(\gamma,\beta)} = \beta^{-1}$, then

$k_\beta = -\beta^{-2}$; $k_{\beta\beta} = 2\beta^{-3}$ and $k_\gamma = k_{\gamma\gamma} = k_{\gamma\beta} = k_{\beta\gamma} = 0$.

$$E_I = E(\beta^{-1}|\underline{x})$$
$$= \hat{\beta}^{\,-1} - \hat{\beta}^{\,-2}\,\hat{\psi}_\gamma\,\hat{\eta}_{\beta\gamma} + (\hat{\beta}^{\,-3} - \hat{\beta}^{\,-2}\,\hat{\psi}_\beta)\,\hat{\eta}_{\beta\beta} - \frac{\hat{\beta}^{\,-2}}{2}[\hat{\eta}_{\gamma\beta}(\,\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta} + \mathcal{L}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma}) + \hat{\eta}_{\beta\beta}(\hat{\mathcal{L}}_{\beta\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\beta}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})]. \quad (48)$$

Now, to compute E_J suppose that $(k(\gamma,\beta))^{-2} = \beta^{-2}$, then

$k_\beta = -\frac{2}{\beta^3}$; $k_{\beta\beta} = \frac{6}{\beta^4}$ and $k_\gamma = k_{\gamma\gamma} = k_{\gamma\beta} = k_{\beta\gamma} = 0$.

$$E_J = E(\beta^{-2}\,|\underline{x})$$
$$= \hat{\beta}^{-2} - 2\hat{\beta}^{-3}\,\hat{\psi}_\gamma\,\hat{\eta}_{\beta\gamma} + (3\hat{\beta}^{-4} - 2\hat{\beta}^{-3}\,\hat{\psi}_\beta)\,\hat{\eta}_{\beta\beta} - \hat{\beta}^{-3}[\hat{\eta}_{\gamma\beta}\,(\,\hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma}) + \hat{\eta}_{\beta\beta}(\hat{\mathcal{L}}_{\beta\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\beta}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})]. \quad (49)$$

So, Bayes estimate of β under SISELF based on Lindley's approximation can be obtained by:

$$\hat{\beta}^L_{BSI} = \frac{E_I}{E_J} = \frac{E(\beta^{-1}|\underline{x})}{E(\beta^{-2}|\underline{x})}. \quad (50)$$

- Suppose that $k(\gamma,\beta) = \epsilon_1(t) = e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)}$.

To compute E_I Suppose that $(k(\gamma,\beta))^{-1} = (\epsilon_1(t))^{-1} = e^{(\gamma t + \frac{\beta}{2}t^2)}$, then

$k_\gamma = t\,e^{(\gamma t + \frac{\beta}{2}t^2)}$, $k_\beta = \frac{t^2}{2}\,e^{(\gamma t + \frac{\beta}{2}t^2)}$; $k_{\gamma\gamma} = t^2\,e^{(\gamma t + \frac{\beta}{2}t^2)}$; $k_{\beta\beta} = \frac{t^4}{4}\,e^{(\gamma t + \frac{\beta}{2}t^2)}$

$$k_{\gamma\beta} = k_{\beta\gamma} = \frac{t^3}{2}\,e^{(\gamma t + \frac{\beta}{2}t^2)}.$$

$$E_I = E\left(e^{\left(\gamma t + \frac{\beta}{2}t^2\right)}|\underline{x}\right).$$
$$= e^{(\hat{\gamma} t + \frac{\hat{\beta}}{2}t^2)} + \frac{1}{2}\left[(\hat{k}_{\gamma\gamma} + 2\hat{k}_\gamma\,\hat{\psi}_\gamma)\,\hat{\eta}_{\gamma\gamma} + (\hat{k}_{\gamma\beta} + 2\hat{k}_\gamma\,\hat{\psi}_\beta)\hat{\eta}_{\gamma\beta} + (\hat{k}_{\beta\gamma} + 2\hat{k}_\beta\,\hat{\psi}_\gamma)\hat{\eta}_{\beta\gamma} + (\hat{k}_{\beta\beta} + 2\hat{k}_\beta\,\hat{\psi}_\beta)\,\hat{\eta}_{\beta\beta}\right] + \frac{1}{2}\left[(\hat{k}_\beta\,\hat{\eta}_{\gamma\beta} + \hat{k}_\gamma\,\hat{\eta}_{\gamma\gamma})(\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma}) + (\hat{k}_\beta\,\hat{\eta}_{\beta\beta} + \hat{k}_\gamma\,\hat{\eta}_{\beta\gamma})(\hat{\mathcal{L}}_{\beta\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\beta}\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})\right] \quad (51)$$

Now, to compute E_J Suppose that $\left(k(\gamma,\beta)\right)^{-2} = (\epsilon_1(t))^{-2} = e^{2\,(\gamma t + \frac{\beta}{2}t^2)}$, then

$k_\gamma = 2te^{2\,(\gamma t + \frac{\beta}{2}t^2)}$; $k_\beta = t^2e^{2\,(\gamma t + \frac{\beta}{2}t^2)}$; $k_{\gamma\gamma} = 4t^2e^{2\,(\gamma t + \frac{\beta}{2}t^2)}$; $k_{\beta\beta} = t^4e^{2\,(\gamma t + \frac{\beta}{2}t^2)}$

$k_{\beta\gamma} = k_{\gamma\beta} = 2t^3e^{2\,(\gamma t + \frac{\beta}{2}t^2)}$.

$$E_J = E\left(e^{2\,(\gamma t + \frac{\beta}{2}t^2)}|\underline{x}\right).$$

$$= e^{2(\hat{\gamma}t+\frac{\hat{\beta}}{2}t^2)} + \frac{1}{2}[(\hat{k}_{\gamma\gamma}+2\hat{k}_\gamma\hat{\psi}_\gamma)\hat{\eta}_{\gamma\gamma} + (\hat{k}_{\gamma\beta}+2\hat{k}_\gamma\hat{\psi}_\beta)\hat{\eta}_{\gamma\beta} + (\hat{k}_{\beta\gamma}+2\hat{k}_\beta\hat{\psi}_\gamma)\hat{\eta}_{\beta\gamma} + (\hat{k}_{\beta\beta}+2\hat{k}_\beta\hat{\psi}_\beta)\hat{\eta}_{\beta\beta}] + \frac{1}{2}[(\hat{k}_\beta\hat{\eta}_{\gamma\beta}+\hat{k}_\gamma\hat{\eta}_{\gamma\gamma})(\hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\gamma}\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\gamma}\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\hat{\eta}_{\gamma\gamma}) + (\hat{k}_\beta\hat{\eta}_{\beta\beta}+\hat{k}_\gamma\hat{\eta}_{\beta\gamma})(\hat{\mathcal{L}}_{\beta\beta\beta}\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\beta}\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\beta}\hat{\eta}_{\gamma\gamma})] \quad (52)$$

So, Bayes estimate of $\epsilon_1(t)$ under SISELF based on Lindley's approximation, is given by:

$$\hat{\epsilon}^{\,L}_{1BSI}(t) = \frac{E_I}{E_J} = \frac{E\left((\epsilon_1(t))^{-1}|\underline{x}\right)}{E\left((\epsilon_1(t))^{-2}|\underline{x}\right)}. \quad (53)$$

- Suppose that $k(\gamma,\beta) = \epsilon_2(t) = \gamma + \beta t$.

To compute E_I Suppose that $\left(k(\gamma,\beta)\right)^{-1} = (\gamma+\beta t)^{-1}$, then

$$k_\gamma = -(\gamma+\beta t)^{-2};\ k_{\gamma\gamma} = 2(\gamma+\beta t)^{-3};\ k_\beta = -t(\gamma+\beta t)^{-2};\ k_{\beta\beta} = 2t^2(\gamma+\beta t)^{-3}$$

$$k_{\gamma\beta} = k_{\beta\gamma} = 2t(\gamma+\beta t)^{-3}$$

$$E_I = E\left((\gamma+\beta t)^{-1}|\underline{x}\right)$$

$$= (\hat{\gamma}+\hat{\beta}t)^{-1} + \frac{1}{2}[(\hat{k}_{\gamma\gamma}+2\hat{k}_\gamma\hat{\psi}_\gamma)\hat{\eta}_{\gamma\gamma} + (\hat{k}_{\gamma\beta}+2\hat{k}_\gamma\hat{\psi}_\beta)\hat{\eta}_{\gamma\beta} + (\hat{k}_{\beta\gamma}+2\hat{k}_\beta\hat{\psi}_\gamma)\hat{\eta}_{\beta\gamma} \ (\hat{k}_{\beta\beta}+2\hat{k}_\beta\hat{\psi}_\beta)\hat{\eta}_{\beta\beta}] + \frac{1}{2}[(\hat{k}_\beta\hat{\eta}_{\gamma\beta}+\hat{k}_\gamma\hat{\eta}_{\gamma\gamma})(\hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\gamma}\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\gamma}\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\hat{\eta}_{\gamma\gamma}) + (\hat{k}_\beta\hat{\eta}_{\beta\beta}+\hat{k}_\gamma\hat{\eta}_{\beta\gamma})(\hat{\mathcal{L}}_{\beta\beta\beta}\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\beta}\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\beta}\hat{\eta}_{\gamma\gamma})]. \quad (54)$$

Now, to compute E_J Suppose that $(k(\gamma,\beta))^{-2} = (\gamma+\beta t)^{-2}$, then

$$k_\gamma = -2(\gamma+\beta t)^{-3},\ k_{\gamma\gamma} = 6(\gamma+\beta t)^{-4};\ k_\beta = -2t(\gamma+\beta t)^{-3},\ k_{\beta\beta} = 6t^2(\gamma+\beta t)^{-4}$$

$$k_{\gamma\beta} = k_{\beta\gamma} = 6t(\gamma+\beta t)^{-4}$$

$$E_J = E\left((\gamma+\beta t)^{-2}|\underline{x}\right)$$

$$= (\hat{\gamma}+\hat{\beta}t)^{-2} + \frac{1}{2}[(\hat{k}_{\gamma\gamma}+2\hat{k}_\gamma\hat{\psi}_\gamma)\hat{\eta}_{\gamma\gamma} + (\hat{k}_{\gamma\beta}+2\hat{k}_\gamma\hat{\psi}_\beta)\hat{\eta}_{\gamma\beta} + (\hat{k}_{\beta\gamma}+2\hat{k}_\beta\hat{\psi}_\gamma)\hat{\eta}_{\beta\gamma} + (\hat{k}_{\beta\beta}+2\hat{k}_\beta\hat{\psi}_\beta)\hat{\eta}_{\beta\beta}] + \frac{1}{2}[(\hat{k}_\beta\hat{\eta}_{\gamma\beta}+\hat{k}_\gamma\hat{\eta}_{\gamma\gamma})(\hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\gamma}\hat{\eta}_{\beta\gamma}+\hat{\ell}_{\mathcal{L}\gamma\beta\gamma}\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\hat{\eta}_{\gamma\gamma}) + (\hat{k}_\beta\hat{\eta}_{\beta\beta}+\hat{k}_\gamma\hat{\eta}_{\beta\gamma})(\hat{\mathcal{L}}_{\beta\beta\beta}\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\beta}\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\beta}\hat{\eta}_{\gamma\gamma})]. \quad (55)$$

So, Bayes estimate of $\boldsymbol{\epsilon_2}(t)$ under SISELF based on Lindley's approximation, is given by:

$$\widehat{\boldsymbol{\epsilon_2}}^{L}_{BSI}(t) = \frac{E_I}{E_J} = \frac{E\left((\epsilon_2(t))^{-1}|\underline{x}\right)}{E\left((\epsilon_2(t))^{-2}|\underline{x}\right)}. \quad (56)$$

3.3 Approximate Bayes estimate under LLF

- Suppose that $k(\gamma,\beta) = \gamma$

$e^{-zk(\gamma,\beta)} = e^{-z\gamma}$ then

$k_\gamma = -z\,e^{-z\gamma}$; $k_{\gamma\gamma} = z^2 e^{-z\gamma}$ and $k_\beta = k_{\beta\beta} = k_{\gamma\beta} = k_{\beta\gamma} = 0$.

So, Bayes estimate of γ under LLF based on Lindley's approximation, is given by:

$$\hat{\gamma}^L_{BL} = -\frac{1}{z}ln\left[E\left(e^{-z\gamma}|\underline{x}\right)\right].$$

Where,

$$E\left(e^{-z\gamma}|\underline{x}\right) = e^{-z\hat{\gamma}} + \frac{1}{2}[(z^2 e^{-z\hat{\gamma}} - 2z\,e^{-z\hat{\gamma}}\hat{\psi}_\gamma)\hat{\eta}_{\gamma\gamma} - 2z\,e^{-z\hat{\gamma}}\hat{\psi}_\beta\hat{\eta}_{\gamma\beta}] - \frac{1}{2}z\,e^{-z\hat{\gamma}}[\hat{\eta}_{\gamma\gamma}(\hat{\mathcal{L}}_{\alpha\beta\beta}\hat{\eta}_{\beta\beta}+\hat{\mathcal{L}}_{\beta\gamma\gamma}\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\gamma}\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\gamma}\hat{\eta}_{\gamma\gamma}) + \hat{\eta}_{\beta\gamma}(\hat{\mathcal{L}}_{\beta\beta\beta}\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\beta}\hat{\eta}_{\beta\gamma}+\hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\gamma\beta}+\hat{\mathcal{L}}_{\gamma\gamma\beta}\hat{\eta}_{\gamma\gamma})]. \quad (57)$$

- Suppose that $k(\gamma,\beta) = \beta$

$e^{-zk(\gamma,\beta)} = e^{-zB}$ then

$k_\beta = -z\,e^{-z\beta}$; $k_{\beta\beta} = z^2 e^{-z\beta}$ and $k_\gamma = k_{\gamma\gamma} = k_{\gamma\beta} = k_{\beta\gamma} = 0$.

So, Bayes estimate of β under LLF based on Lindley's approximation, is given by:

$$\hat{\beta}^{L}_{BL} = -\frac{1}{z} ln\left[E\left(e^{-z\beta}|\underline{x}\right)\right].$$

Where,

$$E\left(e^{-z\beta}|\underline{x}\right)] = e^{-z\hat{\beta}} + \frac{1}{2}[-2z\, e^{-z\hat{\beta}}\ \hat{\psi}_{\gamma}\, \hat{\eta}_{\beta\gamma} + \left(z^2\, e^{-z\hat{\beta}} - 2\, z\, e^{-z\hat{\beta}}\ \hat{\psi}_{\beta}\right) \hat{\eta}_{\beta\beta} - \frac{1}{2} z\, e^{-z\hat{\beta}}[\, \hat{\eta}_{\gamma\beta}\, (\, \hat{\mathcal{L}}_{\gamma\beta\beta}\, \hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\gamma}\, \hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\gamma}\, \hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\, \hat{\eta}_{\gamma\gamma}) + \hat{\eta}_{\beta\beta}\, (\hat{\mathcal{L}}_{\beta\beta\beta}\, \hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\beta}\, \hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\beta}\, \hat{\eta}_{\gamma\gamma})]. \quad (58)$$

- Suppose that $k(\gamma,\beta) = \epsilon_1(t) = e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)}$

$e^{-zk(\gamma,\beta)} = e^{-z\, e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)}}$ then

$$k_{\gamma} = z\, t\ e^{-(\gamma t + \frac{\beta}{2}t^2)}\, e^{-z\, e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)}},\ k_{\beta} = \frac{z\, t^2}{2}\ e^{-(\gamma t + \frac{\beta}{2}t^2)}\, e^{-z\, e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)}}$$

$$k_{\gamma\gamma} = z\, t^2\ e^{-(\gamma t + \frac{\beta}{2}t^2)}\, e^{-z\, e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)}}[\, z\, e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)} - 1]$$

$$k_{\beta\beta} = \frac{z\, t^4}{4}\ e^{-(\gamma t + \frac{\beta}{2}t^2)}\, e^{-z\, e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)}}[\, z\, e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)} - 1]$$

$$k_{\gamma\beta} = k_{\beta\gamma} = \frac{z\, t^3}{2}\ e^{-(\gamma t + \frac{\beta}{2}t^2)}\, e^{-z\, e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)}}\left[\, z\, e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)} - 1\right].$$

So, Bayes estimate of $\epsilon_1(t)$ under LLF based on Lindley's approximation, is given by:

$$\hat{\epsilon}^{L}_{1BL}(t) = -\frac{1}{z} ln\left[E\left(e^{-z\epsilon_1(t)}|\underline{x}\right)\right].$$

Where,

$$E\left(e^{-z\epsilon_1(t)}|\underline{x}\right) = e^{-z\, e^{-\left(\hat{\gamma} t + \frac{\hat{\beta}}{2}t^2\right)}} + \frac{1}{2}\left[\left(\hat{k}_{\gamma\gamma} + 2\hat{k}_{\gamma}\, \hat{\psi}_{\gamma}\right)\hat{\eta}_{\gamma\gamma} + \left(\hat{k}_{\gamma\beta} + 2\hat{k}_{\gamma}\, \hat{\psi}_{\beta}\right)\hat{\eta}_{\gamma\beta}\, (\hat{k}_{\beta\gamma} + 2\hat{k}_{\beta}\, \hat{\psi}_{\gamma})\, \hat{\eta}_{\beta\gamma} + \left(\hat{k}_{\beta\beta} + 2\hat{k}_{\beta}\, \hat{\psi}_{\beta}\right)\hat{\eta}_{\beta\beta}\right] + \frac{1}{2}[(\hat{k}_{\beta}\, \hat{\eta}_{\gamma\beta} + \hat{k}_{\gamma}\, \hat{\eta}_{\gamma\gamma})(\hat{\mathcal{L}}_{\gamma\beta\beta}\, \hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\gamma}\, \hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\gamma}\, \hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\, \hat{\eta}_{\gamma\gamma}) + \left(\hat{k}_{\beta}\, \hat{\eta}_{\beta\beta} + \hat{k}_{\gamma}\, \hat{\eta}_{\beta\gamma}\right)(\hat{\mathcal{L}}_{\beta\beta\beta}\, \hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\beta}\, \hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\beta}\, \hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\beta}\, \hat{\eta}_{\gamma\gamma})]. \quad (59)$$

- Suppose that $k(\gamma,\beta) = \epsilon_2(t) = \gamma + \beta t$

$e^{-zk(\gamma,\beta)} = e^{-z(\gamma+\beta t)}$, then

$k_{\gamma} = -z\, e^{-z(\gamma+\beta t)}$; $k_{\beta} = -z\, t\, e^{-z(\gamma+\beta t)}$; $k_{\gamma\gamma} = z^2\, e^{-z(\gamma+\beta t)}$

$k_{\beta\beta} = z^2 t^2\, e^{-z\,(\gamma+\beta t)}$ and $k_{\gamma\beta} = k_{\beta\gamma} = z^2 t\, e^{-z(\gamma+\beta t)}$.

So, Bayes estimate of $\epsilon_2(t)$ under LLF based on Lindley's approximation, is given by:

$$\hat{\epsilon}^{L}_{2BL}(t) = -\frac{1}{z} ln\left[E\left(e^{-z\epsilon_2(t)}|\underline{x}\right)\right].$$

Where,

$$E\left(e^{-z\epsilon_2(t)}|\underline{x}\right) = e^{-z(\hat{\gamma}+\hat{\beta} t)} + \frac{1}{2}\left[\left(\hat{k}_{\gamma\gamma} + 2\hat{k}_{\gamma}\, \hat{\psi}_{\gamma}\right)\hat{\eta}_{\gamma\gamma} + \left(\hat{k}_{\gamma\beta} + 2\hat{k}_{\gamma}\, \hat{\psi}_{\beta}\right)\hat{\eta}_{\gamma\beta} + \left(\hat{k}_{\beta\gamma} + 2\hat{k}_{\beta}\, \hat{\psi}_{\gamma}\right)\hat{\eta}_{\beta\gamma} + \left(\hat{k}_{\beta\beta} + 2\hat{k}_{\beta}\, \hat{\psi}_{\beta}\right)\hat{\eta}_{\beta\beta}\right] + \frac{1}{2}[(\hat{k}_{\beta}\, \hat{\eta}_{\gamma\beta} + \hat{k}_{\gamma}\, \hat{\eta}_{\gamma\gamma})(\hat{\mathcal{L}}_{\gamma\beta\beta}\, \hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\gamma}\, \hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\gamma}\, \hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\, \hat{\eta}_{\gamma\gamma}) + \left(\hat{k}_{\beta}\, \hat{\eta}_{\beta\beta} + \hat{k}_{\gamma}\, \hat{\eta}_{\beta\gamma}\right)(\hat{\mathcal{L}}_{\beta\beta\beta}\, \hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\beta}\, \hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\beta}\, \hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\beta}\, \hat{\eta}_{\gamma\gamma})]. \quad (60)$$

3.4 Approximate Bayes estimate under PLF

- Suppose that $k(\gamma,\beta) = \gamma$

$(k(\gamma,\beta))^2 = \gamma^2$ then

$k_{\gamma} = 2\gamma;\ k_{\gamma\gamma} = 2$ and $k_{\beta} = k_{\beta\beta} = k_{\gamma\beta} = k_{\beta\gamma} = 0$.

So, Bayes estimate of γ under PLF based on Lindley's approximation, is given by:

$$\hat{\gamma}^{L}_{BP} = \sqrt{E\left(\gamma^2|\underline{x}\right)}\,.$$

Where,

$$E(\gamma^2|\underline{x}) = \hat{\gamma}^2 + (1 + 2\hat{\gamma}\,\hat{\psi}_\gamma)\,\hat{\eta}_{\gamma\gamma} + 2\hat{\gamma}\,\hat{\psi}_\beta\,\hat{\eta}_{\gamma\beta} + \hat{\gamma}\,[\hat{\eta}_{\gamma\gamma}(\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma}) + \hat{\eta}_{\beta\gamma}(\hat{\mathcal{L}}_{\beta\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\beta}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})]. \quad (61)$$

- Suppose that $k(\gamma,\beta) = \beta$

$(k(\gamma,\beta))^2 = \beta^2$ then

$k_\beta = 2\beta$; $k_{\beta\beta} = 2$ and $k_\gamma = k_{\gamma\gamma} = k_{\gamma\beta} = k_{\beta\gamma} = 0.$

So, Bayes estimate of β under PLF based on Lindley's approximation, is given by:

$$\hat{\beta}^L_{BP} = \sqrt{E(\beta^2|\underline{x})}\text{ ,where } E(\beta^2|\underline{x}) = \hat{\beta}^2 + 2\,\hat{\beta}\hat{\psi}_\gamma\,\hat{\eta}_{\beta\gamma} + (1 + 2\hat{\beta}\,\hat{\psi}_\beta)\,\hat{\eta}_{\beta\beta} + \hat{\beta}\,[\hat{\eta}_{\gamma\beta}(\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma}) + \hat{\eta}_{\beta\beta}(\hat{\mathcal{L}}_{\beta\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\beta}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})]. \quad (62)$$

- Suppose that $k(\gamma,\beta) = \epsilon_1(t) = e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)}$

$(k(\gamma,\beta))^2 = (e^{-\left(\gamma t + \frac{\beta}{2}t^2\right)})^2 = e^{-2\left(\gamma t + \frac{\beta}{2}t^2\right)}$, then

$k_\gamma = -2\,t\;e^{-2\left(\gamma t + \frac{\beta}{2}t^2\right)}$; $k_\beta = -t^2\,e^{-2\,(\gamma t + \frac{\beta}{2}t^2)}$; $k_{\gamma\gamma} = 4\,t^2\;e^{-2\,(\gamma t + \frac{\beta}{2}t^2)}$

$k_{\beta\beta} = t^4\;e^{-2\,(\gamma t + \frac{\beta}{2}t^2)}$; $k_{\gamma\beta} = k_{\beta\gamma} = 2\,t^3\,e^{-2\,(\gamma t + \frac{\beta}{2}t^2)}$.

So, Bayes estimate of $\epsilon_1(t)$ under PLF based on Lindley's approximation, is given by:

$\hat{\epsilon}^{\,L}_{1BP}(t) = \sqrt{E((\epsilon_1(t))^2|\underline{x})}$, where

$$E((\epsilon_1(t))^2|\underline{x}) = e^{-2\left(\hat{\gamma}t + \frac{\hat{\beta}}{2}t^2\right)} + \frac{1}{2}\,[(\hat{k}_{\gamma\gamma} + 2\hat{k}_\gamma\,\hat{\psi}_\gamma)\,\hat{\eta}_{\gamma\gamma} + (\hat{k}_{\gamma\beta} + 2\hat{k}_\gamma\,\hat{\psi}_\beta)\,\hat{\eta}_{\gamma\beta} + (\hat{k}_{\beta\gamma} + 2\hat{k}_\beta\,\hat{\psi}_\gamma)\,\hat{\eta}_{\beta\gamma} + (\hat{k}_{\beta\beta} + 2\hat{k}_\beta\,\hat{\psi}_\beta)\,\hat{\eta}_{\beta\beta}] + \frac{1}{2}[(\hat{k}_\beta\,\hat{\eta}_{\gamma\beta} + \hat{k}_\gamma\,\hat{\eta}_{\gamma\gamma})(\hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma}) + (\hat{k}_\beta\,\hat{\eta}_{\beta\beta} + \hat{k}_\gamma\,\hat{\eta}_{\beta\gamma})(\hat{\mathcal{L}}_{\beta\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\beta}\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})]. \quad (63)$$

- Suppose that $k(\gamma,\beta) = \epsilon_2(t) = \gamma + \beta t$

$(k(\gamma,\beta))^2 = (\gamma + \beta t)^2$, then

$k_\gamma = 2(\gamma + \beta t)$; $k_\beta = 2t\,(\gamma + \beta t)$; $k_{\gamma\gamma} = 2$; $k_{\beta\beta} = 2t^2$; $k_{\gamma\beta} = k_{\beta\gamma} = 2t.$

So, Bayes estimate of $\epsilon_2(t)$ under PLF based on Lindley's approximation, is given by:

$\hat{\epsilon}^{\,L}_{2BP}(t) = \sqrt{E((\epsilon_2(t))^2|\underline{x})}$, where

$$E\left((\epsilon_2(t))^2|\underline{x}\right) = (\gamma + \beta t)^2 + \frac{1}{2}\,[(\hat{k}_{\gamma\gamma} + 2\hat{k}_\gamma\,\hat{\psi}_\gamma)\,\hat{\eta}_{\gamma\gamma} + (\hat{k}_{\gamma\beta} + 2\hat{k}_\gamma\,\hat{\psi}_\beta)\,\hat{\eta}_{\gamma\beta} + (\hat{k}_{\beta\gamma} + 2\hat{k}_\beta\,\hat{\psi}_\gamma)\,\hat{\eta}_{\beta\gamma} + (\hat{k}_{\beta\beta} + 2\hat{k}_\beta\,\hat{\psi}_\beta)\,\hat{\eta}_{\beta\beta}] + \frac{1}{2}[(\hat{k}_\beta\,\hat{\eta}_{\gamma\beta} + \hat{k}_\gamma\,\hat{\eta}_{\gamma\gamma})\,(\hat{\mathcal{L}}_{\gamma\beta\beta}\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\gamma}\,\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\gamma}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\gamma}\,\hat{\eta}_{\gamma\gamma}) + (\hat{k}_\beta\,\hat{\eta}_{\beta\beta} + \hat{k}_\gamma\,\hat{\eta}_{\beta\gamma})(\hat{\mathcal{L}}_{\beta\beta\beta}\,\hat{\eta}_{\beta\beta} + \hat{\mathcal{L}}_{\beta\gamma\lambda}\hat{\eta}_{\beta\gamma} + \hat{\mathcal{L}}_{\gamma\beta\beta}\,\hat{\eta}_{\gamma\beta} + \hat{\mathcal{L}}_{\gamma\gamma\beta}\,\hat{\eta}_{\gamma\gamma})]. \quad (64)$$

4. Numerical experiments

In this manuscript, a Monte Carlo style study was performed to generate an independent identical distributed random sample [10], according to ER distribution through the adoption of inverse transformation method with size $n = 20, 50$ and 200 to take care of small, medium and large data sets. The number of samples replicated chosen to be ($L = 1000$). The modified maximum likelihood and approximation Bayes estimators of the unknown parameters along with the reliability and hazard rate functions of ER distribution have been computed with five different cases as shown below,

Parameters	Case1	Case2	Case3	Case4	Case5

γ	0.4	0.9	1	0.9	1
β	0.8	0.8	0.8	1	1

The initial values required for proceeding with the Modified maximum likelihood chosen to be the mean and the iterative process stops when the absolute difference between two successive iterations becomes less than $er = 0.0001$ and $\sigma = 0.1$. The values of hyperparameter associated with prior distributions are chosen to be $a = b = c = 0.0001$. Choose the value of Linex loss function constant $z = 2$. The comparisons between parameter estimates were based on values from Mean Square Error (MSE) while it was based on values from Integrated Mean Square Error ($IMSE$) for the estimates of the reliability function and it were based on Integrated mean absolute percentage error ($IMAPE$) of the hazard rate function [11]:

$$MSE(\gamma) = \sum_{m=1}^{L} \frac{(\hat{\gamma}_m - \gamma)^2}{L}, \tag{65}$$

$\hat{\gamma}_m$ = is the estimate of γ at the m^{th} replicate.

$$MSE(\beta) = \sum_{m=1}^{L} \frac{\left(\hat{\beta}_m - \beta\right)^2}{L}. \tag{66}$$

$\hat{\beta}_m$ = is the estimate of β at the m^{th} replicate

$$IMSE\left(\widehat{\epsilon_1}(t)\right) = \sum_{m=1}^{L} \left\{ \frac{\left(\sum_{i=1}^{r_t} \frac{\left(\widehat{\epsilon_1 m}(t_i) - \epsilon_1(t_i)\right)^2}{r_t} \right)}{L} \right\}, \tag{67}$$

r_t = is the number of times chosen to be 4 (where $t = 1, 2, 3, 4$)

$\widehat{\epsilon_{1m}}(t_i)$ = is the estimate of$\epsilon_1(t)$ at the m^{th} replicateand i^{th} time

$$IMAPE\left(\widehat{\epsilon_2}(t)\right) = \frac{1}{L} \sum_{m=1}^{L} \left\{ \frac{\sum_{i=1}^{r_t} \left| \frac{\left(\epsilon_{2m}(t_i) - \widehat{\epsilon_{2m}}(t_i)\right)}{\epsilon_{2m}(t_i)} \right|}{r_t} \right\}, \tag{68}$$

$\widehat{\epsilon_{2m}}(t_i)$ = is the estimate of$\epsilon_2(t)$ at the m^{th} replicateand i^{th} time.

Table 1: MSE of the estimates the Parameter γ of ER distribution for different sample sizes

n	***Cases***		***Estimates of γ***					***Best Estimate of γ***
	γ	β	$\hat{\gamma}_{ER}^{MLE}$	$\hat{\gamma}_{BS}^{L}$	$\hat{\gamma}_{BSI}^{L}$	$\hat{\gamma}_{BL}^{L}$	$\hat{\gamma}_{BP}^{L}$	
20	0.4	0.8	0.0050831	0.0099610	0.0100323	0.0285540	0.0099500	MLE
	0.9	0.8	0.4681152	1.0151821	1.0190918	0.9009856	1.0142139	MLE
	1	0.8	0.6324724	1.3571389	1.3599063	1.2298908	1.3564509	MLE
	0.9	1	0.2378978	0.5546275	0.5616290	0.4334765	0.5529055	MLE
	1	1	0.3824130	0.8610120	0.8688998	0.7099256	0.8590662	MLE
50	0.4	0.8	0.0037083	0.0049876	0.0049918	0.0240514	0.0049870	MLE
	0.9	0.8	0.4340271	0.9548039	0.9555191	0.8411406	0.9546254	MLE
	1	0.8	0.6082837	1.3129878	1.3141600	1.1855733	1.3126955	MLE
	0.9	1	0.2059748	0.4952687	0.4958907	0.3763031	0.4951134	MLE
	1	1	0.3458621	0.7931901	0.7945588	0.6472083	0.7928487	MLE
200	0.4	0.8	0.0024198	0.0012717	0.0012720	0.0209602	0.0012716	PLF
	0.9	0.8	0.4276180	0.9434166	0.9434630	0.8297639	0.9434050	MLE
	1	0.8	0.5970435	1.2915602	1.2916707	1.1643740	1.2915326	MLE
	0.9	1	0.1181990	0.4791555	0.4792328	0.3611808	0.4791361	MLE
	1	1	0.3315027	0.7660132	0.7661015	0.6198825	0.7659912	MLE

Table 2: MSE of the estimates the parameter β of ER distribution for different sample sizes

n	Cases		Estimates of β					Best Estimate of β
	γ	β	$\hat{\beta}_{ER}^{MLE}$	$\hat{\beta}_{BS}^{L}$	$\hat{\beta}_{BSI}^{L}$	$\hat{\beta}_{BL}^{L}$	$\hat{\beta}_{BP}^{L}$	
20	0.4	0.8	0.1498505	0.1791646	0.1754924	0.2743217	0.1802931	MLE
	0.9	0.8	0.3415608	0.5642187	0.5534060	0.6601824	0.5665655	MLE
	1	0.8	0.3547434	0.5946436	0.5879103	0.6884973	0.5961756	MLE
	0.9	1	0.3440228	0.4945332	0.4808370	0.6429151	0.4977292	MLE
	1	1	0.3824130	0.5854082	0.5598214	0.7213730	0.5794723	MLE
50	0.4	0.8	0.1363115	0.1537411	0.1525546	0.2476700	0.1540339	MLE
	0.9	0.8	0.3124351	0.5079313	0.5063886	0.6037469	0.5083091	MLE
	1	0.8	0.3367209	0.5593682	0.5568883	0.6530982	0.5599668	MLE
	0.9	1	0.3062567	0.4197749	0.4187146	0.5669869	0.4200380	MLE
	1	1	0.3458621	0.5030750	0.5009649	0.6487923	0.5035944	MLE
200	0.4	0.8	0.1289290	0.1387902	0.1387174	0.2330422	0.1388083	MLE
	0.9	0.8	0.3069435	0.4971512	0.4970554	0.5930308	0.4971751	MLE
	1	0.8	0.3281777	0.5418177	0.5415988	0.6354602	0.5418722	MLE
	0.9	1	0.2970338	0.4019835	0.4018542	0.5483369	0.4020158	MLE
	1	1	0.3315027	0.4738677	0.4737371	0.6199817	0.4739003	MLE

Table 3: IMSE of the estimates the reliability function of ER distribution for different sample sizes

n	Cases		Estimates of $\epsilon_1(t)$					Best Estimate of $\epsilon_1(t)$
	γ	β	$\widehat{\epsilon_1}_{ER}^{MLE}(t)$	$\widehat{\epsilon_1}_{BS}^{L}(t)$	$\widehat{\epsilon_1}_{BSI}^{L}(t)$	$\widehat{\epsilon_1}_{BL}^{L}(t)$	$\widehat{\epsilon_1}_{BP}^{L}(t)$	
20	0.4	0.8	0.0066091	0.0051270	0.0051830	0.0051135	0.0051081	PLF
	0.9	0.8	0.1059359	0.0941876	0.0933256	0.0941358	0.0942289	SISELF
	1	0.8	0.1212146	0.1107062	0.1102064	0.1106746	0.1107514	SISELF
	0.9	1	0.0318309	0.0280943	0.0281349	0.0280303	0.0280437	LLF
	1	1	0.0443648	0.0397714	0.0397264	0.0395038	0.0397068	LLF
50	0.4	0.8	0.0043947	0.0034652	0.0034817	0.0034615	0.0034607	PLF
	0.9	0.8	0.0860386	0.0805243	0.0804929	0.0805120	0.0805288	SISELF
	1	0.8	0.1092829	0.1020770	0.1019675	0.1020609	0.1020935	SISELF
	0.9	1	0.0240267	0.0223576	0.0223731	0.0223531	0.0223535	LLF
	1	1	0.0347095	0.0326035	0.0326313	0.0325930	0.0325956	LLF
200	0.4	0.8	0.0031601	0.0024021	0.0024032	0.0024019	0.0024018	PLF
	0.9	0.8	0.0822716	0.0777460	0.0777450	0.0777452	0.0777463	SISELF
	1	0.8	0.1025472	0.0966006	0.0965949	0.0965989	0.0966019	SISELF
	0.9	1	0.0222763	0.0207861	0.0207881	0.0207855	0.0207856	LLF
	1	1	0.0312589	0.0296133	0.0296154	0.0296127	0.0296128	LLF

Table 4: IMAPE of the estimates the hazard function of ER distribution for different sample sizes

n	Cases		Estimates of $\epsilon_2(t)$					Best Estimate of $\epsilon_2(t)$
	γ	β	$\widehat{\epsilon_{2}}_{ER}^{MLE}(t)$	$\widehat{\epsilon_{2}}_{BS}^{L}(t)$	$\widehat{\epsilon_{2}}_{BSI}^{L}(t)$	$\widehat{\epsilon_{2}}_{BL}^{L}(t)$	$\widehat{\epsilon_{2}}_{BP}^{L}(t)$	
20	0.4	0.8	0.3733574	0.2919368	0.2795575	0.2842135	0.2949865	SISELF
	0.9	0.8	0.7388867	0.6976275	0.6842282	0.6938133	0.7005718	SISELF
	1	0.8	0.7618303	0.7272667	0.7193601	0.7250947	0.7290798	SISELF
	0.9	1	0.5666635	0.5020543	0.4861704	0.4923933	0.5057665	SISELF
	1	1	0.6123020	0.5588770	0.5399736	0.5467110	0.5603530	SISELF
50	0.4	0.8	0.3503542	0.2734094	0.2700109	0.2712790	0.2742447	SISELF
	0.9	0.8	0.7090616	0.6747264	0.6727215	0.6740593	0.6752177	SISELF
	1	0.8	0.7439795	0.7120920	0.7090919	0.7111822	0.7128177	SISELF
	0.9	1	0.5364756	0.4822055	0.4809474	0.4813848	0.4825172	SISELF
	1	1	0.5854914	0.5371673	0.5349307	0.5358238	0.5377168	SISELF
200	0.4	0.8	0.3389581	0.2640841	0.2638735	0.2639506	0.2641367	SISELF
	0.9	0.8	0.7035472	0.6705270	0.6704017	0.6704843	0.6705583	SISELF
	1	0.8	0.7361720	0.7058381	0.7055708	0.7057538	0.7059047	SISELF
	0.9	1	0.5289994	0.4754265	0.4752728	0.4753255	0.4754648	SISELF
	1	1	0.5749528	0.5278662	0.5277273	0.5277817	0.5279009	SISELF

5. Concluding remarks from numerical experiments

Modified maximum likelihood and approximate Bayes Estimates have been used to estimate of the two unknown parameters γ and β, reliability and hazard rate functions of ER distribution. We offer a numerical comparison through using a Monte Carlo style to obtained estimates the two unknown parameters γ and β, reliability and hazard rate functions. By MSE, IMSE and IMAPE, respectively.

The most essential concluding observations of the numerical results are:

- Concluding from Table (1): The modified maximum likelihood estimates showed the best performance in comparison with other estimates with all cases and for all sample sizes except for large sample size with parameters, $\gamma = 0.4, \beta = 0.8$, where approximate Bayes estimate under precautionary loss function was the best; also, we found approximate Bayes estimate under Linex loss function gave the best performance in comparison with other approximate Bayes estimates for all sample sizes and in all cases except for the case, $\gamma = 0.4, \beta = 0.8$, where approximate Bayes estimate under precautionary loss function was the best.
- Concluding from Table (2): The modified maximum likelihood estimates showed the best performance in comparison in other estimates with all cases and for all sample sizes also. We found approximate Bayes estimate under scale invariant squared error loss function give the best performance in comparison with other approximate Bayes estimates for all sample sizes and with all cases.
- Concluding from Table (3): Approximate Bayes estimate under precautionary loss function showed the best performance in comparison in other estimates for all sample sizes with case, $\gamma = 0.4, \beta = 0.8$.We found approximate Bayes estimate under scale invariant squared error loss function gave the best performance in comparison with other estimates for all sample sizes with two cases, $\gamma = 0.9, \beta = 0.8$ and $\gamma = 1, \beta = 0.8$. and approximate Bayes estimate under Linex loss function gave the best performance in comparison with other estimates for all sample sizes and with two cases, $\gamma = 0.9, \beta = 1$ and $\gamma = 1, \beta = 1$.

Concluding from Table (4): Approximate Bayes estimate under scale invariant squared error loss function showed the best performance in comparison with other estimates for all sample sizes under all scenarios.

Based on our findings, we recommend primarily employing modified maximum likelihood estimates for estimating the two parameters of the ER distribution. For estimating the reliability function of the ER distribution, we suggest using approximate Bayes estimates under the precautionary loss function with γ=0.4 and β=0.8, and under the scale-invariant squared error with γ=0.9,1 and β=0.8. For the hazard rate function of the ER distribution, we recommend using approximate Bayes estimates under the scale-invariant squared error loss function.

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