Influence of MHD for Newtonian Fluid and Heat Transfer in Microchannals between Two Parallel Plates Using HAM

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Abstract

The aim of this paper is the study of the influence magnetic field on steady state flows and heat transfer in microchannels between two parallel plates.

It is found that the motion equations are controlled by many dimensionless parameter, namely magnetic field parameter M Reynolds number Re, physical quantity at wall W and Knudsen number Kn also found that the energy equations are controlled by many dimensionless parameter, namely magnetic field parameter M Reynolds number Re, physical quantity at wall W and Knudsen number Kn , Prinkman number Br and Peclet number Pe.

The equations which controlled this type of fluid flow are complicated, so finding an analytical solution is not easy.

We obtained the velocity and energy distribution by using homotopy analysis method (HAM).

We have been studied the influence of all the physical parameters, that mentioned above on the velocity And heat transfer distribution .

This study is done through drawing about (30) graph by using the Mathematica package.

Keywords: Temperature jump, Viscous dissipation, Homotopy analysis method.

تأثير الحقل المغناطيسي على مائع نيوتيني وانتقال الحراره بين صفيحتين متوازيتين باستخدام HAM

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الخلاصة:

الهدف من هذا البحث هو دراسة تأثير الحقل المغناطيسي على جريان مستقر لمائع ثابت اللزوجة وانتقال الحراره في الانابيب الدقيقه بين صفيحتين متوازيتين .

لقد تبين أن معادلة الحركة تحكمها بعض المعلمات اللابعدية ، مثل معلمة الحقل المغناطيسي M، وعدد رينولدز Re ، و الكمية الفعلية في الجدار Wو عدد نودسن kn وهذه المعادلات الغير خطيه تم حلها باستخدام طريقه الهوموتوبي التحليليه .

وكذالك تبين ان معادله الطاقه تحكمها اي ضا" بعض المعلمات اللابعدية ، مثل معلمة الحقل المغناطيسي M ، وعدد رينولد Re ، و الكمية الفعلية في الجدار Wو عدد نودسن kn و عدد برنك مان Br ، وعدد بكلت Pe

ان المعادلات التي تحكم هذا النوع من المسائل ذات طبيعه معقده وبالتالي ايجاد الحل التحليلي غير سهل ، قمنا بدراسة تأثير كل من الاعداد الفيزيائية المذكورة اعلاه على توزيع السرعة وانتقال الحراره

هذه الدراسة قد تمت من خلال رسم حوالي (30) بيان بأستخدام البرنامج الجاهز Mathematica .

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1. Introduction

Fluid is that state of matter, which is capable of changing shape and is capable of flowing. Fluids may be classified as real "viscous" and ideal "perfect" according to whether the fluid is capable of exerting shearing stress or not. Real fluid is called Newtonian if the relation between stress and rate of strain is linear, otherwise is called non-Newtonian fluid [1].

Magnetofluiddynamics (MFD) is that branch of applied mathematics which deals with the flow of electrically conducting fluids in electric and magnetic fields. It unified in a common framework the electromagnetic and fluid-dynamic theories to yield a description of the concurrent effects of the magnetic field on the flow and the flow on the magnetic field [2].

The magnetohydrodynamic (MHD) phenomenon is characterized by an interaction between the hydrodynamic and boundary layer and the electromagnetic field. The studies of boundary layer flows of viscous and non-Newtonian fluids over a stretching surface have received much attention because of their extensive applications in the field of metallurgy and chemical engineering, for example, in the extrusion of polymer sheet from a dye . Such investigations of magnetohydrodynamic (MHD) flows are very important industrially and have applications in different areas of researches such as petroleum production and metallurgical processes, it is now well known that in technological applications the non-Newtonian fluids are more appropriate than the Newtonian fluids [2].

A systematic research on micro devices and MEMS started in the late 1980's. Micro ducts, micro nozzles, micro pumps, micro turbines and micro valves are the examples of the devices involving liquid and gas flows. Modeling mass, momentum and energy transport may necessitate including slip, rarefaction, compressibility, intermolecular forces and other unconventional effects. The Knudsen number (Kn) can classify the gas flow in micro channel into four flow regimes: continuum flow (Kn < 0.001), slip flow (0.001 < Kn < 0.1), transition flow (0.1 < Kn < 10) and free molecular flow (Kn > 10) [3]. Since Navier–Stokes (N–S) equations are not valid for Kn beyond 0.1, the lattice Boltzmann method (LBM) was developed as an alternative numerical scheme [4] and [5].

However, for flows in continuum and slip regimes, Eckert and Drake [6] have indicated that there is strong evidence to use the N–S equations modified by boundary conditions. Tsien [7] originally designated the regime next to continuum flow as the "slip flow", following Maxwell and Smoluchowski in assuming that the first failure of continuum theory would occur at gas–solid interfaces, where the empirical conditions of continuity of tangential velocity and temperature should give way to the slip and temperature-jump boundary conditions. Studies of the continuum theory warn that in principle the N–S-plus-slip theory lacks internal consistency, but the try-it-and-see approach has yielded a substantial body of practically satisfactory results[8]. Liu et al. [9] and Arkilic et al.

In this paper , we attempt to obtain analytical solutions for the imposed problem. The HAM proposed by Liao[10-16] is employed to solve the problem. Many types of nonlinear problems were solved with HAM in the literatures [17-21] which verify the validity of the method. For latest development, Please refer to [22].

In the Previous study, they derive the similarity solutions for flows and heat transfer in micro channels. By using similarity transformation, they change the governing equations into ordinary differential equations . The homotopy analysis method (HAM), an analytical method originally from a basic concept in topology, is employed to solve the non-linear coupled ODEs. The rarefied effects on velocity profile and friction constant are obtained. Both the constant heat flux (CHF) and the constant wall temperature (CWT) boundary conditions are considered. The combined effects of the Br and Kn on Nu are exhibited.

Now, we study, effect of MHD on flow and heat transfer in micro channels between two parallel plates using HAM.

2. Basic ideas of HAM:-

This method is proposed by Liao [23-25]. Below the outline of the HAM will be presented.

Consider a non-linear equation governed by

$$A(u) + f(r) = 0 \quad (1)$$

where A is a non-linear operator, $f(r)$ is a known function and u is an unknown function. By means of homotopy analysis method, one first construct a family of equations

$$(1-p)\ell[v(r,p) - u_0(r)] = ph\{A[v(r,p)] - f(r)\}, \quad (2)$$

where ℓ is an auxiliary linear operator ℓ (u)=0 then u=0, $u_0(r)$ is an initial guess, h is an auxiliary parameter, $p \in [0,1]$ is an embedding parameter, $v(r,p)$ is an unknown function of r and p. Liao [20,21] expanded $v(r,p)$ in Taylor series about the embedding parameter

$$\mathrm{v(r,p)} = \mathrm{u}_0(r) + \sum_{m=1}^{\infty} u_m(r)\, p^m, \tag{3}$$

where

$$u_m(r) = \frac{1}{m!} \left. \frac{\partial^m v(r,p)}{\partial p^m} \right|_{p=0}. \tag{4}$$

The convergence of the series (3) depends upon the auxiliary parameter h. If it is convergent at $p=1$, one has

$$u(r) = u_0(r) + \sum_{m=1}^{\infty} u_m(r) \tag{5}$$

Differentiating the zeroth order deformation equation (2) m-time with respect to p and then dividing them by $m!$ and finally setting $p=0$ we obtain the following m-th order deformation problem:

$$\ell[u_m(r) - \chi_m u_{m-1}(r)] = hR_m(r), \tag{6}$$

in which

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases} \tag{7}$$

$$R_m(r) = \frac{1}{(m-1)!} \left\{ \frac{d^{m-1}}{dp^{m-1}} A\left[u_0(r) + \sum_{m=1}^{\infty} u_m(r)\, p^m \right] \right\} \Bigg|_{p=0} \tag{8}$$

There are many different ways to get the higher order deformation equation. However, according to the fundamental theorem in calculus, the term $u_m(r)$ in the series (3) is unique. Note that the HAM contains an auxiliary parameter h, which provides us with a simple way to control and adjust the series solution (5).

3. Governing equation:-

As depicted in figure-1, the inlet velocity and temperature profile are assumed to be uniform, the distance between the two pare-1, allel plates is 2H. The governing equations based on the Naver-Stokes equations with slip-flow boundary conditions at the walls are used to describe the physical processes. The process is assumed to be two-dimensional steady laminar flow. The body forces and the effect of compressibility are neglected. The tangential accommodation coefficient and thermal accommodation coefficient are assumed to be unity. Then, the mathematical model for slip flow between two parallel plates can be given by

Continuity Equation $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{9}$$

N-S Equation $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon \nabla^2 u - \frac{\sigma B_0^2 u}{\rho} \tag{10}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon \nabla^2 v \tag{11}$$

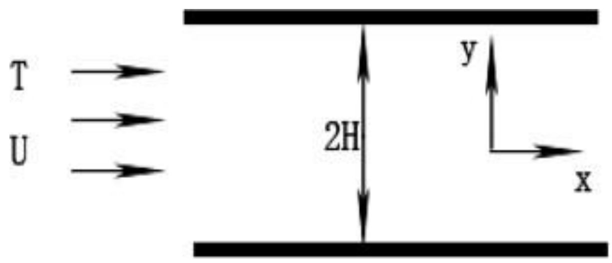

Figure 1- Microchannel between two parallel plates

By taking the viscous dissipation effects into account, energy equation is written as

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial x^2}+\frac{\upsilon}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 \qquad (12)$$

The boundary conditions are

$$y=0\begin{cases}u=l\dfrac{\partial u}{\partial y}\quad , \qquad v=0 & (a)\\ T-T_w=l\dfrac{\partial t}{\partial y} & (b)\end{cases} \qquad (13)$$

$$y=H\ :\qquad \frac{\partial u}{\partial y}=0\ ,\ \frac{\partial T}{\partial y}=0 \qquad (14)$$

The variables can be separated using similarity transformations based on the stream function ψ is defined as

$$u=\frac{\partial\psi}{\partial y}\ ,\ v=-\frac{\partial\psi}{\partial x}$$

$$\eta=\frac{y}{H}\quad ,\ \psi=Ux\,f(\eta) \qquad (15)$$

Using the expressions (15), Eq. (11) can be simplified as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right)=0 \qquad (16)$$

It can be concluded from the above that the function $\frac{\partial p}{\partial x}$ is independent of variable y, which means we can assume $\frac{\partial p}{\partial x}$ is of the form:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=C_x \qquad (17)$$

Here C_x is a constant

For CWT energy equations, the dimensionless temperatures are given by

$$\theta(\eta)=\frac{H^2}{x^2}\frac{T-T_w}{T_m-T_w} \qquad (18)$$

Substituting the Eqs.(15) and (17) into (10) and plug Eqs. (18) into (12), we obtain following ordinary differential equations with boundary conditions:

$$f^{'''}(\eta)-\mathrm{Re}\,f^{'2}(\eta)+\mathrm{Re}\,f(\eta)\,f^{''}(\eta)+W-M\ f^{'}(\eta)=0 \qquad (19)$$

$$f^{'}(0)-kn\,f^{''}(0)=0,\ f(0)=0,\ f^{''}(1)=0 \qquad (20)$$

Energy equation with corresponding boundary condition reduces to:

$$\theta^{''}(\eta)+Br.\,f^{''2}(\eta).2Pe.\,f^{'}(\eta).\theta(\eta)-Pe.f(\eta).\theta^{'}(\eta)=0 \qquad (21)$$

$$\theta(0)-kn.\theta(0)=0\,,\ \theta^{''}(1)=0 \tag{22}$$

Here, $\mathrm{Re}=\dfrac{UH}{\nu}\,, W=\dfrac{UH^3}{\nu U}\,, M=\dfrac{\sigma B_0^2 u H^2}{\rho\nu}\,,\ Pe=\dfrac{UH}{\alpha}\,, Br=\dfrac{1}{\lambda}\dfrac{\mu U^2}{T_w-T_m}$ (23)

4. Solution of the governing equation:-

In this section, we attempt to obtain analytical solutions for the imposed problem. using the HAM.

4.1. Basic procedure

For the HAM solving procedure, we first select initial guess solutions as follows:

$$f_0(\eta)=\frac{1}{6}\eta^3-\frac{1}{2}\eta^2-kn.\eta \tag{24}$$

$$\theta_0(\eta)=kn+\eta-\frac{1}{2}\eta^2 \tag{25}$$

Then we define the linear operators

$$L_1\Phi(\xi,\eta)=\Phi^{'''}(\xi,\eta),\ \ L_2\Theta(\xi,\eta)=\Theta^{''}(\xi,\eta) \tag{26}$$

Here, either for the constant heat flux or constant wall temperature case the initial solutions of energy equation is in second order, we can use the same linear operator $\theta^{''}(\xi,\eta)$. Further more, for the stated two cases having the same energy equation, the nonlinear operator also could be the same. The nonlinear operators can be defined as

$$N_1\Phi(\xi,\eta)=\Phi^{'''}(\xi,\eta)-\mathrm{Re}\Phi^{'2}(\xi,\eta)+\mathrm{Re}\Phi(\xi,\eta)\Phi^{''}(\xi,\eta)+W-M\ f^{'}(\eta,\xi) \tag{27}$$

$$N_2\Theta(\xi,\eta)=\Theta^{''}(\xi,\eta)-Br\left[\Phi^{''}(\xi,\eta)\right]^2+Pe\Phi(\xi,\eta)\Theta^{'}(\xi,\eta)-2Pe\Phi^{'}(\xi,\eta)\Theta(\xi,\eta) \tag{28}$$

where $\xi\in[0,1]$ is an embedding parameter, as ξ increases from 0 to 1, $\Phi(\xi,\eta)$ and $\Theta(\xi,\eta)$ vary from the initial guess $f_0(\eta)$ and $\theta_0(\eta)$ to the exact solution $f(\eta)$ and $\theta(\eta)$, respectively.

We develop the so called zeroth-order deformation equations and corresponding boundary conditions: $(1-\xi)L_1[\Phi(\xi;\eta)-f_0(\eta)]=ph\,N_1[\Phi(\xi,\eta)]$ (29)

$$(1-\xi)L_2[\Theta(\xi;\eta)-\theta_0(\eta)]=ph\,N_2[\Theta(\xi,\eta)] \tag{30}$$

$$f^{'}_k(0)-kn.\,f^{''}_k(0)=0,\quad f_k(0)=0,\quad f^{''}_k(1)=0 \tag{31}$$

$$\theta(0)-kn.\theta^{'}(0)=0,\quad \theta(0)=0 \tag{32}$$

Differentiating the zeroth-order deformation Eqs. (29) and (30) k-times with respect to ξ and then dividing them by k!, finally setting $\xi=0$, we obtain the following kth-order deformation equations as

$$L_1[f_k(\eta)-\chi_k f_{k-1}(\eta)=hR_k(f_{k-1}(\eta)) \tag{33}$$

$$L_2[\theta_k(\eta)-\chi_k\theta_{k-1}(\eta)]=hG_{k-1} \tag{34}$$

$$f^{'}_k(0)-kn.\,f^{''}_k(0)=0,\quad f_k(0)=0,\quad f^{''}_k(1)=0 \tag{35}$$

$$\theta(0)-kn.\theta^{'}(0)=0,\quad \theta^{'}(1)=0 \tag{36}$$

for both boundary conations.In which h is an auxiliary parameter.

$$R_k(\eta)=f^{'''}_{k-1}(\eta)+\mathrm{Re}\sum_{i=0}^{k-1}f^{''}_i(\eta)\,f_{k-1-i}(\eta)-\mathrm{Re}\sum_{i=0}^{k-1}f^{'}_i(\eta)\,f^{'}_{k-1-i}(\eta)+W(1-\chi_k)-M\ f^{'}(\eta) \tag{37}$$

$$G_{k-1}(\eta)=\theta^{''}_{k-1}(\eta)+Br\sum_{i=0}^{k-1}f^{''}(\eta)\,f^{''}_{k-1-i}(\eta)+Pe\sum_{i=0}^{k-1}f_k(\eta)\,\theta^{'}_{k-1-i}(\eta)-2Pe\sum_{i=0}^{k-1}\theta_k(\eta)\,f^{'}_{k-1-i} \tag{38}$$

and

$$\chi_k = \begin{cases} 0 & when\ \ k \le 1 \\ 1 & when\ k > 1 \end{cases}$$

We use the symbolic calculation software MATHEMATICA and solve the set of linear differential Eqs.(33) and (34) with boundary conditions (35) and (36) up to first few order of approximation. It is found that $f(\eta)$ and $\theta(\eta)$ can be written as

$$f(\eta) = \sum_{i=0}^{k} f_i(\eta) \quad , \quad \theta(\eta) = \sum_{i=0}^{k} \theta_i(\eta)$$

4.2. Convergence of the solutions

It is noticed that the explicit, analytical expression, figure-2, contain auxiliary parameter h. As pointed out by Liao [23],the convergence region and rate of approximations given by the HAM are strongly dependent on h, figure-2, portray the h-curve of the velocity profile. For the velocity distribution, tables (1) illustrate the values of the second derivatives for different order of the approximations and for different values of the parameter h where $-0.8 \le h \le 0.8$ .It is noted that the best value for h is -0.2, since the less difference between the second order derivatives, for different order of the approximations, occurs at that value.

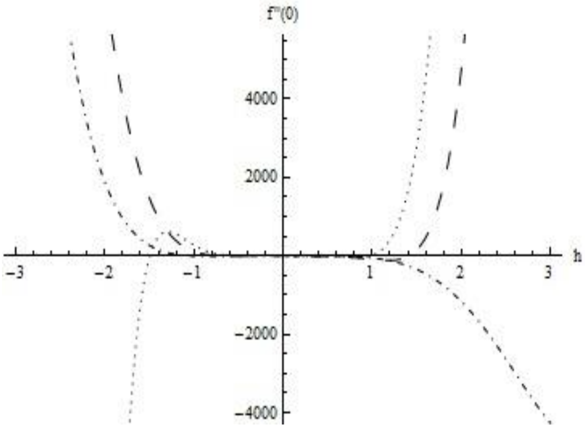

Figure 2- 9th-order of approximation h curve for $f''(0)$

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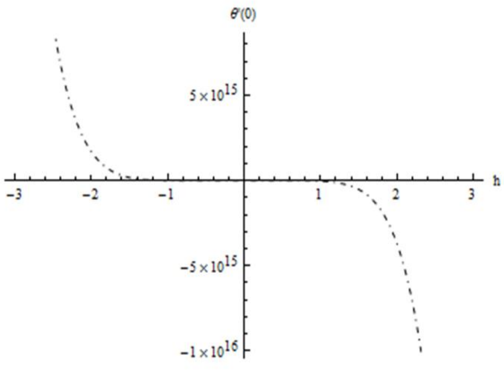

Figure 3- 3th-order of approximation h curve for $\theta'(\eta)$

Table 1- the values of the convergence parameter h using the first derivative. At $\eta = 0$.

$value of\ h$	$f''(\eta) = f_0 + \ldots + f_8$	$f''(\eta) = f_0 + \ldots + f_6$	$f''(\eta) = f_0 + \ldots + f_4$
-0.2	0.426903	0.443616	0.23225
-0.4	-0.717462	-0.051250	0.708562
-0.6	4.35533	-2.26583	0.437963
-0.8	30.8713	-2.91213	-1.0726
0.4	4.2385	1.35421	0.437963
0.6	7.4449	6.45256	-0.131426
0.8	3.67869	11.4263	-1.0726

Table 2- the values of the convergence parameter h using the second derivative. At $\eta = 0$.

$value of\ h$	$\theta'(\eta) = \theta_0 + \theta_1$	$\theta'(\eta) = \theta_0 + \theta_1 + \theta_2$
-1	-4.93584	-20.7829
-0.8	-4.00394	-9.68941
-0.6	-3.1444	-4.79118
-0.4	-2.35723	-2.897
-0.3	-1.99079	-2.40765
-0.2	-1.64243	-1.99291
0.2	-0.429934	0.471792
0.4	0.0677656	2.36515
0.6	0.493098	4.40297
0.8	0.846062	6.29314
1	1.12666	7.9017

5. Results and discussion

Utilizing the analytical solutions, calculations are performed to investigate the effect of MHD parameter "M", Reynolds number "Re", Knudsen number "kn", Prinkman number "Br", Peclet number "Pe", Reynolds number "Re" and physical quantity at wall "W". The following results are made of

5.1.Velocity distribution:

1- As MHD parameter "M" increases, there is small decreasing in the velocity rang. See Figure- 4)
2- As Reynolds number "Re" increases, there is small decreasing in the velocity rang. See Figure- 5)
3- As Knudsen number "kn" increases, there is small decreasing in the velocity rang. See Figure- 6)
4-As physical quantity at wall "W" increases, there is small decreasing in the velocity rang. See Fig(7)

5.2. Heat distribution:

1- As MHD parameter "M" increases, there is small decreasing in the velocity rang. See Figure- 8)
2- As Reynolds number "Re" increases, there is small decreasing in the velocity rang. See Figure- 9)
3-As Knudsen number "Kn" increases, there is small decreasing in the velocity rang. See Figure- 10)
4-As physical quantity at wall "W" increases, there is small decreasing in the velocity rang. See Figure- 11)
5-As Prinkman number "Br" increases, there is small decreasing in the velocity rang. See Figure- 12)
6- As Peclet number "Pe" increases, there is small decreasing in the velocity rang. See Figure- 13)

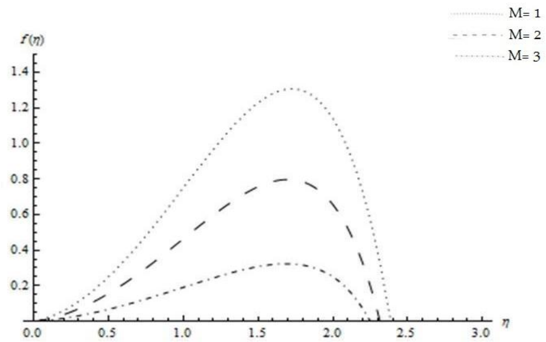

Figure 4- Velocity distribution with , $\mathrm{Re}=7, h=-0.2, W=1$, $kn=0.1$. $M=1,2,3$.

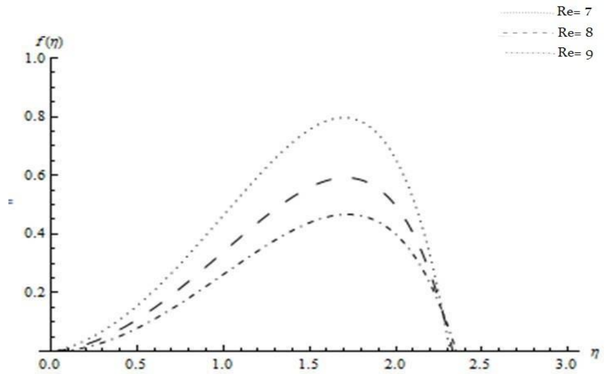

Figure 5- Velocity distribution with , $M=2, h=-0.2, W=1$, $kn=0.1$. $\mathrm{Re}=7,8,9$.

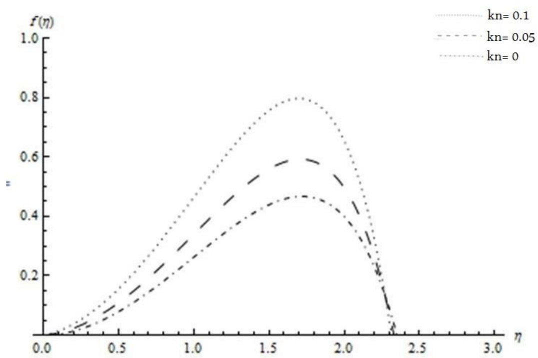

Figure 6- Velocity distribution with , $M=2, h=-0.2, W=1$, $\mathrm{Re}=7$. $kn=0.1,0.05,0$.

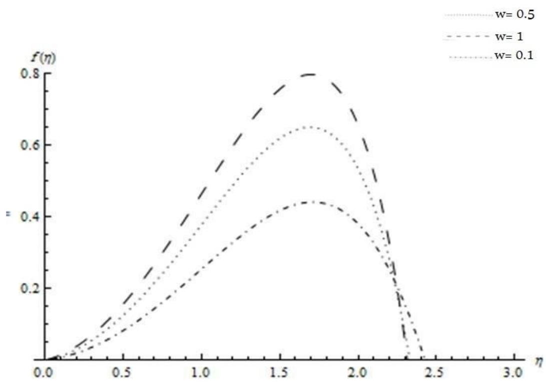

Figure 7- Velocity distribution with , $M=2, h=-0.2, kn=0.1$, $\mathrm{Re}=7$. $W=0.5,1,0.2$

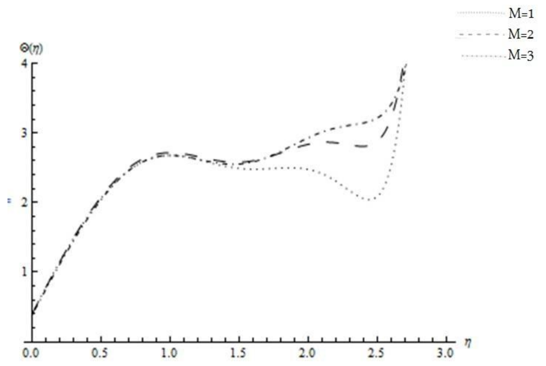

Figure 8- Heat distribution with , $\mathrm{Re}=7, h=-0.2, Kn=0.1, W=1, Pe=9$, $Br=6$. $M=1,2,3$.

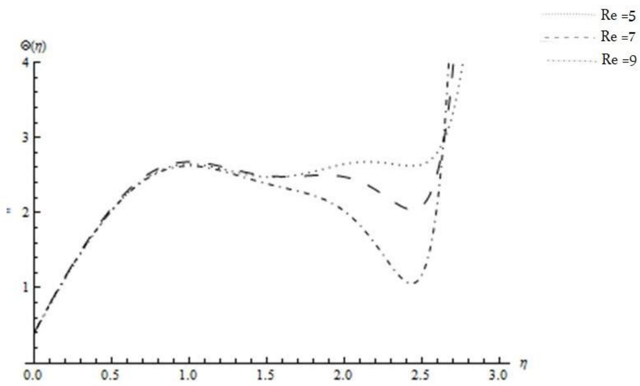

Figure 9- Heat distribution with , , $M=1, h=-0.2, Kn=0.1, W=1, Pe=9$, $Br=6$. $\mathrm{Re}=5,7,9$.

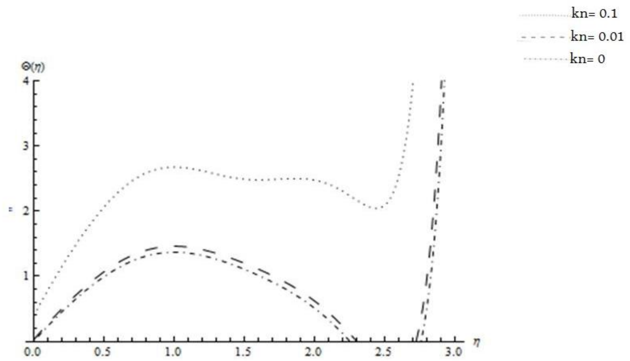

Figure 10- Heat distribution with , , $M=1, h=-0.2, \mathrm{Re}=7.1, W=1, Pe=9$, $Br=6$. $kn=0.1,0.01,0$.

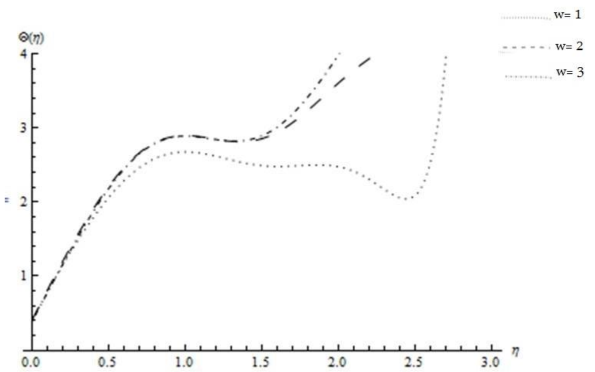

Figure 11- Heat distribution with $M = 1, h = -0.2, \mathrm{Re} = 7.1, Kn = o.1, Pe = 9$, $Br = 6$. $W = 1,2,3$.

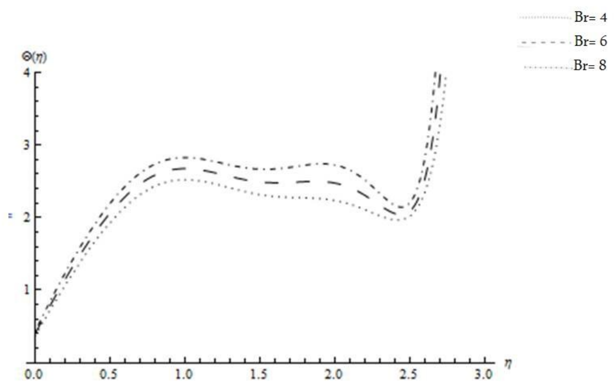

Figure 12- Heat distribution with $M = 1, h = -0.2, \mathrm{Re} = 7.1, Kn = 0.1, Pe = 9$, $W = 1$. $Br = 4,6,8$.

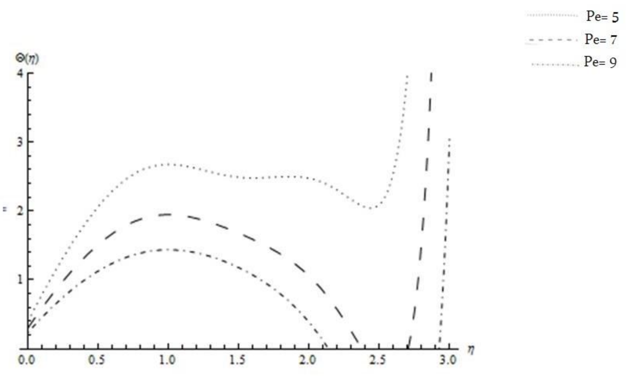

Figure 13- Heat distribution with , , $M = 1, h = -0.2, \mathrm{Re} = 7.1, W = 1, kn = 0.1$, $Br = 6$. $pe = 5,7,9$.

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