The Dynamics of Four-Species Ecological Model

Raid Kamel Naji, Rami Raad Saady
Department of Mathematic, College of Science, University of Baghdad, Baghdad, Iraq.

Abstract
In this paper, a four species mathematical models involving different types of ecological interactions is proposed and analyzed. Holling type – II functional response is a doubted to describes the behavior of predation. The existence, uniqueness and boundedness of the solution are discussed. The existences and the stability analysis of all possible equilibrium points are studied. suitable Lyapunov functions are used to study the global dynamics of the system. Numerical simulations are also carried out to investigate the influence of certain parameters on the dynamical behavior of the model, to support the analytical results of the model.

Keywords: equilibrium point, stability, Competition, Commensalisms..

ديناميكية النموذج الايكولوجي لاربعة انواع

رائد كامل ناجي و رامي رعد سعدي
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصة:
هذا البحث ، تم اقتراح ودراسة نموذج رياضي يتضمن الانواع الاربعة من التفاعلات البيئية المختلفة . الدالة المستخدمة لوصف سلوك الافتراس هي من النوع هولنك من النوع الثاني . ناقشنا وجود، وحدانية وقيد الحل. قمنا بدراسة وجود و تحليل الاستقرارية لجميع نقاط التوازن الممكنة. كما أستخدمة دوال ليابونوف المناسبة لدراسة الديناميكية الشاملة للنماذج المقترحة كذالك تم استخـدام المحاكاة العددية لبحث السلوك الديناميكي الشامل للنظام.

1. Introduction

Mathematical modeling is an important interdisciplinary activity which involves the study of some aspects of diverse disciplines. Biology, Epidemiodology, Physiology, Ecology, Immunology, Bio-economics, Genetics, Pharmacokinetics are some of those disciplines. This mathematical modeling has taken a lot of attentions in recent years and spread to all branches of life and drew the attention of every one [9]. Ecology relates to study of living beings in relation with their living styles. Research in the branch of theoretical ecology was initiated by Lotka [1] and by Volterra [2]. Since then many scientists and researchers gave a lot of time and interest to this branch of study, see for example Meyer [3], Cushing [4], Paul Colinvaux[5], Freedman [6], Kapur [7, 8]. The ecological interactions can be broadly classified as prey-predator, competition, mutualism, commensalism and so on. Srinivas [9] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Later, Narayan [10] has investigated the two species prey-predator models with a partial covers for the prey and alternative food for the predator. Recently stability analysis of competitive species was investigated by Reddy [11]. Local stability analysis for a two species ecological mutualism model has been investigated by Reddy et al., [12, 13].

*Email: rknaji@gmail.com, r_my80@yahoo.com

In this paper however , investigation is devoted to an analytical study of a four species Syn-ecological system , with Holling type-II functional response involving a predator (say N_2) preys upon the prey (say N_1) : the prey is a commensal to the host N_3 which itself is in competition with the fourth species N_4 ; N_2 and N_4 are natural. Figure-1, shows the schematic sketch of the system under investigation. The model equations of the system constitute a set of four first order non-linear ordinary differential equations.

2. The mathematical model:-

Consider the four species Syn-Ecosymbiosis, comprising of prey-predator, commensalisms and competition, model that consists of a prey (for example, Anemone) whose population density at time T denoted by N_1, the predator (for example, Butterfly fish) whose population density at time T denoted by N_2, the host (for example, Hermit crabs) whose population density at time T denoted by N_3, and the host's competitor species (for example, other type of Hermit crabs) whose population density at time T denoted by N_4. Now in order to formulates the mathematical model of the above Syn-Ecosymbiosis system, the following assumptions are adopted:

1. The predator species preys upon the prey species according to Holling type-II functional response with maximum attack rate $a_1 > 0$ and half saturation constant $b > 0$. While, in the absence of the predator the prey species grows logistically with carrying capacity $k_1 > 0$ and intrinsic growth rate $r_1 > 0$. Moreover in the absence of the prey the predator decay exponential with natural death rate $d_1 > 0$, however in the existence of prey the predator individuals competes each other with intraspecific competition constant rate $d_2 > 0$
2. The existence of the host N_3 enhance the existence of the prey species N_1 with the commensal constant rate $c > 0$, while the existence of N_1 do not affect (positively or negatively) the existence of N_3.
3. Both the species N_3 and N_4 growth logistically with intrinsic growth rates $r_i > 0$ for $i = 2,3$ and carrying capacities $k_i > 0$ for $i = 2,3$ respectively.
4. Finally there is an interspecific competition interaction between the species N_3 and N_4 with competition intensity rates $\alpha_1 > 0$ and $\alpha_2 > 0$ respectively.

Therefore the dynamics of the above proposed model can be represented by the following set of the first order nonlinear differential equations while the block diagram of this model system can be illustrated in Figure-1.

$$\begin{aligned}
\frac{dN_1}{dT} &= r_1 N_1\left(1-\frac{N_1}{k_1}\right) - \frac{a_1 N_1}{b+N_1} N_2 + cN_1 N_3 \\
\frac{dN_2}{dT} &= e\frac{a_1 N_1}{b+N_1} N_2 - d_1 N_2 - d_2 N_2^2 \\
\frac{dN_3}{dT} &= r_2 N_3\left(1-\frac{N_3}{k_2}\right) - \alpha_1 N_3 N_4 \\
\frac{dN_4}{dT} &= r_3 N_4\left(1-\frac{N_4}{k_3}\right) - \alpha_2 N_3 N_4
\end{aligned} \quad (1)$$

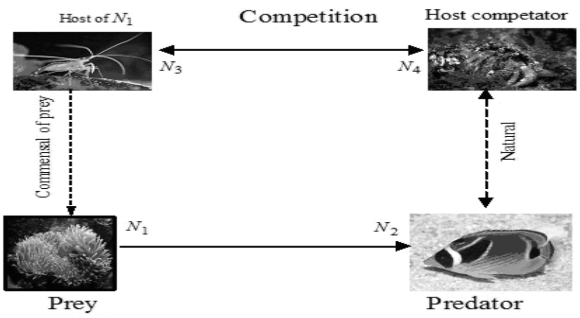

Figure 1- The block diagram of system (1).

Note that the above proposed model has fourteen parameters in all, which make the analysis difficult. So, in order to simplify the system, the number of parameters is reduced by using the following dimensionless variables and parameters:

$t = r_1 T$, $x = \frac{N_1}{k_1}$, $y = \frac{N_2}{k_1}$, $z = \frac{cN_3}{r_1}$, $w = \frac{\alpha_1 N_4}{r_1}$,

$u_1 = \frac{a_1}{r_1}$, $u_2 = \frac{b}{k_1}$, $u_3 = \frac{d_1}{r_1}$, $u_4 = \frac{d_2 k_1}{r_1}$, $u_5 = \frac{r_2}{r_1}$,

$u_6 = \frac{r_1}{ck_2}$, $u_7 = \frac{r_3}{r_1}$, $u_8 = \frac{r_1}{\alpha_1 k_3}$, $u_9 = \frac{\alpha_2}{c}$

Then the non-dimensional form of system (1) can be written as:

$$\begin{aligned}
\frac{dx}{dt} &= x\left[(1-x) - \frac{u_1 y}{u_2 + x} + z\right] = xf_1(x,y,z,w) \\
\frac{dy}{dt} &= y\left[\frac{eu_1 x}{u_2 + x} - u_3 - u_4 y\right] = yf_2(x,y,z,w) \\
\frac{dz}{dt} &= z[u_5(1-u_6 z) - w] = zf_3(x,y,z,w) \\
\frac{dw}{dt} &= w[u_7(1-u_8 w) - u_9 z] = wf_4(x,y,z,w)
\end{aligned} \qquad (2)$$

with $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$ and $w(0) \geq 0$. It is observed that the number of parameters have been reduced from fourteen in the system (1) to ten in the system (2). Obviously the interaction functions of the system (2) are continuous and have continuous partial derivatives on the following positive four dimensional space:

$R_+^4 = \{(x,y,z,w) \in R^4 : x(0) \geq 0, y(0) \geq 0, z(0) \geq 0, w(0) \geq 0\}$.

Therefore these functions are Lipschitzian on R_+^4 , and hence the solution of the system (2) exists and is unique. Further, in the following theorem, the boundedness of the solution of the system (2) in R_+^4 is established.

Theorem (1): All the solutions of system (2) which initiate in R_+^4 are uniformly bounded.

Proof:

Let $(x(t), y(t), z(t), w(t))$ be any solution of the system (2) with non-negative initial condition $(x_0, y_0, z_0, w_0) \in R_+^4$. Now according to the third equation of system (2) we have

$$\frac{dz}{dt} \leq u_5 z(1 - u_6 z)$$

So, by using the comparison theorem on the above differential inequality with the initial point $z(0) = z_0$ we get:

$$z(t) \leq \frac{z_0}{z_0 u_6 + (1 - z_0 u_6)e^{-u_5 t}}$$

Thus, $\lim_{t\to\infty} z(t) \leq \frac{1}{u_6}$ and hence, $Sup.z(t) \leq \frac{1}{u_6}$, $\forall t > 0$.

Similarly, from the forth equation of system (2) we obtain that $\lim_{t\to\infty} w(t) \leq \frac{1}{u_8}$, and hence $Sup.w(t) \leq \frac{1}{u_8}$, $\forall t > 0$.

Finally, according to the first equation of system (2) we have

$$\frac{dx}{dt} \leq x(1-x) + xz$$

So, again by using the comparison theorem on the above differential inequality with the initial point $x(0) = x_0$ and the upper bound of $z(t)$ we get:

$\lim_{t\to\infty} x(t) \le L$, where $L = 1 + \frac{1}{u_6}$

Therefore, $Sup.x(t) \le L$, $\forall t > 0$.

Now define the function: $M(t) = x(t) + \frac{1}{e} y(t) + z(t) + w(t)$, and then take the time derivative of $M(t)$ along the solution of the system (2) we get:

$$\frac{dM}{dt} \le 2L + \frac{L}{u_6} + 2\frac{u_5}{u_6} + 2\frac{u_7}{u_8} - sM$$

where $s = \min\{1, u_3, u_5, u_7\}$. Then

$$\frac{dM}{dt} + sM \le H \text{ , where } H = 2\left(L + \frac{L}{2u_6} + \frac{u_5}{u_6} + \frac{u_7}{u_8} \right)$$

Again by solving this differential inequality for the initial value $M(0) = M_0$, we get:

$$M(t) \le \frac{H}{s} + \left(M_0 - \frac{H}{s} \right) e^{-st}$$

Then,

$$\lim_{t\to\infty} M(t) \le \frac{H}{s}$$

So, $0 \le M(t) \le \frac{H}{s}$, $\forall t > 0$. Hence all the solutions of system (2) are uniformly bounded and the proof is complete. ■

3. The existence of equilibrium points:-

In this section, the existence of all possible equilibrium points of system (2) is discussed. It is observed that, system (2) has at most twelve equilibrium points, which are mentioned in the following: The equilibrium points $E_0 = (0,0,0,0)$, which known as the washout point, and the single species points $E_1 = (1,0,0,0)$, $E_2 = \left(0,0,\frac{1}{u_6},0\right)$, $E_3 = \left(0,0,0,\frac{1}{u_8}\right)$ are always exists. The first planar equilibrium point $E_4 = (\hat{x}, \hat{y}, 0, 0)$ exists uniquely in $Int.R_+^2$ of $xy-$plane if there is a positive solution to the following set of equations:

$$(1-x) - \frac{u_1 y}{u_2 + x} = 0 \tag{3a}$$

$$\frac{e u_1 x}{u_2 + x} - u_3 - u_4 y = 0 \tag{3b}$$

From equation (3a) we have,

$$y = \frac{(1-x)(u_2 + x)}{u_1} \tag{4}$$

Clearly, $y > 0$ when $x < 1$. Now by substituting (4) in (3b) and then simplifying the resulting term we obtain that

$$f(x) = \gamma_1 x^3 + \gamma_2 x^2 + \gamma_3 x + \gamma_4 = 0 \tag{5}$$

where

$$\gamma_1 = u_2 > 0$$
$$\gamma_2 = 2u_2 u_4 - u_4$$
$$\gamma_3 = e u_1^2 - u_1 u_3 - 2u_2 u_4 + u_2^2 u_4$$
$$\gamma_4 = -\left(u_1 u_2 u_3 + u_2^2 u_4\right) < 0$$

Therefore the first planar equilibrium point $E_4 = (\hat{x}, \hat{y}, 0, 0)$, where $\hat{x}$ is a positive root of equation (5) and $\hat{y} = y(\hat{x})$ that results from (4), exists uniquely in the $Int.R_+^2$ of $xy-$plane if in addition to the condition $\hat{x} < 1$ at least one of the following conditions are satisfied:

$$u_2 > \frac{1}{2} \tag{6a}$$

$eu_1^2 + u_2^2 u_4 < u_1 u_3 + 2u_2 u_4$ (6b)

The second planar equilibrium point $E_5 = (0,0,\tilde{z},\tilde{w})$ exists uniquely in the $Int.R_+^2$ of $zw-$plane if there is a positive solution to the following set of equations:

$$u_5(1-u_6 z) - w = 0 \quad (7a)$$

$$u_7(1-u_8 w) - u_9 z = 0 \quad (7b)$$

Straightforward computation gives that

$$\tilde{z} = \frac{u_7(u_5 u_8 - 1)}{u_5 u_6 u_7 u_8 - u_9} \text{ and } \tilde{w} = \frac{u_5(u_6 u_7 - u_9)}{u_5 u_6 u_7 u_8 - u_9} \quad (7c)$$

Clearly $\tilde{z}$ and $\tilde{w}$ are positive and hence E_5 exists uniquely in $Int.R_+^2$ of $zw-$plane provided that one set of the following sets of conditions is satisfied:

$u_5 u_8 > 1$ and $u_6 u_7 > u_9$ (8a)

$u_5 u_8 < 1$ and $u_6 u_7 < u_9$ (8b)

The third planar equilibrium point $E_6 = (\bar{x},0,\bar{z},0) = \left(\frac{u_6+1}{u_6},0,\frac{1}{u_6},0\right)$ always exists in $Int.R_+^2$ of $xz-$plane where $\bar{x}$ and $\bar{z}$ represent the positive solution of the following system:

$$1 - x + z = 0 \quad (9a)$$

$$u_5(1-u_6 z) = 0 \quad (9b)$$

The fourth planar equilibrium point $E_7 = (\bar{\bar{x}},0,0,\bar{\bar{w}}) = \left(1,0,0,\frac{1}{u_8}\right)$ always exists in $Int.R_+^2$ of $xw-$plane where $\bar{\bar{x}}$ and $\bar{\bar{w}}$ represent the positive solution of the following system:

$$1 - x = 0 \quad (10a)$$

$$u_7(1-u_8 w) = 0 \quad (10b)$$

Now, the first three species equilibrium point $E_8 = (\breve{x},\breve{y},\breve{z},0)$ exists uniquely in $Int.R_+^3$ of $xyz-$space if there is a positive solution to the following set of equations:

$$(1-x) - \frac{u_1 y}{u_2 + x} + z = 0 \quad (11a)$$

$$\frac{e u_1 x}{u_2 + x} - u_3 - u_4 y = 0 \quad (11b)$$

$$u_5(1-u_6 z) = 0 \quad (11c)$$

From equation (11c) we have,

$$\breve{z} = \frac{1}{u_6} \quad (11d)$$

Substituting (11d) in (11a) and then simplifying the resulting term we get:

$$y = \frac{u_6[(u_2 + x)(1-x)] + (u_2 + x)}{u_1 u_6} \quad (11e)$$

Now, by substituting (11e) in (11b) and then simplifying the resulting term we obtain that

$$f(x) = \beta_1 x^3 + \beta_2 x^2 + \beta_3 x + \beta_4 = 0 \quad (12)$$

where

$$\beta_1 = u_4 u_6 > 0$$

$$\beta_2 = 2u_2 u_4 u_6 - u_4 u_6 - u_4$$

$$\beta_3 = e u_1^2 u_6 - u_1 u_3 u_6 - 2u_2 u_4 u_6 + u_2^2 u_4 u_6 - 2u_2 u_4$$

$$\beta_4 = -\left(u_1 u_2 u_3 u_6 + u_2^2 u_4 u_6 + u_2^2 u_4\right) < 0$$

Note that Eq.(12) has a unique positive root, namely $\breve{x}$, provided that at least one of the following conditions are satisfied:

$$2u_2 u_6 > u_6 + 1 \quad (13a)$$

$$u_6\left(e u_1^2 + u_2^2 u_4\right) < u_6(u_1 u_3 + 2u_2 u_4) + 2u_2 u_4 \quad (13b)$$

Consequently, the first three species equilibrium point $E_8 = (\breve{x}, \breve{y}, \breve{z}, 0)$ where $\breve{y} = y(\breve{x})$ given by Eq.(11e), exists uniquely in the $Int.R_+^3$ of xyz – space if in addition to conditions (13a) – (13b) the following condition holds

$$u_6 + 1 > u_6 \breve{x} \quad (14)$$

The second three species equilibrium point $E_9 = (\hat{x}, \hat{y}, 0, \hat{w})$ exists uniquely in $Int.R_+^3$ of xyw – space if there is a positive solution to the following set of equations:

$$(1 - x) - \frac{u_1 y}{u_2 + x} = 0 \quad (15a)$$

$$\frac{e u_1 x}{u_2 + x} - u_3 - u_4 y = 0 \quad (15b)$$

$$u_7 (1 - u_8 w) = 0 \quad (15c)$$

From equation (15c) we have,

$$\hat{w} = \frac{1}{u_8} \quad (15d)$$

Also, from equation (15a) we have,

$$y = \frac{(1 - x)(u_2 + x)}{u_1} \quad (15e)$$

By substituting (15e) in (15b) and then simplifying the resulting term we obtain that

$$f(x) = \sigma_1 x^3 + \sigma_2 x^2 + \sigma_3 x + \sigma_4 = 0 \quad (16)$$

where

$$\sigma_1 = u_4 > 0$$

$$\sigma_2 = 2u_2 u_4 - u_4$$

$$\sigma_3 = e u_1^2 - u_1 u_3 - 2u_2 u_4 + u_2^2 u_4$$

$$\sigma_4 = -(u_1 u_2 u_3 + u_2^2 u_4) < 0$$

Not that Eq.(16) has a unique positive root, namely $\hat{x}$, provided that at least one of the following conditions are satisfied:

$$u_2 > \frac{1}{2} \quad (17a)$$

$$e u_1^2 + u_2^2 u_4 < u_1 u_3 + 2u_2 u_4 \quad (17b)$$

Consequently, the second three species equilibrium point $E_9 = (\hat{x}, \hat{y}, 0, \hat{w})$ where $\hat{y} = y(\hat{x})$ is given by Eq.(15e), exists uniquely in the $Int.R_+^3$ of xyw – space if in addition to conditions (17a) – (17b) the following condition holds

$$\hat{x} < 1 \quad (18)$$

The third three species equilibrium point $E_{10} = (x^\bullet, 0, z^\bullet, w^\bullet)$ exists uniquely in the $Int.R_+^3$ of xzw – space if there is a positive solution to the following set of equations:

$$1 - x + z = 0 \quad (19a)$$

$$u_5 (1 - u_6 z) - w = 0 \quad (19b)$$

$$u_7 (1 - u_8 w) - u_9 w = 0 \quad (19c)$$

Straightforward computation shows that these three equations give that

$$x^\bullet = \frac{u_5 u_6 u_7 u_8 - u_9 + u_7 (u_5 u_8 - 1)}{u_5 u_6 u_7 u_8 - u_9}, z^\bullet \equiv \tilde{z}, w^\bullet \equiv \tilde{w} \quad (20)$$

here $\tilde{z}$ and $\tilde{w}$ are given in Eq. (7c). Clearly $x^\bullet$, $z^\bullet$ and $w^\bullet$ are positive and hence E_{10} exists uniquely in the $Int.R_+^3$ of xzw – space provided that condition (8a) or (8b) is satisfied:

Finally the positive (coexistence) equilibrium point $E_{11} = (x^*, y^*, z^*, w^*)$ exists if there is a positive solution to the following set of equations:

$$(1 - x) - \frac{u_1 y}{u_2 + x} + z = 0 \quad (21a)$$

$$\frac{eu_1 x}{u_2 + x} - u_3 - u_4 y = 0 \tag{21b}$$
$$u_5(1 - u_6 z) - w = 0 \tag{21c}$$
$$u_7(1 - u_8 w) - u_9 z = 0 \tag{21d}$$

From equation (21c) and equation (21d) we get

$$z^* \equiv \tilde{z} \text{ and } w^* \equiv \tilde{w} \tag{21e}$$

Clearly $z^* > 0$ and $w^* > 0$ provided that condition (8a) or (8b) holds.
Substituting (21e) in (21a) and then simplifying the resulting term we obtain that

$$y = \frac{(u_2 + x)[s_2(1 - x) + u_7 s_1]}{u_1 s_2} \tag{21f}$$

where $s_1 = u_5 u_6 - 1$ and $s_2 = u_5 u_6 u_7 u_8 - u_9$.
Now by substituting (21f) in (21b) and then simplifying the resulting term we obtain that

$$f(x) = \delta_1 x^3 + \delta_2 x^2 + \delta_3 x + \delta_4 = 0 \tag{22}$$

where

$\delta_1 = u_4 s_2$

$\delta_2 = 2u_2 u_4 s_2 - u_4 u_7 s_1 - u_4 s_2$

$\delta_3 = eu_1^2 s_2 - u_1 u_3 s_2 + u_4 u_2^2 s_2 - 2u_2 u_4 s_2 - 2u_2 u_4 u_7 s_1$

$\delta_4 = -\left(u_1 u_2 u_3 s_2 + u_2^2 u_4 s_2 + u_2^2 u_4 u_7 s_1\right)$

Clearly, by using discard rule of sign, Eq.(22) has a unique positive root, denoted by x^*, provided that in addition to condition (8a) at least one of the following conditions hold

$$2u_2 s_2 > u_7 s_1 + s_2 \tag{23a}$$
$$\left(eu_1^2 + u_2^2 u_4\right)s_2 < u_1 u_3 s_2 + 2u_2 u_4 (s_2 + u_7 s_1) \tag{23b}$$

or else in addition to condition (8b) at least one of the following conditions hold

$$2u_2 s_2 < u_7 s_1 + s_2 \tag{23c}$$
$$\left(eu_1^2 + u_2^2 u_4\right)s_2 > u_1 u_3 s_2 + 2u_2 u_4 (s_2 + u_7 s_1) \tag{23d}$$

Consequently, the positive equilibrium point $E_{11} = (x^*, y^*, z^*, w^*)$, where $y^* = y(x^*)$ as given in Eq.(21f), exists uniquely in $Int.R_+^4$ if and only if in addition to the above conditions the following condition is satisfied.

$$x^* < \frac{s_2 + u_7 s_1}{s_2} \tag{24}$$

4. The stability analysis of system (2):-

In this section the stability analysis of all feasible equilibrium points of system (2) is studied analytically with the help of linearization method as bellow.
Note that, from now onward the symbols $\lambda_{ix}, \lambda_{iy}, \lambda_{iz}$ and λ_{iw} represent the eigenvalues of the Jacobian matrix $J(E_i); i = 0,1,2,\ldots,11$ that describe the dynamics in the $x-direction$, $y-direction$, $z-direction$ and $w-direction$ respectively,
It is easy to verify that, the Jacobian matrix of system (2) at the trivial equilibrium point $E_0 = (0,0,0,0)$ can be written in the form:

$$J(E_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -u_3 & 0 & 0 \\ 0 & 0 & u_5 & 0 \\ 0 & 0 & 0 & u_7 \end{bmatrix}$$

Thus the eigenvalues of $J(E_0)$ are $\lambda_{0x} = 1 > 0$, $\lambda_{0y} = -u_3 < 0$, $\lambda_{0z} = u_5 > 0$ and $\lambda_{0w} = u_7 > 0$, then E_0 is a saddle point.
The Jacobian matrix of system (2) at the first single species equilibrium point $E_1 = (1,0,0,0)$ can be written as:

$$J(E_1)=\begin{bmatrix} -1 & \frac{-u_1}{u_2+1} & 1 & 0 \\ 0 & \frac{eu_1}{u_2+1}-u_3 & 0 & 0 \\ 0 & 0 & u_5 & 0 \\ 0 & 0 & 0 & u_7 \end{bmatrix}$$

Hence the eigenvalues of $J(E_1)$ are $\lambda_{1x}=1>0$, $\lambda_{1y}=\frac{eu_1}{u_2+1}-u_3$, $\lambda_{1z}=u_5>0$ and $\lambda_{1w}=u_7>0$, then E_1 is a saddle point.

The Jacobian matrix of system (2) at the second single species equilibrium point $E_2=\left(0,0,\frac{1}{u_6},0\right)$ can be written as:

$$J(E_2)=\begin{bmatrix} 1+\frac{1}{u_6} & 0 & 0 & 0 \\ 0 & -u_3 & 0 & 0 \\ 0 & 0 & -u_5 & \frac{-1}{u_6} \\ 0 & 0 & 0 & u_7-\frac{u_9}{u_6} \end{bmatrix}$$

Thus the eigenvalues of $J(E_2)$ are $\lambda_{2x}=1+\frac{1}{u_6}>0$, $\lambda_{2y}=-u_3<0$ $\lambda_{2z}=-u_5<0$ and $\lambda_{2w}=u_7-\frac{u_9}{u_6}$, then E_2 is a saddle point.

The Jacobian matrix of system (2) at the third single species equilibrium point $E_3=\left(0,0,0,\frac{1}{u_8}\right)$ can be written as:

$$J(E_3)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -u_3 & 0 & 0 \\ 0 & 0 & u_5-\frac{1}{u_8} & 0 \\ 0 & 0 & \frac{-u_9}{u_8} & -u_7 \end{bmatrix}$$

Thus the eigenvalues of $J(E_3)$ are $\lambda_{3x}=1>0$, $\lambda_{3y}=-u_3<0$, $\lambda_{3z}=u_5-\frac{1}{u_8}$ and $\lambda_{3w}=-u_7<0$, then E_3 is a saddle point.

The Jacobian matrix of system (2) at the first two species equilibrium point $E_4=(\hat{x},\hat{y},0,0)$ can be written as:

$$J(E_4)=\begin{bmatrix} \hat{x}\left[-1+\frac{u_1\hat{y}}{(u_2+\hat{x})^2}\right] & \frac{-u_1\hat{x}}{u_2+\hat{x}} & \hat{x} & 0 \\ \frac{eu_1u_2\hat{y}}{(u_2+\hat{x})^2} & -u_4\hat{y} & 0 & 0 \\ 0 & 0 & u_5 & 0 \\ 0 & 0 & 0 & u_7 \end{bmatrix}$$

Hence the characteristic equation of $J(E_4)$ is given by:

$$\left[\lambda^2+A_1\lambda+A_2\right](u_5-\lambda)(u_7-\lambda)=0$$

where $A_1=\hat{x}-\frac{u_1\hat{x}\hat{y}}{(u_2+\hat{x})^2}+u_4\hat{y}$

$$A_2=u_4\hat{x}\hat{y}\left(1-\frac{u_1\hat{y}}{(u_2+\hat{x})^2}\right)+\frac{eu_1^2u_2\hat{x}\hat{y}}{(u_2+\hat{x})^3}$$

So, either

$$(u_5-\lambda)(u_7-\lambda)=0 \qquad (25a)$$

Or

$$\lambda^2 + A_1\lambda + A_2 = 0 \quad (25b)$$

Hence from equation (25a) we obtain that:

$$\lambda_{4z} = u_5 > 0 \ , \ \lambda_{4w} = u_7 > 0$$

Thus E_4 is unstable.

The Jacobian matrix of system (2) at the second two species equilibrium point $E_5 = (0,0,\tilde{z},\tilde{w}) = \left(0,0,\frac{u_7(u_5u_8-1)}{u_5u_6u_7u_8-u_9},\frac{u_5(u_6u_7-u_9)}{u_5u_6u_7u_8-u_9}\right)$ can be written as:

$$J(E_5) = \begin{bmatrix} 1+\tilde{z} & 0 & 0 & 0 \\ 0 & -u_3 & 0 & 0 \\ 0 & 0 & -u_5u_6\tilde{z} & -\tilde{z} \\ 0 & 0 & -u_9\tilde{w} & -u_7u_8\tilde{w} \end{bmatrix}$$

Therefore the characteristic equation is:

$$(1+\tilde{z}-\lambda)(-u_3-\lambda)\left[\lambda^2 + (u_5u_6\tilde{z}+u_7u_8\tilde{w})\lambda + (u_5u_6u_7u_8-u_9)\tilde{z}\,\tilde{w}\right] = 0$$

So, either

$$(1+\tilde{z}-\lambda)$$

$$(-u_3-\lambda) = 0 \quad (26a)$$

Or

$$\lambda^2 + (u_5u_6\tilde{z}+u_7u_8\tilde{w})\lambda + (u_5u_6u_7u_8-u_9)\tilde{z}\,\tilde{w} = 0 \quad (26b)$$

Hence from equation (26a) we obtain that:

$$\lambda_{5x} = 1+\tilde{z} > 0 \ , \ \lambda_{5y} = -u_3 < 0 .$$

Thus E_5 is unstable.

The Jacobian matrix of system (2) at the third two species equilibrium point $E_6 = (\bar{x},0,\bar{z},0) = \left(\frac{u_6+1}{u_6},0,\frac{1}{u_6},0\right)$ can be written as:

$$J(E_6) = \begin{bmatrix} -\bar{x} & \frac{-u_1\bar{x}}{u_2+\bar{x}} & \bar{x} & 0 \\ 0 & \frac{eu_1\bar{x}-u_3(u_2+\bar{x})}{u_2+\bar{x}} & 0 & 0 \\ 0 & 0 & -u_5 & \frac{-1}{u_6} \\ 0 & 0 & 0 & \frac{u_6u_7-u_9}{u_6} \end{bmatrix}$$

Thus the eigenvalues of $J(E_6)$ are

$\lambda_{6x} = -\bar{x} < 0$, $\lambda_{6y} = \frac{eu_1+eu_1u_6-u_2u_3u_6-u_3-u_3u_6}{u_2u_6+u_6+1}$,

$\lambda_{6z} = -u_5 < 0$ and $\lambda_{6w} = \frac{u_6u_7-u_9}{u_6}$

Therefore, if the following conditions hold

$$eu_1(1+u_6) < u_3(u_2u_6+u_6+1) \quad (27a)$$

$$u_6u_7 < u_9 \quad (27b)$$

Then E_6 is locally asymptotically stable. However, it is a saddle point otherwise.

The Jacobian matrix of system (2) at the forth two species equilibrium point $E_7 = \left(1,0,0,\frac{1}{u_8}\right)$ can be written as:

$$J(E_7)=\begin{bmatrix} -1 & \frac{-u_1}{u_2+1} & 1 & 0 \\ 0 & \frac{eu_1-u_2u_3-u_3}{u_2+1} & 0 & 0 \\ 0 & 0 & \frac{u_5u_8-1}{u_8} & 0 \\ 0 & 0 & \frac{-u_9}{u_8} & -u_7 \end{bmatrix}$$

Thus the eigenvalues of $J(E_7)$ are given by;

$\lambda_{7x}=-1<0$, $\lambda_{7y}=\frac{eu_1-u_3(u_2+1)}{u_2+1}$ $\lambda_{7z}=\frac{u_5u_8-1}{u_8}$ and $\lambda_{7w}=-u_7<0$.

Therefore, if the following conditions hold

$$eu_1<u_3(u_2+1) \tag{28a}$$

$$u_5u_8<1 \tag{28b}$$

Then E_7 is locally asymptotically stable. However, it is a saddle point otherwise.

The Jacobian matrix of system (2) at the first three species equilibrium point $E_8=(\breve{x},\breve{y},\breve{z},0)=(\breve{x},\breve{y},\frac{1}{u_6},0)$ can be written as:

$$J(E_8)=\begin{bmatrix} -\breve{x}+\frac{u_1\breve{x}\breve{y}}{(u_2+\breve{x})^2} & \frac{-u_1\breve{x}}{u_2+\breve{x}} & \breve{x} & 0 \\ \frac{eu_1u_2\breve{y}}{(u_2+\breve{x})^2} & -u_4\breve{y} & 0 & 0 \\ 0 & 0 & -u_5 & \frac{-1}{u_6} \\ 0 & 0 & 0 & \frac{u_6u_7-u_9}{u_6} \end{bmatrix}$$

Hence the characteristic equation of $J(E_8)$ is given by:

$$\left[\lambda^2+\tilde{A}_1\lambda+\tilde{A}_2\right](-u_5-\lambda)\left(\frac{u_6u_7-u_9}{u_6}-\lambda\right)=0$$

where

$$\breve{A}_1=-\breve{x}\left(-1+\frac{u_1\breve{y}}{(u_2+\breve{x})^2}\right)+u_4\breve{y}$$

$$\breve{A}_2=u_4\breve{x}\breve{y}\left(1-\frac{u_1\breve{y}}{(u_2+\breve{x})^2}\right)+\frac{eu_1^2u_2\breve{x}\breve{y}}{(u_2+\breve{x})^3}$$

So, either

$$(-u_5-\lambda)\left(\frac{u_6u_7-u_9}{u_6}-\lambda\right)=0 \tag{29a}$$

which gives two of the eigenvalues of $J(E_8)$ by

$$\lambda_{8z}=-u_5<0 \text{ and } \lambda_{8w}=\frac{u_6u_7-u_9}{u_6} .$$

Or

$$\lambda^2+\breve{A}_1\lambda+\breve{A}_2=0 \tag{29b}$$

which gives the other two eigenvalues of $J(E_8)$ by

$$\lambda_{8x}=-\frac{\breve{A}_1}{2}+\frac{1}{2}\sqrt{\breve{A}_1^{\,2}-4\breve{A}_2}$$

$$\lambda_{8y}=-\frac{\breve{A}_1}{2}-\frac{1}{2}\sqrt{\breve{A}_1^{\,2}-4\breve{A}_2}$$

Straightforward computations show that all the above eigenvalues have negative real parts provided that the following conditions are satisfied

$$\frac{u_1 \breve{y}}{(u_2+\breve{x})^2}<1 \tag{30a}$$

$$u_6 u_7 < u_9 \tag{30b}$$

So, E_8 is locally asymptotically stable in the R_+^3. However, it is a saddle point otherwise.

The Jacobin matrix of system (2) at the second three species equilibrium point $E_9 = (\hat{x}, \hat{y}, 0, \hat{w}) = (\hat{x}, \hat{y}, 0, \frac{1}{u_9})$ can be written as:

$$J(E_9)=\begin{bmatrix} -\hat{x}+\frac{u_1\hat{x}\hat{y}}{(u_2+\hat{x})^2} & \frac{-u_1\hat{x}}{u_2+\hat{x}} & \hat{x} & 0 \\ \frac{eu_1u_2\hat{y}}{(u_2+\hat{x})^2} & -u_4\hat{y} & 0 & 0 \\ 0 & 0 & \frac{u_5u_8-1}{u_8} & 0 \\ 0 & 0 & \frac{-u_9}{u_8} & -u_7 \end{bmatrix}$$

Hence the characteristic equation of $J(E_9)$ is given by:

$$\left[\lambda^2 + B_1\lambda + B_2\right]\left(\frac{u_5u_8-1}{u_8}-\lambda\right)(-u_7-\lambda)=0$$

where

$$B_1 = -\hat{x}\left(-1+\frac{u_1\hat{y}}{(u_2+\hat{x})^2}\right)+u_4\hat{y}$$

$$B_2 = u_4\hat{x}\hat{y}\left(1-\frac{u_1\hat{y}}{(u_2+\hat{x})^2}\right)+\frac{eu_1^2u_2\hat{x}\hat{y}}{(u_2+\hat{x})^3}$$

So, either

$$\left(\frac{u_5u_8-1}{u_8}-\lambda\right)(-u_7-\lambda)=0 \tag{31a}$$

which gives two of the eigenvalues of $J(E_9)$ by:

$$\lambda_{9z}=\frac{u_5u_8-1}{u_8}\ ,\quad \lambda_{9w}=-u_7<0$$

Or

$$\lambda^2 + B_1\lambda + B_2 = 0 \tag{31b}$$

which gives the other two eigenvalues of $J(E_9)$ by:

$$\lambda_{9x} = -\tfrac{B_1}{2}+\tfrac{1}{2}\sqrt{B_1{}^2-4B_2}$$

$$\lambda_{9y} = -\tfrac{B_1}{2}-\tfrac{1}{2}\sqrt{B_1{}^2-4B_2}$$

Straightforward computations show that all the above eigenvalues have negative real parts provided that the following conditions are satisfied:

$$\frac{u_1\hat{y}}{(u_2+\hat{x})^2}<1 \tag{32a}$$

$$u_5u_8<1 \tag{32b}$$

So, E_9 is locally asymptotically stable in the R_+^4. However, it is a saddle point otherwise.

The Jacobin matrix of system (2) at the third three species equilibrium point $E_{10} = (x^\bullet, 0, z^\bullet, w^\bullet)$ can be written as:

$$J(E_{10})=\begin{bmatrix} -x^{\bullet} & \frac{-u_1 x^{\bullet}}{u_2+x^{\bullet}} & x^{\bullet} & 0 \\ 0 & \frac{eu_1 x^{\bullet}}{u_2+x^{\bullet}}-u_3 & 0 & 0 \\ 0 & 0 & -u_5 u_6 z^{\bullet} & -z^{\bullet} \\ 0 & 0 & -u_9 w^{\bullet} & -u_7 u_8 w^{\bullet} \end{bmatrix}$$

Hence the characteristic equation of $J(E_{10})$ is given by:

$$\left(-x^{\bullet}-\lambda\right)\left(\left(\frac{eu_1 x^{\bullet}}{u_2+x^{\bullet}}-u_3\right)-\lambda\right)\left[\lambda^2+B_1^{\bullet}\lambda+B_2^{\bullet}\right]=0$$

where

$$B_1^{\bullet}=u_5 u_6 z^{\bullet}+u_7 u_8 w^{\bullet}$$
$$B_2^{\bullet}=\left(u_5 u_6 u_7 u_8-u_9\right)z^{\bullet} w^{\bullet}$$

So, either

$$\left(-x^{\bullet}-\lambda\right)\left(\left(\frac{eu_1 x^{\bullet}}{u_2+x^{\bullet}}-u_3\right)-\lambda\right)=0 \qquad (33a)$$

which gives two of the eigenvalues of $J(E_{10})$ by:

$$\lambda_{10x}=-x^{\bullet}<0\ ,\quad \lambda_{10y}=\frac{eu_1 x^{\bullet}-u_3\left(u_2+x^{\bullet}\right)}{u_2+x^{\bullet}}$$

Or

$$\lambda^2+B_1^{\bullet}\lambda+B_2^{\bullet}=0 \qquad (33b)$$

which gives the other two eigenvalues of $J(E_{10})$ by:

$$\lambda_{10z}=-\frac{B_1^{\bullet}}{2}+\frac{1}{2}\sqrt{B_1^{\bullet 2}-4B_2^{\bullet}}$$
$$\lambda_{10w}=-\frac{B_1^{\bullet}}{2}-\frac{1}{2}\sqrt{B_1^{\bullet 2}-4B_2^{\bullet}}$$

Straightforward computations show that all the above eigenvalues have negative real parts provided that the following conditions are satisfied

$$u_5 u_6 u_7 u_8>u_9 \qquad (34a)$$
$$eu_1 x^{\bullet}<u_3\left(u_2+x^{\bullet}\right) \qquad (34b)$$

So, E_{10} is locally asymptotically stable in the R_+^4. However, it is a saddle point otherwise.

The Jacobian matrix of system (2) at the positive equilibrium point $E_{11}=\left(x^*,y^*,z^*,w^*\right)$ can be written as:

$$J(E_{11})=\begin{bmatrix} -x^*+\frac{u_1 x^* y^*}{\left(u_2+x^*\right)^2} & \frac{-u_1 x^*}{u_2+x^*} & x^* & 0 \\ \frac{eu_1 u_2 y^*}{\left(u_2+x^*\right)^2} & -u_4 y^* & 0 & 0 \\ 0 & 0 & -u_5 u_6 z^* & -z^* \\ 0 & 0 & -u_9 w^* & -u_7 u_8 w^* \end{bmatrix}$$

Hence the characteristic equation of $J(E_{11})$ is given by:

$$\left[\lambda^2+R_1\lambda+R_2\right]\left[\lambda^2+R_3\lambda+R_4\right]=0$$

where

$$R_1=-x^*\left(-1+\frac{u_1 y^*}{\left(u_2+x^*\right)^2}\right)+u_4 y^*$$

$$R_2 = u_4 x^* y^* \left(1 - \frac{u_1 y^*}{(u_2 + x^*)^2}\right) + \frac{e u_1^2 u_2 x^* y^*}{(u_2 + x^*)^3}$$
$$R_3 = u_5 u_6 z^* + u_7 u_8 w^*$$
$$R_4 = (u_5 u_6 u_7 u_8 - u_9) z^* w^*$$

So, either

$$\lambda^2 + R_1 \lambda + R_2 = 0 \quad (35a)$$

which gives the first two eigenvalues of $J(E_{11})$ as:

$$\lambda_{11x} = -\tfrac{R_1}{2} + \tfrac{1}{2}\sqrt{R_1^{\,2} - 4R_2}$$
$$\lambda_{11y} = -\tfrac{R_1}{2} - \tfrac{1}{2}\sqrt{R_1^{\,2} - 4R_2}$$

Straightforward computations show that the above eigenvalues have negative real parts provided that the following condition is satisfied.

$$\frac{u_1 y^*}{(u_2 + x^*)^2} < 1 \quad (35b)$$

Or

$$\lambda^2 + R_3 \lambda + R_4 = 0 \quad (35c)$$

which gives the other two eigenvalues of $J(E_{11})$ as:

$$\lambda_{11z} = -\tfrac{R_3}{2} + \tfrac{1}{2}\sqrt{R_3^{\,2} - 4R_4}$$
$$\lambda_{11w} = -\tfrac{R_3}{2} - \tfrac{1}{2}\sqrt{R_3^{\,2} - 4R_4}$$

Again straightforward computations show that the above eigenvalues have negative real parts provided that the following condition is satisfied

$$u_5 u_6 u_7 u_8 > u_9 \quad (35d)$$

So, E_{11} is locally asymptotically stable in the R_+^4 under the conditions (35b) and (35d). However, it is a saddle point otherwise.

5. Global Stability Analysis:-

In this section the global stability analysis for the equilibrium points, which are locally asymptotically stable, of system (2) is studied analytically with the help of Lyapunov method as shown in the following theorems

Theorem (2): Assume that, the equilibrium point $E_6 = (\bar{x}, 0, \bar{z}, 0)$ of system (2) is locally asymptotically stable and the following conditions hold

$$\bar{x} < \frac{u_2 u_3}{e u_1} \quad (36a)$$
$$1 < 4 u_5 u_6 \quad (36b)$$
$$\frac{(\bar{z} + u_7)^2}{4 u_7 u_8} < \left[(x - \bar{x}) - \sqrt{u_5 u_6}\,(z - \bar{z})\right]^2 \quad (36c)$$

Then the equilibrium point E_6 of system (2) is globally asymptotically stable.

Proof: Consider the following function

$$V_1(x, y, z, w) = c_1\left(x - \bar{x} - \bar{x}\ln\frac{x}{\bar{x}}\right) + c_2 y + c_3\left(z - \bar{z} - \bar{z}\ln\frac{z}{\bar{z}}\right) + c_4 w$$

where c_1, c_2, c_3 *and* c_4 are positive constants to be determine.

Clearly $V_1 : R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_1 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_1}{dt} = -c_1(x-\bar{x})^2 + c_1(x-\bar{x})(z-\bar{z}) - c_3u_5u_6(z-\bar{z})^2 - \frac{u_1 x\, y}{u_2+x}(c_1 - c_2 e) + c_1\frac{u_1\bar{x}\, y}{u_2} - c_2u_3 y + (c_3\bar{z} + c_4u_7)w - c_4u_7u_8w^2$$

by choosing $c_1 = 1$, $c_2 = \frac{1}{e}$, $c_3 = c_4 = 1$ we get

$$\frac{dV_1}{dt} = -(x-\bar{x})^2 + (x-\bar{x})(z-\bar{z}) - u_5u_6(z-\bar{z})^2 - \left(\frac{u_3}{e} - \frac{u_1\bar{x}}{u_2}\right)y + (\bar{z}+u_7)w\left[1 - \frac{u_7\,u_8\,w}{\bar{z}+u_7}\right]$$

Now since the function $f(w) = (\bar{z}+u_7)w\left[1 - \frac{u_7u_8\,w}{\bar{z}+u_7}\right]$ in the last term represents a logistic function with respect to w and hence it is bounded above by the constant $\frac{(\bar{z}+u_7)^2}{4u_7u_8}$ then according to the conditions (36a) – (36b) we have

$$\frac{dV_1}{dt} < -\left[(x-\bar{x}) - \sqrt{u_5\,u_6}\,(z-\bar{z})\right]^2 + \frac{(\bar{z}+u_7)^2}{4u_7\,u_8}$$

So, if condition (36c) holds then we obtain that $\frac{dV_1}{dt}$ is negative definite and hence V_1 is a Lyapunov function. Thus E_6 is a globally asymptotically stable and the proof is complete. ■

Theorem (3): Assume that, the equilibrium point $E_7 = (\bar{\bar{x}}, 0, 0, \bar{\bar{w}})$ of system (2) is locally asymptotically stable and the following conditions hold

$$\bar{\bar{x}} < \frac{u_2\,u_3}{e u_1} \qquad (37a)$$

$$\frac{(1+u_5)^2}{4u_5\,u_6} < (x-\bar{\bar{x}})^2 + \frac{u_7\,u_8\,\bar{\bar{x}}}{u_9\,\bar{\bar{w}}}(w-\bar{\bar{w}})^2 \qquad (37b)$$

Then the equilibrium point E_7 of system (2) is globally asymptotically stable.

Proof: Consider the following function

$$V_2(x,y,z,w) = c_1\left(x - \bar{\bar{x}} - \bar{\bar{x}}\ln\frac{x}{\bar{\bar{x}}}\right) + c_2 y + c_3 z + c_4\left(w - \bar{\bar{w}} - \bar{\bar{w}}\ln\frac{w}{\bar{\bar{w}}}\right)$$

where c_1, c_2, c_3 *and* c_4 are positive constants to be determine

Clearly $V_2 : R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_2 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_2}{dt} = -c_1(x-\bar{\bar{x}})^2 - \frac{u_1 x\, y}{u_2+x}(c_1 - c_2 e) - \left(c_2u_3 - c_1\frac{u_1\bar{\bar{x}}}{u_2}\right)y + c_1 z - (c_1\bar{\bar{x}} - c_4u_9\bar{\bar{w}})z + c_3u_5 z - c_3u_5u_6\,z^2 - c_4u_7u_8(w-\bar{\bar{w}})^2$$

by choosing $c_1 = 1, c_2 = \frac{1}{e}, c_3 = 1, c_4 = \frac{\bar{\bar{x}}}{u_9\bar{\bar{w}}}$ we get

$$\frac{dV_2}{dt} = -(x-\bar{\bar{x}})^2 - \frac{u_7\,u_8\,\bar{\bar{x}}}{u_9\,\bar{\bar{w}}}(w-\bar{\bar{w}})^2 - \left(\frac{u_3}{e} - \frac{u_1\bar{\bar{x}}}{u_2}\right)y + (1+u_5)z\left[1 - \frac{u_5\,u_6\,z}{1+u_5}\right]$$

Now since the function $f(z) = (1+u_5)z\left[1 - \frac{u_5u_6\,z}{1+u_5}\right]$ in the last term represents a logistic function with respect to z and hence it is bounded above by the constant $\frac{(1+u_5)^2}{4u_5\,u_6}$ then according to the condition (37a) we have

$$\frac{dV_2}{dt} < -\left[\left(x - \overline{\overline{x}} \right)^2 + \frac{u_7 u_8 \overline{\overline{x}}}{u_9 \overline{\overline{w}}} \left(w - \overline{\overline{w}} \right)^2 \right] + \frac{(1+u_5)^2}{4u_5 u_6}$$

So, if the condition (37b) holds then we obtain that $\frac{dV_2}{dt}$ is negative definite and hence V_2 is a Lyapunov function. Thus E_7 is a globally asymptotically stable and the proof is complete. ■

Theorem (4): Assume that, the equilibrium point $E_8 = (\breve{x}, \breve{y}, \breve{z}, 0)$ of system (2) is locally asymptotically stable and the following conditions hold

$$\frac{u_1 \breve{y}}{u_2 (u_2 + \breve{x})} < 1 \qquad (38a)$$

$$\left(\frac{e u_1 u_2 - u_1 u_2 - u_1 \breve{x}}{(u_2 - x)(u_2 + \breve{x})} \right)^2 < 2u_4 \left(1 - \frac{u_1 \breve{y}}{u_2 (u_2 + \breve{x})} \right) \qquad (38b)$$

$$1 < 2u_5 u_6 \left(1 - \frac{u_1 \breve{y}}{u_2 (u_2 + \breve{x})} \right) \qquad (38c)$$

$$\frac{(\breve{z} + u_7)^2}{4u_7 u_8} < \beta_1 + \beta_2 \qquad (38d)$$

here

$$\beta_1 = \left[\frac{1}{\sqrt{2}} \sqrt{1 - \frac{u_1 \breve{y}}{u_2 (u_2 + \breve{x})}} (x - \breve{x}) - \sqrt{u_4} (y - \breve{y}) \right]^2$$

$$\beta_2 = \left[\frac{1}{\sqrt{2}} \sqrt{1 - \frac{u_1 \breve{y}}{u_2 (u_2 + \breve{x})}} (x - \breve{x}) - \sqrt{u_5 u_6} (z - \breve{z}) \right]^2$$

Then the equilibrium point E_8 of system (2) is globally asymptotically stable.

Proof: Consider the following function

$$V_3(x, y, z, w) = \left(x - \breve{x} - \breve{x} \ln \frac{x}{\breve{x}} \right) + \left(y - \breve{y} - \breve{y} \ln \frac{y}{\breve{y}} \right) + \left(z - \breve{z} - \breve{z} \ln \frac{z}{\breve{z}} \right) + w$$

Clearly $V_3 : R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_3 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_3}{dt} \le -\left(1 - \frac{u_1 \breve{y}}{u_2 (u_2 + \breve{x})} \right) (x - \breve{x})^2 + \left(\frac{e u_1 u_2 - u_1 u_2 - u_1 \breve{x}}{(u_2 - x)(u_2 + \breve{x})} \right) (x - \breve{x})(y - \breve{y}) - u_4 (y - \breve{y})^2 + (x - \breve{x})(z - \breve{z}) - u_5 u_6 (z - \breve{z})^2 + (\breve{z} + u_7) w \left[1 - \frac{u_7 u_8 w}{\breve{z} + u_7} \right]$$

Now since the function $f(w) = (\breve{z} + u_7) w \left[1 - \frac{u_7 u_8 w}{\breve{z} + u_7} \right]$ in the last term represents a logistic function with respect to w and hence it is bounded above by the constant $\frac{(\breve{z} + u_7)^2}{4u_7 u_8}$ then by using the conditions (38a) – (38c) we get

$$\frac{dV_3}{dt} < -\beta_1 - \beta_2 + \frac{(\breve{z} + u_7)^2}{4u_7 u_8}$$

So, if the condition (38d) holds then we obtain that $\frac{dV_3}{dt}$ is negative definite and hence V_3 is a Lyapunov function. Thus E_8 is a globally asymptotically stable and the proof is complete. ■

Theorem (5): Assume that, the equilibrium point $E_9 = (\hat{x}, \hat{y}, 0, \hat{w})$ of system (2) is locally asymptotically stable and the following conditions hold

$$\frac{u_1 \hat{y}}{u_2 (u_2 + \hat{x})} < 1 \qquad (39a)$$

$$\left(\frac{eu_1u_2 - u_1u_2 - u_1\hat{x}}{(u_2 - x)(u_2 + \hat{x})}\right)^2 < 4u_4\left(1 - \frac{u_1\hat{y}}{u_2(u_2 + \hat{x})}\right) \quad (39b)$$

$$\frac{u_5 + \hat{x} + u_9\hat{w}}{u_6} < \delta_1 + \delta_2 \quad (39c)$$

where $\delta_1 = \left[\sqrt{1 - \frac{u_1\,\hat{y}}{u_2\,(u_2 + \hat{x})}}\,(x - \hat{x}) - \sqrt{u_4}\,(y - \hat{y})\right]^2$; $\delta_2 = u_7u_8(w - \hat{w})^2$. Then the equilibrium point E_9 of system (2) is globally asymptotically stable.

Proof: Consider the following function

$$V_4(x,y,z,w) = \left(x - \hat{x} - \hat{x}\ln\frac{x}{\hat{x}}\right) + \left(y - \hat{y} - \hat{y}\ln\frac{y}{\hat{y}}\right) + z + \left(w - \hat{w} - \hat{w}\ln\frac{w}{\hat{w}}\right)$$

Clearly $V_4 : R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_4 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_4}{dt} \le -\left(1 - \frac{u_1\hat{y}}{u_2(u_2 + \hat{x})}\right)(x - \hat{x})^2 + \left(\frac{eu_1u_2 - u_1\hat{x} - u_1u_2}{(u_2 - x)(u_2 + \hat{x})}\right)(x - \hat{x})(y - \hat{y}) - u_4(y - \hat{y})^2 + (u_5 + \hat{x} + u_9\,\hat{w})z - u_7u_8(w - \hat{w})^2$$

by using the condition (39a) – (39b) we get

$$\frac{dV_4}{dt} < -\delta_1 - \delta_2 + \frac{u_5 + \hat{x} + u_9\hat{w}}{u_6}$$

Then $\frac{dV_4}{dt}$ is negative definite due to condition (39c) and hence V_4 is a Lyapunov function. Thus E_9 is a globally asymptotically stable and the proof is complete.

Theorem (6): Assume that, the equilibrium point $E_{10} = (x^\bullet, 0, z^\bullet, w^\bullet)$ of system (2) is locally asymptotically stable and the following conditions hold

$$1 < 2u_5u_6 \quad (40a)$$

$$x^\bullet < \frac{u_2u_3}{eu_1} \quad (40b)$$

$$2 < \frac{u_5u_6u_7u_8}{u_9} \quad (40c)$$

Then the equilibrium point E_{10} of system (2) is globally asymptotically stable.

Proof: Consider the following function

$$V_5(x,y,z,w) = c_1\left(x - x^\bullet - x^\bullet\ln\frac{x}{x^\bullet}\right) + c_2y + c_3\left(z - z^\bullet - z^\bullet\ln\frac{z}{z^\bullet}\right) + c_4\left(w - w^\bullet - w^\bullet\ln\frac{w}{w^\bullet}\right)$$

where c_1, c_2, c_3 *and* c_4 are positive constants to be determine.

Clearly $V_5 : R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_5 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_5}{dt} \le -c_1(x - x^\bullet)^2 - \frac{u_1xy}{u_2 + x}(c_1 - c_2e) + c_1\frac{u_1x^\bullet y}{u_2} + c_1(x - x^\bullet)(z - z^\bullet) - c_2u_3y - c_3u_5u_6(z - z^\bullet)^2 - c_3(z - z^\bullet)(w - w^\bullet) - c_4u_7u_8(w - w^\bullet)^2 - c_4u_9(w - w^\bullet)(z - z^\bullet)$$

by choosing $c_1 = 1, c_2 = \frac{1}{e},\ c_3 = 1,\ c_4 = \frac{1}{u_9}$ we get

$$\frac{dV_5}{dt} = -\left(x - x^{\bullet}\right)^2 + \left(x - x^{\bullet}\right)\left(z - z^{\bullet}\right) - u_5 u_6 \left(z - z^{\bullet}\right)^2 - \left(\frac{u_3}{e} - \frac{u_1 x^{\bullet}}{u_2}\right) y - 2\left(z - z^{\bullet}\right)\left(w - w^{\bullet}\right) - \frac{u_7 u_8}{u_9}\left(w - w^{\bullet}\right)^2$$

by using the conditions (40a) – (40c) we get

$$\frac{dV_5}{dt} \le -\left[\left(x - x^{\bullet}\right) - \sqrt{\frac{u_5 u_6}{2}}\left(z - z^{\bullet}\right)\right]^2 - \left(\frac{u_3}{e} - \frac{x^{\bullet} u_1}{u_2}\right) y - \left[\sqrt{\frac{u_5 u_6}{2}}\left(z - z^{\bullet}\right) - \sqrt{\frac{u_7 u_8}{u_9}}\left(w - w^{\bullet}\right)\right]^2$$

Then $\frac{dV_5}{dt}$ is negative definite and hence V_5 is a Lyapunov function. Thus E_{10} is a globally asymptotically stable and the proof is complete. ■

Theorem (7): Assume that, the equilibrium point $E_{11} = \left(x^*, y^*, z^*, w^*\right)$ of system (2) is locally asymptotically stable and the following conditions hold

$$\frac{u_1 y^*}{u_2\left(u_2 + x^*\right)} < 1 \qquad (41a)$$

$$\left(\frac{eu_1 u_2 - u_1 u_2 - u_1 x^*}{(u_2 - x)\left(u_2 + x^*\right)}\right)^2 < 2u_4\left(1 - \frac{u_1 y^*}{u_2\left(u_2 + x^*\right)}\right) \qquad (41b)$$

$$1 < u_5 u_6\left(1 - \frac{u_1 y^*}{u_2\left(u_2 + x^*\right)}\right) \qquad (41c)$$

$$(1 + u_9)^2 < 2u_5 u_6 u_7 u_8 \qquad (41d)$$

Then the equilibrium point E_{11} of system (2) is globally asymptotically stable.

Proof: Consider the following function

$$V_6(x, y, z, w) = \left(x - x^* - x^* \ln\frac{x}{x^*}\right) + \left(y - y^* - y^* \ln\frac{y}{y^*}\right) + \left(z - z^* - z^* \ln\frac{z}{z^*}\right) + \left(w - w^* - w^* \ln\frac{w}{w^*}\right)$$

Clearly $V_6 : R_+^4 \to R$ is a C^1 positive definite function. Now by differentiating V_6 with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_6}{dt} \le -\left(1 - \frac{u_1 y^*}{u_2(u_2 + x)}\right)\left(x - x^*\right)^2 + \left(\frac{eu_1 u_2 - u_1 u_2 - u_1 x^*}{(u_2 - x)\left(u_2 + x^*\right)}\right)\left(x - x^*\right)\left(y - y^*\right) + \left(x - x^*\right)\left(z - z^*\right) - u_4\left(y - y^*\right)^2 - u_5 u_6\left(z - z^*\right)^2 - u_7 u_8\left(w - w^*\right)^2 - (1 + u_9)\left(z - z^*\right)\left(w - w^*\right)$$

by using the conditions (41a) – (41e) we get

$$\frac{dV_6}{dt} \leq -\left[\sqrt{\frac{1}{2}\left(1-\frac{u_1 y^*}{u_2(u_2+x^*)}\right)}\left(x-x^*\right)-\sqrt{u_4}\left(y-y^*\right)\right]^2 - \left[\sqrt{\frac{1}{2}\left(1-\frac{u_1 y^*}{u_2(u_2+x^*)}\right)}\left(x-x^*\right)-\sqrt{\frac{u_5 u_6}{2}}\left(z-z^*\right)\right]^2 - \left[\sqrt{\frac{u_5 u_6}{2}}\left(z-z^*\right)-\sqrt{u_7 u_8}\left(w-w^*\right)\right]^2$$

Then $\frac{dV_6}{dt}$ is negative definite and hence V_6 is a Lyapunov function. Thus E_{11} is a globally asymptotically stable and the proof is complete. ■

6. Numerical Simulation:-

In this paper the dynamical behavior of system (2) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions (35a) and (35d) of the positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point as shown in Figure-2.

Note that, from now on ward the solid, dash, dot and dash-dot are used to describing the trajectories of the prey x, the predator y, the Host z and the Host competitor w respectively.

$$u_1 = 0.6,\ u_2 = 0.25,\ u_3 = 0.1,\ u_4 = 0.05,\ u_5 = 2,\ u_6 = 0.5$$
$$u_7 = 2,\ u_8 = 0.75,\ u_9 = 0.8,\ e = 0.5 \qquad (42)$$

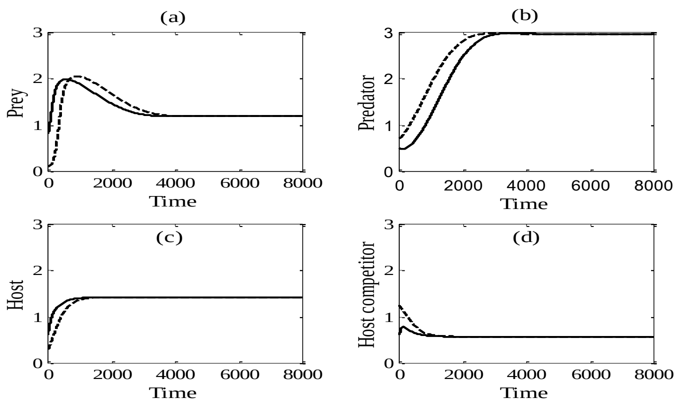

Figure 2- Time series of the solution of system (2) that started from two different initial points $(0.8,0.7,0.6,0.9)$ and $(1.0,0.5,0.3,1.25)$ for the data given by Eq. (42). (a) trajectories of x as a function of time, (b) trajectories of y as a function of time, (c) trajectories of z as a function of time,(d) trajectories of w as a function of time.

Clearly, Figure-2 shows that system (2) has a globally asymptotically stable as the solution of system (2) approaches asymptotically to the positive equilibrium point $E_{11} = (1.2, 2.96, 1.42, 0.57)$ starting from two different initial points and this is confirming our obtained analytical results.

Now in order to discuss the effect of the parameters values of system (2) on the dynamical behavior of the system, the system is solved numerically for the data given in Eq. (42) with varying one parameter each time. It is observed that for the data as given in Eq. (42) with $u_1 < 0.23$, the solution of system (2) approaches asymptotically to $E_{10} = (x^\bullet, 0, z^\bullet, w^\bullet)$ in the $xzw-$space as shown in Figure-3, however for $0.23 \leq u_1 \leq 0.61$ the system approaches to the positive equilibrium point, finally for $0.61 < u_1$ it is observed as given in Figure-4, that system (2) has a periodic dynamics in the $Int.R_+^4$.

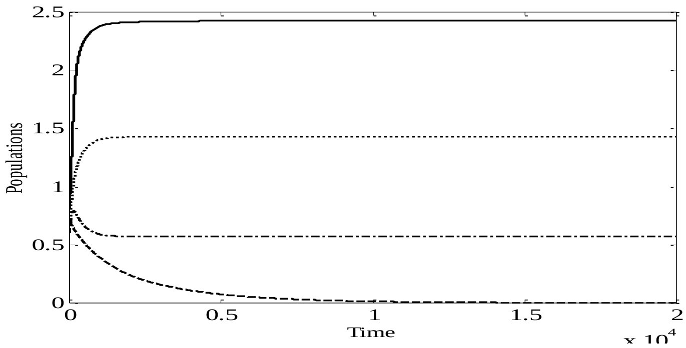

Figure 3- Time series of the solution of system (2) for the data given by Eq. (42) with $u_1 = 0.15$, which approaches to $(2.42,0.0,1.42,0.57)$ in $xzw-$space

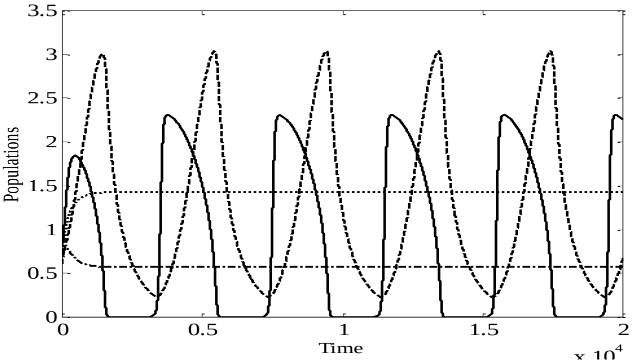

Figure 4- Time series of the solution of system (2) for the data given by Eq. (42) with $u_1 = 0.7$, which approaches to periodic dynamics in $Int.R_+^4$.

By varying the parameter u_2 keeping the rest of parameters values as in Eq. (42), it observed that for $u_2 < 0.24$ system (2) approaches to periodic dynamics in $Int.R_+^4$, while for $0.24 \le u_2$ the solution still has a stable positive equilibrium point. On other hand varying the parameter u_3 keeping the rest of parameters values as in Eq. (42), it observed that for $u_3 \le 0.09$ system (2) approaches to periodic dynamics in $Int.R_+^4$, while for $0.09 < u_3 < 0.27$ the solution still has a stable positive equilibrium point, further for $0.27 \le u_3$ the solution of system (2) approaches asymptotically to the equilibrium point $E_{10} = (x^\bullet, 0, z^\bullet, w^\bullet)$ in the $xzw-$space. Moreover, varying the parameter u_4 keeping the rest of parameters values as in Eq. (42), showed that for $u_4 \le 0.04$ system (2) approaches to periodic dynamics in $Int.R_+^4$, while for $0.04 < u_4 < 1$ the solution still has a stable positive equilibrium point.

For the parameters values given in Eq. (42) with u_5 varying in the range $u_5 \le 1.33$ the solution of system (2) approaches asymptotically to the periodic dynamics in the interior of positive octant of $xyw-$space as shown in Figure-5, however for $1.33 < u_5 < 1.86$ it is observed that system (2) has a periodic dynamics in $Int.R_+^4$, finally for $1.86 < u_5$ the solution approaches to a positive equilibrium point.

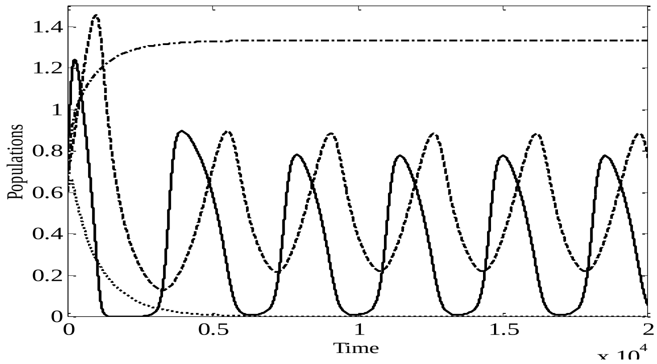

Figure 5- Time series of the solution of system (2) for the data given by Eq. (42) with $u_5 = 1.25$, which approaches to periodic dynamics in the interior of positive octant of xyw – space.

For the parameters values given in Eq. (42) with u_6 varying in the range $u_6 \leq 0.4$ the solution of system (2) approaches asymptotically to the equilibrium point $E_8 = (\breve{x}, \breve{y}, \breve{z}, 0)$ in the interior of positive octant of xyz – space as shown in Figure-6, however for $0.4 < u_6$ it is observed that system (2) approaches asymptotically to a positive equilibrium point .

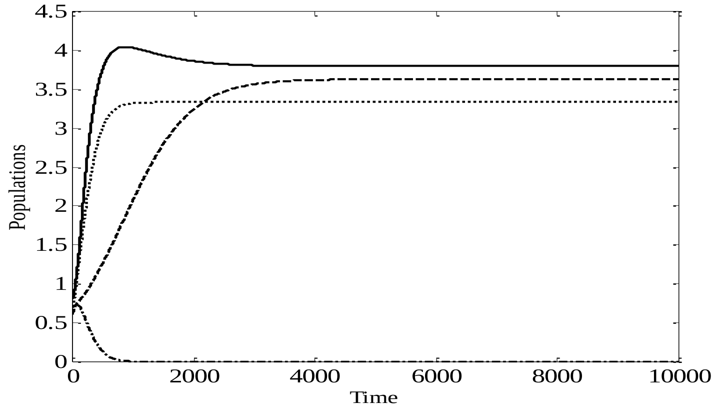

Figure 6- Time series of the solution of system (2) for the data given by Eq. (42) with $u_6 = 0.3$, which approaches asymptotically to $(3.79, 3.62, 3.33, 0)$ in the interior of positive octant of xyz – space.

For the parameters values given in Eq. (42) with u_7 varying in the range $u_7 \leq 1.6$ the solution of system (2) approaches asymptotically to the equilibrium point $E_8 = (\breve{x}, \breve{y}, \breve{z}, 0)$ in the interior of positive octant of xyz – space, however for $1.6 < u_7 \leq 2.1$ it is observed that the solution of system (2) approaches asymptotically to a positive equilibrium point, finally for $2.1 < u_7$ system (2) has a periodic dynamics in $Int.R_+^4$ as shown in Figure-7.

For the parameters values given in Eq. (42) with u_8 varying in the range $u_8 \leq 0.5$ system (2) has a periodic dynamics in the interior of positive octant of xyw – space, however for $0.5 < u_8$ it is observed that system (2) approaches asymptotically to a positive equilibrium point .

For the parameters values given in Eq. (42) with u_9 varying in the range $u_9 < 0.75$ system (2) has a periodic dynamics in $Int.R_+^4$ as shown in Figure-8, however for $0.75 \leq u_9 < 0.99$ it is observed that the solution of system (2) approaches asymptotically to a positive equilibrium point, finally for $0.99 \leq u_9$ the solution of system (2) approaches asymptotically to the equilibrium point $E_8 = (\breve{x}, \breve{y}, \breve{z}, 0)$ in the interior of positive octant of xyz – space.

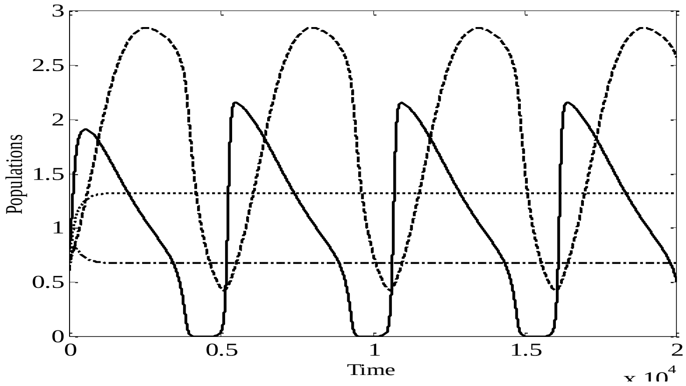

Figure 7- Time series of the solution of system (2) for the data given by Eq. (42) with $u_7 = 2.15$, which approaches to periodic dynamics in $Int.R_+^4$.

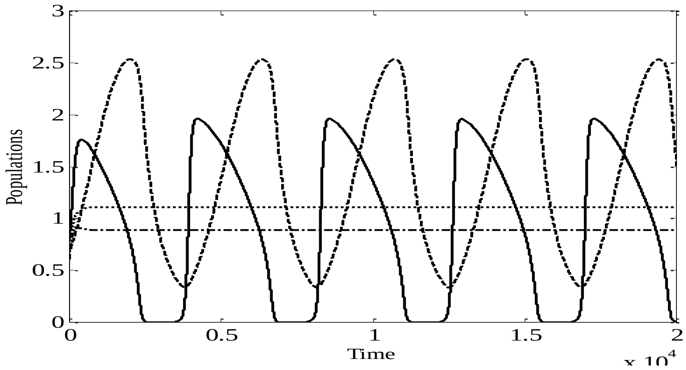

Figure 8- Time series of the solution of system (2) for the data given by Eq. (42) with $u_9 = 0.6$, which approaches to periodic dynamics in $Int.R_+^4$.

For the parameters values given in Eq. (42) with e varying in the range $e \leq 0.17$ the solution of system (2) approaches asymptotically to $E_{10} = \left(x^{\bullet}, 0, z^{\bullet}, w^{\bullet}\right)$ in the interior of positive octant of $xzw-$space as shown in Figure-9, however for $0.17 < e \leq 0.51$ it is observed that the solution of system (2) approaches asymptotically to a positive equilibrium point, finally for $0.51 < e$ system (2) has a periodic dynamics in $Int.R_+^4$.

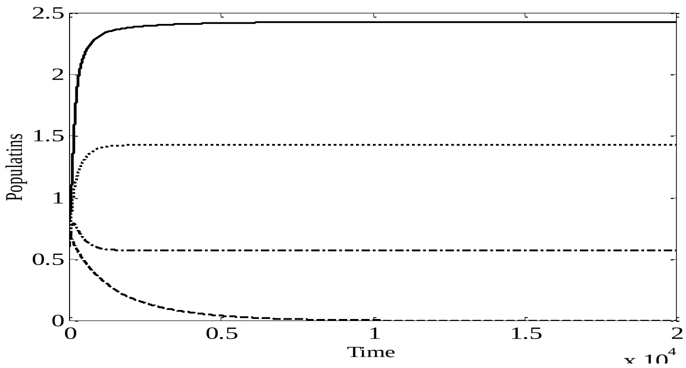

Figure 9- Time series of the solution of system (2) for the data given by Eq. (42) with $e = 0.1$, which approaches asymptotically to $(2.42, 0, 1.42, 0.57)$ in the interior of positive octant of $xzw-$space.

Moreover, for the parameters values given in Eq. (42) with $u_5 = 1.25$ and $u_1 = 0.4$ the solution of system (2) approaches asymptotically to $E_9 = (\hat{x}, \hat{y}, 0, \hat{w})$ in the interior of positive octant of $xyw-$space as shown in Figure-10, however decreases the parameter u_1 further, say $u_1 = 0.2$, then the solution of system (2) approaches asymptotically to $E_7 = \left(\overline{\overline{x}}, 0, 0, \overline{\overline{w}}\right)$ as shown in Figure-11.

Finally for the parameters values given in Eq. (42) with $u_7 = 1$ and $u_1 = 0.1$ the solution of system (2) approaches asymptotically to $E_6 = (\bar{x}, 0, \bar{z}, 0)$ in the interior of positive quadrant of $xz-$plane as shown in Figure-12.

Straightforward computation shows that the data used in figures-(10,11,12) satisfy the stability conditions of the equilibrium points $E_9 = (\hat{x}, \hat{y}, 0, \hat{w})$, $E_7 = (\bar{\bar{x}}, 0, 0, \bar{\bar{w}})$ and $E_6 = (\bar{x}, 0, \bar{z}, 0)$ respectively which confirm our analytical results too.

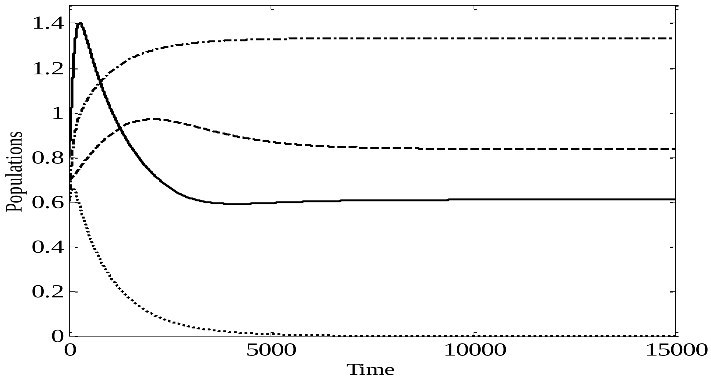

Figure 10- Time series of the solution of system (2) for the data given by Eq. (42) with $u_5 = 1.25$ and $u_1 = 0.4$, which approaches asymptotically to $(0.61, 0.83, 0, 1.33)$ in the interior of positive octant of $xyw-$space.

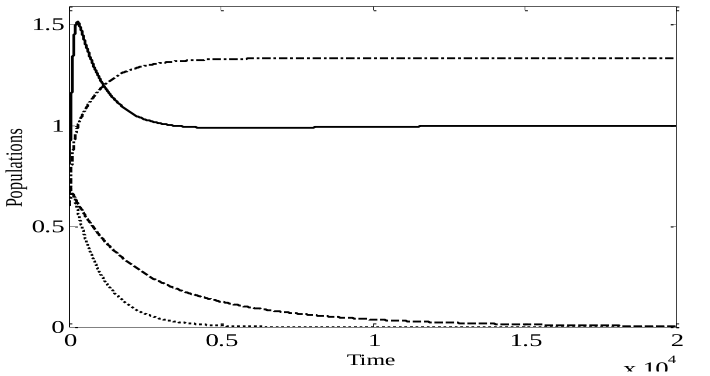

Figure 11- Time series of the solution of system (2) for the data given by Eq. (42) with $u_5 = 1.25$ and $u_1 = 0.2$, which approaches asymptotically to $(1, 0, 0, 1.33)$ in the interior of positive quadrant of $xw-$plane.

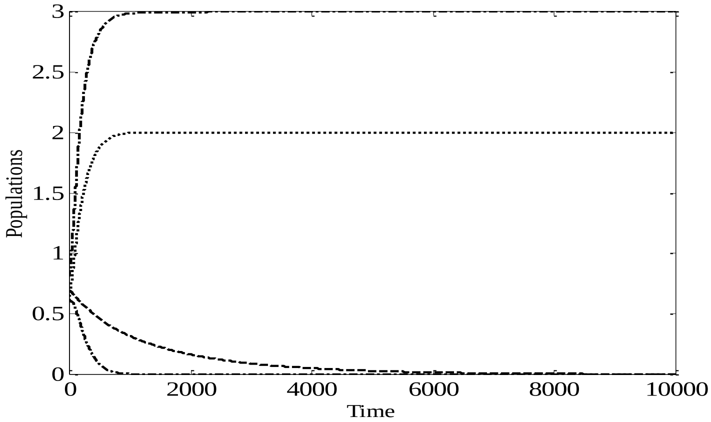

Figure 12- Time series of the solution of system (2) for the data given by Eq. (42) with $u_7 = 1$ and $u_1 = 0.1$, which approaches asymptotically to $(3, 0, 2, 0)$ in the interior of positive quadrant of $xz-$plane.

7. Conclusion and Discussion:-

In this paper, four species Syn-Ecosymbiosis model, comprising of prey-predator, commensalisms and competition is proposed for study. It is assumed that the predator species preys upon the prey according to Holling type-II functional response. The existence, uniqueness and boundedness of the solution of the system are discussed. The existence of all possible equilibrium points is studied. The local and global dynamical behaviors of the system are studied analytically as well as numerically. Finally to understand the effect of varying each parameter on the global dynamics of system (2) and to confirm our obtained analytical results, system (2) has been solved numerically for a biological feasible set of hypothetical parameters values and the following results are obtained:

1. System has only two types of dynamical behavior in the $Int.R_+^4$, approaches to either positive equilibrium point or else approaches to a limit cycle.
2. For the set of data given by Eq. (42), system (2) has a globally asymptotically stable positive point in the $Int.R_+^4$. However as the attack rate u_1 decreases then the predator species will faces extinction and the solution of system (2) approaches to $E_{10}=(x^\bullet,0,z^\bullet,w^\bullet)$ in the first octant of $xzw-$space. While increasing u_1 will causes destabilizing of system (2) and the solution approaches to a limit cycle in $Int.R_+^4$. It is observed that the conversion rate parameter e has the same effect as u_1.
3. As the half saturation rate u_2 decreases keeping the rest of parameters as in Eq. (42), the positive equilibrium point will be unstable and the solution of system (2) approaches asymptotically to a limit cycle in the $Int.R_+^4$. Otherwise the system still have a globally asymptotically stable positive point in $Int.R_+^4$. It is observed that the intraspecific competition rate parameter u_4 has the same effect as u_2.
4. As the predator's natural death rate u_3 decreases keeping the rest of parameters as in Eq. (42), the positive equilibrium point will be unstable and the solution of system (2) approaches asymptotically to a limit cycle in the $Int.R_+^4$. However increasing the parameter u_3 causes extinction in predator species and the solution of system (2) approaches to $E_{10}=(x^\bullet,0,z^\bullet,w^\bullet)$ in the first octant of $xzw-$space.
5. As the host's intrinsic growth rate u_5 decreases slightly keeping the rest of parameters as in Eq. (42), the positive equilibrium point will be unstable and the solution of system (2) approaches asymptotically to a limit cycle in the $Int.R_+^3$. However, further decreases of u_5 causes extinction in the host species and the solution of system (2) approaches asymptotically to a limit cycle in the positive octant of $xyw-$space.
6. As the inverse of the carrying capacity rate u_6 of the host species decreases keeping the rest of parameters as in Eq. (42), the competitor host faces extinction and the solution of system (2) approaches asymptotically to the equilibrium point $E_8=(\breve{x},\breve{y},\breve{z},0)$ in the first octant of $xyz-$space. Otherwise the system still have a globally asymptotically stable positive point in $Int.R_+^4$.
7. As the competitor host intrinsic growth rate u_7 decreases keeping the rest of parameters as in Eq. (42), the competitor host faces extinction and the solution of system (2) approaches asymptotically to the equilibrium point $E_8=(\breve{x},\breve{y},\breve{z},0)$ in the first octant of $xyz-$space. However, increasing u_7 will causes destabilizing of system (2) and the solution approaches to a limit cycle in $Int.R_+^4$.
8. As the inverse of the carrying capacity rate u_8 of the host competitor species decreases keeping the rest of parameters as in Eq. (42), the host species faces extinction and the solution of system (2) approaches asymptotically to the limit cycle in the first octant of $xyw-$space.
9. As the host competitor intensity of competition rate u_9 decreases keeping the rest of parameters as in Eq. (42), the positive equilibrium point will be unstable and the solution of system (2) approaches asymptotically to a limit cycle in the $Int.R_+^4$. However increasing the parameter u_9

causes extinction in the host competitor species and the solution of system (2) approaches to $E_8 = (\breve{x}, \breve{y}, \breve{z}, 0)$ in the first octant of $xyz-$space.

10. For the parameters values given by Eq. (42) with $u_5 = 1.25, u_1 = 0.4$ it is observed that all the stability conditions of E_9 are satisfied and the solution approaches asymptotically to $E_9 = (\hat{x}, \hat{y}, 0, \hat{w})$ in the first octant of $xyw-$space. However further decreasing the attack rate parameter u_1 causes extinction in predator species too and the solution of system (2) approaches asymptotically to $E_7 = (\bar{\bar{x}}, 0, 0, \bar{\bar{w}})$ in the first quadrant of $xw-$plane.

11. Finally, for the parameters values given by Eq. (42) with $u_7 = 1.0, u_1 = 0.1$ it is observed that all the stability conditions of E_6 are satisfied and the solution approaches asymptotically to $E_6 = (\bar{x}, 0, \bar{z}, 0)$ in the first quadrant of $xz-$plane.

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