Modeling and Stability of Lotka-Volterra Prey-Predator System Involving Infectious Disease in Each Population

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Abstract
In this paper, a mathematical model consisting of the prey- predator model with disease in both the population is proposed and analyzed. The existence, uniqueness and boundedness of the solution are discussed. The existences and the stability analysis of all possible equilibrium points are studied. Numerical simulation is carried out to investigate the global dynamical behavior of the system.

Keywords: eco-epidemiological model, *SIS* epidemics disease, prey-predator model, stability analysis.

نمذجة واستقرارية نظام لتكا–فولتيرا للفريسة والمفترس والمتضمن مرض معدي في كل مجتمع

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الخلاصة:
في هذا البحث، تم اقتراح ودراسة نموذج رياضي يتكون من فريسة والمفترس عند وجود مرض في كليهما. ناقشنا وجود، وحدانية وقيد الحل. قمنا بدراسة وجود و تحليل الاستقرارية لجميع نقاط التوازن الممكنة. كما تم استخدام المحاكاة العددية لبحث السلوك الديناميكي الشامل للنظام.

1. Introduction:

The effect of disease in ecological system is now becoming an important issue of research as infectious disease becomes an important factor to regulate human and animal population size. Anderson and May [1]; Chattopadhyay and Arino [2]; Hadeler and Freedman [3]; Venturino [4]; have been devoted to observe the dynamics of such system when prey population is infected with some transmissible diseases. Temple [5]; and Van Dobben [6] observed that the predator take a disproportionately high number of parasite infected prey. Hethcote et al [7] explained how a prey-predator model with logistic growth in the prey population is modified to include an SIS parasitic infection in the prey population with infected prey being more vulnerable to predation. They observed that the infection in prey population could promote coexistence.

In the last few decades; mathematical models have become extremely important tools in understanding and analyzing the spread and control of infectious disease. To the best of our knowledge, Anderson and May [1], Hadelor and Freedman [3]; Venturino [4]; Chattopadhyay and Arino [2]; Han et al [8]; Xiao and Chen [9]; Hethcote et al [7]; Greenhalgh and Haque [10]; Haque and Venturino [11] and others have been studied the influence of transmissible diseases in a prey-predator system. Hilker and Schmitz [12] also studied a prey-predator system with disease in predator population. But the studies with disease in both the populations are rare. Hadeler and Freedman [3] had previously studied a prey-predator model with parasitic infection where the disease is allowed to

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cross the species barrier. They also assumed that the predators could get infected by eaten a prey and the prey could obtain the disease from parasites spread in to the environment by predators. They obtained a threshold condition above which an endemic equilibrium or an endemic periodic solution could arise in the case where there is coexistence of the predator with the uninfected prey. Furthermore, they also showed that in the case where the predator cannot survive only on the prey in a disease–free environment, the perasitization could lead to persistence of the predator. Hsieh and Hsiao [13] proposed a prey-predator model with disease in both populations. They observed that ecological threshold number for the prey-predator ecosystem always determines the coexistence of predator and prey whereas disease basic reproduction number dictates whether the disease would become endemic in the ecosystem or not. Under one of the coupled conditions, a highly infectious disease could drive the predators to extinction, when predators and prey would have coexisted without the disease. For another combination of the conditions, the predation of the more vulnerable infected prey could cause the disease to be eradicated in the ecosystem.

In this paper a consideration is given to prey-predator model where both the prey and predator population are infected simultaneously by same or different diseases infection. On contrast to all of the above studies, in this chapter a prey-predator model involving SIS infectious disease in both the prey and the predator species is proposed and analyzed. It is assumed that the disease doesn't spreads outside the specific species (prey and predator). Instead the disease transmitted within the same species by contact, between susceptible individuals and infected individuals, in additional to the external sources from the environment. Further, in this model, linear type of functional response as well as linear incidence rate for describing the transition of disease are used.

2. Mathematical Model:

In this section, an eco-epidemiological model is proposed for study. The model consists of a prey, whose total population density at time T is denoted by $N(T)$, interacting with predator whose total population density at time T is denoted by $P(T)$. It is assumed that both the prey and the predator populations are infected by different infectious diseases. Further, the following assumptions are made in formulating the basic eco-epidemiological model:

1. There is an SIS epidemic disease in prey population divides the prey population into two classes namely $X(T)$ that represents the density of susceptible prey species at time T and $Y(T)$ which represents the density of infected prey species at time T. Therefore at any T, we have $N(T)=X(T)+Y(T)$.

2. There is an SIS epidemic disease in predator population divides the predator population into two classes namely $Z(T)$ that represents the density of susceptible predator species at time T and $W(T)$ which represents the density of infected predator species at time T. Therefore at any T we have $P(T)=Z(T)+W(T)$.

3. The susceptible and the infected prey are capable of reproducing in logistic fashion with carrying capacity $K>0$, intrinsic growth rates $r_1>0$ and $r_2>0$ respectively. In addition the disease prevents the infected individual to compete with the susceptible however the susceptible individuals have the capability for competition.

4. Disease dose not spread outside the specific species (prey or predator) instead the disease transmitted within the same species by contact with an infected individual at infection rates $\lambda_1>0$ and $\lambda_3>0$ for the prey and predator respectively. In addition, there is an external source of disease causes incidence with the disease within the specific population at an external infection rates $\lambda_2>0$ and $\lambda_4>0$ for the prey and predator respectively. Further the disease disappears and infected individuals become susceptible again at the recover rates $\gamma>0$ and $\theta>0$ for prey and predator species, respectively.

5. In the absence of the prey the susceptible and infected predator decay exponentially with natural death rates $\mu>0$ and $\delta>0$ respectively.

6. The disease in prey may causes mortality with a constant mortality rate represented by $m>0$.

7. The predator (susceptible and infected) consume the prey (susceptible and infected) according to Lotka-Volterra type of functional response at constant consumption rats $\alpha > 0$ (from susceptible prey) and $\beta > 0$ (from infected prey), respectively.

Considering the above basic assumptions the prey-predator model can be represented in the following set of differential equations.

$$\begin{aligned}
\frac{dX}{dT} &= r_1 X\left(1-\frac{X}{K}\right)-(\lambda_1 Y+\lambda_2)X-\alpha X(Z+W)+\gamma Y \\
\frac{dY}{dT} &= r_2 Y\left(1-\frac{X+Y}{K}\right)+(\lambda_1 Y+\lambda_2)X-\beta Y(Z+W)-mY-\gamma Y \\
\frac{dZ}{dT} &= e_1\alpha XZ+e_2\beta YZ-(\lambda_3 W+\lambda_4)Z-\mu Z+\theta W \\
\frac{dW}{dT} &= e_3\alpha XW+e_4\beta YW+(\lambda_3 W+\lambda_4)Z-\delta W-\theta W
\end{aligned} \qquad (1)$$

with $X(0)>0; Y(0)>0; Z(0)>0; W(0)>0$ and $0<e_i<1$; $i=1,2,3,4$ represent the conversion rate constants. Consequently, the flow of the food and disease in system (1) can be describe in the following block diagram.

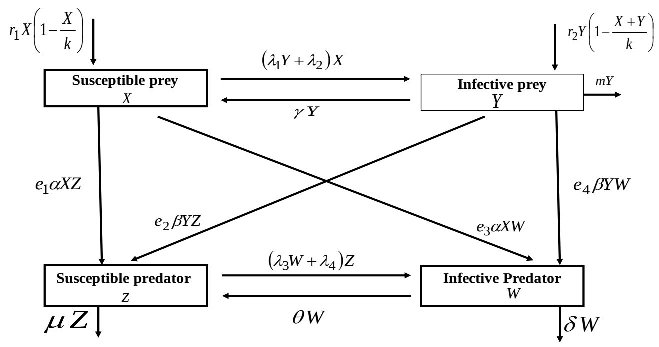

Figure 1- Block diagram of the prey-predator model given by system (1).

Cleary, system (1) included (18) parameters which make the analysis difficult. So, in order to simplify the system the number of parameters is reduced by using the following dimensionless variables

$$t=r_1T,\ x=\frac{X}{K},\ y=\frac{Y}{K},\ z=\frac{Z}{K},\ w=\frac{W}{K}.$$

Thus we obtain the following dimensionless form of the system(1):

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x)-(s_1y+s_2)x-s_3(z+w)x+s_4y = xf_1(x,y,z,w) \\
\frac{dy}{dt} &= s_5y(1-(x+y))+(s_1y+s_2)x-s_6(z+w)y-s_7y-s_4y = yf_2(x,y,z,w) \\
\frac{dz}{dt} &= e_1s_3xz+e_2s_6yz-(s_8w+s_9)z-s_{10}z+s_{11}w = zf_3(x,y,z,w) \\
\frac{dw}{dt} &= e_3s_3xw+e_4s_6yw+(s_8w+s_9)z-s_{11}w+s_{12}w = wf_4(x,y,z,w)
\end{aligned} \qquad (2)$$

here:

$$s_1=\frac{\lambda_1 k}{r_1}>0, s_2=\frac{\lambda_2}{r_1}>0, s_3=\frac{\alpha k}{r_1}>0, s_4=\frac{\gamma}{r_1}>0, s_5=\frac{r_2}{r_1}>0, s_6=\frac{\beta k}{r_1}>0,$$

$$s_7=\frac{m}{r_1}>0, s_8=\frac{\lambda_3 k}{r_1}>0, s_9=\frac{\lambda_4}{r_1}>0, s_{10}=\frac{\mu}{r_1}>0, s_{11}=\frac{\theta}{r_1}>0, s_{12}=\frac{\delta}{r_1}>0$$

represent the dimensionless parameters of the system (2). Moreover the initial condition of system (2) may be taken as any point in the region R_+^4. Obviously, the interaction functions in the right hand side of system (2) are continuously differentiable function on R_+^4, hence they are Lipschitizian. Therefore the solution of system (2) exists and is unique. Further, all the solutions of system (2) with non-negative initial condition are uniformly bounded as shown in the following theorem.

Theorem (1) All solutions of system (2) are uniformly bounded.

Proof: Let $x(t), y(t), z(t), w(t)$ be any solution of the system (2). Define the function $M(t) = x(t) + y(t) + z(t) + w(t)$, then the time derivative of $M(t)$ along the solution of the system (2), gives

$$\frac{dM}{dt} \le 1 + s_5 - nM = H - nM$$

where $n = \min\{1, s_5 + s_7, s_{10}, s_{12}\}$ and $H = 1 + s_5$. Now, by using Gromwell lemma, it obtains that:

$$0 < M(t) \le M(0)e^{-nt} + \frac{H}{n}(1 - e^{-nt})$$

which yields $\lim_{t\to\infty} M(t) \le \frac{H}{n}$ that is independent of the initial conditions. Thus the proof is complete■

3. Existence of equilibrium points:

It is observed that, system (2) has at most three biologically feasible equilibrium points, namely $E_i = (x, y, z, w)$, $i = 0,1,2$. The existence conditions for each of these equilibrium points are discussed in the following:

1- **The vanishing equilibrium point** $E_0 = (0,0,0,0)$ always exists.

2- **The predator free equilibrium point** $E_1 = (\hat{x}, \hat{y}, 0, 0)$ where

$$\hat{y} = \frac{\hat{x}(1 - (\hat{x} + s_2))}{s_1\hat{x} - s_4}; s_1\hat{x} \ne s_4 \quad (3)$$

while $\hat{x}$ represents a positive root of the following third order polynomial equation

$$B_1x^3 + B_2x^2 + B_3x + B_4 = 0 \quad (4)$$

where:

$$B_1 = s_5(s_1 - 1) - s_1{}^2;$$
$$B_2 = s_1(s_1 + 2s_4 + s_2 + s_2s_5) + 2s_5 - s_5(2s_1 + s_4 + 2s_2);$$
$$B_3 = s_5(s_1 + 2(s_4 + s_2)) + s_1s_7(s_2 - 1) - s_5(s_2(s_1 + s_2 + s_4) + 1) - s_4(2s_1 + s_4 + s_7);$$
$$B_4 = s_4((s_2s_5 + s_4 + s_7) - (s_2s_7 + s_5))$$

Consequently, straightforward computation shows that E_1 exists uniquely in the $Int.R_+^4$ if and only if the following conditions are hold.

$$\left.\begin{array}{l} \hat{x} + s_2 < 1 \text{ and } s_4 < s_1\hat{x} \\ \text{OR} \\ \hat{x} + s_2 > 1 \text{ and } s_4 > s_1\hat{x} \end{array}\right\} \quad (5a)$$

$$\left.\begin{array}{l} B_1 > 0, B_2 > 0 \text{ and } B_4 < 0 \\ OR \\ B_1 > 0, B_3 < 0 \text{ and } B_4 < 0 \\ OR \\ B_1 < 0, B_3 > 0 \text{ and } B_4 > 0 \\ \text{OR} \\ B_1 < 0, B_2 < 0 \text{ and } B_4 > 0 \end{array}\right\} \quad (5b)$$

3. **The positive equilibrium point** $E_2 = (x^*, y^*, z^*, w^*)$

The positive equilibrium point E_2 exists in the $Int.R_+^4$ if and only if there is a positive solution of the following set of algebraic equations

$$f_1(x,y,z,w) = (1-x) - (s_1 y + s_2) - s_3(z+w)x + \frac{s_4 y}{x} = 0 \quad (6a)$$

$$f_2(x,y,z,w) = s_5\big(1-(x+y)\big) + s_1 x + \frac{s_2 x}{y} - s_6\big(z+w\big) = 0$$

(6b) $f_3(x,y,z,w) = e_1 s_3 x + e_2 s_6 y - s_8 w - (s_9 + s_{10}) + \frac{s_{11} w}{z} = 0$ (6c)

$$f_4(x,y,z,w) = e_3 s_3 x + e_4 s_6 y + s_8 z - (s_{11} + s_{12}) + \frac{s_9 z}{w} = 0 \quad (6d)$$

By solving (6c) and (6d), we obtain that

$$z(x,y) = \frac{B}{s_8 A};\ A \neq 0 \quad (6e)$$

$$w(x,y) = \frac{(A - s_9)B}{s_8(B - s_{11}A)};\ B \neq s_{11}A \quad (6f)$$

where:

$$A = e_1 s_3 x + e_2 s_6 y - s_{10}$$

$$B = (s_{11} + s_{12})A - (e_3 s_3 x + e_4 s_6 y)(A - s_9) - s_9 s_{12}$$

Then by using (6e) and (6f) in (6a) and (6b) yield the following two isoclines.

$$g_1(x,y) = x\left[1 - x - (s_1 y + s_2) - \frac{s_3 B}{S_8}\left(\frac{1}{A} + \frac{A - s_9}{B - s_{11}A}\right)\right] + s_4 y = 0 \quad (6g)$$

$$g_2(x,y) = y\left[s_5\big(1-(x+y)\big) + s_1 x - \frac{s_6 B}{s_8}\left(\frac{1}{A} + \frac{A - s_9}{B - s_{11}A}\right) - (s_4 + s_7)\right] + s_2 x = 0 \quad (6h)$$

Now from (6g) we notice that, when $y \to 0$, then either $x = 0$ or x represents a positive root of the following fourth order polynomial equation.

$$M_1 x^4 + M_2 x^3 + M_3 x^2 + M_4 x + M_5 = 0 \quad (7)$$

here:

$$M_1 = \gamma_1 s_3 [e_1(s_8 + e_1 s_3) - \gamma_1]$$

$$M_2 = -s_8[e_1 s_3 \gamma_4 + \gamma_1(e_1 s_3(1 - s_2) + s_{10})] + s_3[\gamma_1(\gamma_2 + \gamma_4) - e_1 s_3(\gamma_5 + s_{10}\gamma_1)]$$

$$M_3 = s_3 s_{12}(s_9 + s_{10})(e_1 s_8 - \gamma_1) + s_3[\gamma_4(e_1 s_8(1 - s_2) - \gamma_2) - \gamma_1\gamma_3 + e_1 s_3 \gamma_6] + s_{10}[s_8(\gamma_4 + \gamma_1(1 - s_2) + s_3\gamma_5)]$$

$$M_4 = (s_9 + s_{10})\left[-s_8 s_{12}\left(e_1 s_3 (1-s_2) + s_{10}\right) + s_3 (s_{12}\gamma_2 - e_1 s_3 \gamma_3)\right] + s_3\gamma_3\gamma_4 - s_{10}\left[s_8\gamma_4(1-s_2) + s_3 s_6\right]$$

$$M_5 = (s_9 + s_{10})\left[s_8 s_{10} s_{12}(1-s_2) - s_3\gamma_3(s_{10} + s_{12})\right]$$

and

$$\gamma_1 = e_1 e_3 s_3^2$$
$$\gamma_2 = s_3\left[e_1(s_{11} + s_{12}) + e_3(s_9 + s_{10})\right]$$
$$\gamma_3 = s_{10}(s_{11} + s_{12}) + s_9 s_{12}$$
$$\gamma_4 = s_3\left[e_1 s_{12} + e_3(s_9 + s_{10})\right]$$
$$\gamma_5 = e_1 s_3 \gamma_2 + \gamma_1(s_9 + s_{10})$$
$$\gamma_6 = e_1 s_3 \gamma_3 + \gamma_2\left(s_9 + s_{10}\right)$$

Straightforward computation shows that Eq. (7) has a unique positive root namely x_1 if and only if one set of the following sets of conditions holds.

$$\left.\begin{array}{l} M_1 > 0, M_2 > 0, M_3 > 0 \text{ and } M_5 < 0 \\ OR \\ M_1 > 0, M_2 > 0, M_4 < 0 \text{ and } M_5 < 0 \\ OR \\ M_1 > 0, M_3 < 0, M_4 < 0 \text{ and } M_5 < 0 \\ OR \\ M_1 < 0, M_3 > 0, M_4 > 0 \text{ and } M_5 > 0 \\ OR \\ M_1 < 0, M_2 < 0, M_4 > 0 \text{ and } M_5 > 0 \\ OR \\ M_1 < 0, M_2 < 0, M_3 < 0 \text{ and } M_5 > 0 \end{array}\right\} \quad (8)$$

Moreover from Eq. (6g) we have $\frac{dx}{dy} = -\left(\frac{\partial g_1}{\partial y}\right)/\left(\frac{\partial g_1}{\partial x}\right)$. So, $\frac{dx}{dy} < 0$ if one set of the following sets of conditions holds.

$$\frac{\partial g_1}{\partial x} > 0, \frac{\partial g_1}{\partial y} > 0 \text{ OR } \frac{\partial g_1}{\partial x} < 0, \frac{\partial g_1}{\partial y} < 0 \quad (9)$$

Further, from (6h) we notice that, when $y \to 0$ then $x = 0$, in addition since we have $\frac{dx}{dy} = -\left(\frac{\partial g_2}{\partial y}\right)/\left(\frac{\partial g_2}{\partial x}\right)$. So, $\frac{dx}{dy} > 0$ if one set of the following sets of conditions holds.

$$\frac{\partial g_2}{\partial x} > 0, \frac{\partial g_2}{\partial y} < 0 \text{ OR } \frac{\partial g_2}{\partial x} < 0, \frac{\partial g_2}{\partial y} > 0 \quad (10)$$

Then the two isoclines (6g) and (6h) intersect at a unique positive point (x^*, y^*) in the $Int.R_+^2$ of xy-plane. Substituting the value of x^* and y^* in Eq. (6e) and (6f) yield that $z(x^*, y^*) = z^*$ and $w(x^*, y^*) = w^*$ which are positive if and only if one set of the following sets of conditions holds.

$$\left.\begin{array}{l} A > s_9 \text{ and } B > s_{11}A \\ OR \\ 0 < A < s_9 \text{ and } 0 < B < s_{11}A \\ OR \\ s_{11}A > B \end{array}\right\} \qquad (11)$$

Accordingly, the positive equilibrium point E_2 exists uniquely in the $Int.R^4_+$ if in addition to the conditions (8) - (11) the isoclinic $g_1(x,y)=0$ intersect the x-axis at the positive value namely x_1

4. The local Stability Analysis:

In this section, the local stability analyses of system (2) around each of the above equilibrium points are discussed through computing the Jacobian matrix $J(x,y,z,w)$ of the system (2) at each of them. The general Jacobian matrix of system (2) at the point (x,y,z,w) can be written

$$J(x,y,z,w)=\begin{pmatrix} f_1+x\frac{\partial f_1}{\partial x} & x\frac{\partial f_1}{\partial y} & x\frac{\partial f_1}{\partial z} & x\frac{\partial f_1}{\partial w} \\ y\frac{\partial f_2}{\partial x} & f_2+y\frac{\partial f_2}{\partial y} & y\frac{\partial f_2}{\partial z} & y\frac{\partial f_2}{\partial w} \\ z\frac{\partial f_3}{\partial x} & z\frac{\partial f_3}{\partial y} & f_3+z\frac{\partial f_3}{\partial z} & z\frac{\partial f_3}{\partial w} \\ w\frac{\partial f_4}{\partial x} & w\frac{\partial f_4}{\partial y} & w\frac{\partial f_4}{\partial z} & f_4+w\frac{\partial f_4}{\partial w} \end{pmatrix} \qquad (12)$$

where $f_i, i=1,2,3,4$ are given in system (2) and

$$\frac{\partial f_1}{\partial x}=-\left(1+\frac{s_4 y}{x^2}\right);\ \frac{\partial f_1}{\partial y}=-s_1+\frac{s_4}{x};\ \frac{\partial f_1}{\partial z}=-s_3;\ \frac{\partial f_1}{\partial w}=-s_3;$$

$$\frac{\partial f_2}{\partial x}=s_1+\frac{s_2}{y}-s_5;\ \frac{\partial f_2}{\partial y}=-\left(s_5+\frac{s_2 x}{y^2}\right);\ \frac{\partial f_2}{\partial z}=-s_6;\ \frac{\partial f_2}{\partial w}=-s_6;$$

$$\frac{\partial f_3}{\partial x}=e_1 s_3;\ \frac{\partial f_3}{\partial y}=e_2 s_6;\ \frac{\partial f_3}{\partial z}=-\frac{-s_{11}w}{z^2};\frac{\partial f_3}{\partial w}=\frac{s_{11}}{z}-s_8;$$

$$\frac{\partial f_4}{\partial x}=e_3 s_3;\ \frac{\partial f_4}{\partial y}=e_4 s_6;\ \frac{\partial f_4}{\partial z}=s_8+\frac{s_9}{w};\ \frac{\partial f_4}{\partial w}=-\frac{s_9 z}{w^2},$$

Consequently, the Jacobian matrix of system (2) at E_0 can be written as

$$J_0=J(E_0)=\begin{pmatrix} 1-s_2 & s_4 & 0 & 0 \\ s_2 & s_5-(s_4+s_7) & 0 & 0 \\ 0 & 0 & -(s_9+s_{19}) & s_{11} \\ 0 & 0 & s_9 & -(s_{11}+s_{12}) \end{pmatrix} \qquad (13)$$

Note that the characteristic equation of this Jacobian matrix is given by

$$\left[\mu^2-(1+s_5-(s_2+s_4+s_7))\mu+(s_5+s_2 s_7-(s_4+s_7+s_2 s_5))\right]$$
$$\left[\mu^2+(s_9+s_{10}+s_{11}+s_{12})\mu+(s_9 s_{12}+s_{10}(s_{11}+s_{12}))\right]=0$$

Hence, straightforward computations show that, the eigenvalues of $J(E_0)$ satisfy the following relations

$$\mu_x+\mu_y=1+s_5-(s_2+s_4+s_7) \qquad (14a)$$

$$\mu_x.\mu_y=s_5+s_2 s_7-(s_4+s_7+s_2 s_5) \qquad (14b)$$

$\mu_z + \mu_w = -(s_9 + s_{10} + s_{11} + s_{12}) < 0$ (14c)

$\mu_z.\mu_w = s_9 s_{12} + s_{10}(s_{11} + s_{12}) > 0$ (14d)

Here μ_x, μ_y, μ_z and μ_w denote to the eigenvalues in the x- direction y-direction, z-direction and w-direction, respectively. So it easy to verify that, all the eigenvalues have negative real parts and hence the equilibrium point E_0 is locally asymptotically stable in R_+^4 if and only if the following conditions hold.

$1 - s_2 + s_5 < s_4 + s_7 < (1 - s_2)s_5 + s_2 s_7$ (15)

However it is unstable otherwise.

The Jacobian matrix of system (2.2) at E_1 can be written as:

$J_1 = J(E_1) = \lfloor b_{ij} \rfloor_{4\times4}$ (16)

here:

$$b_{11} = -\left(\hat{x} + \frac{s_4 \hat{y}}{\hat{x}}\right); b_{12} = s_4 - s_1\hat{x}; b_{13} = b_{14} = -s_3\hat{x}; b_{21} = s_2 + \hat{y}(s_1 - s_5);$$

$$b_{22} = -\left(s_5\hat{y} + \frac{s_2\hat{x}}{\hat{y}}\right); b_{23} = b_{24} = -s_6\hat{y}; b_{31} = b_{32} = 0; b_{33} = e_1 s_3 \hat{x} + e_2 s_6 \hat{y} - (s_9 + s_{10});$$

$$b_{34} = s_{11}; b_{41} = b_{42} = 0; b_{34} = s_9; b_{44} = e_3 s_3 \hat{x} + e_4 s_6 \hat{y} - (s_{11} + s_{12}).$$

Clearly the characteristic equation of J_1 can be written as:

$$\lfloor \lambda^2 - (b_{11} + b_{22})\lambda + b_{11}b_{22} - b_{12}b_{21} \rfloor \lfloor \lambda^2 - (b_{33} + b_{44})\lambda + b_{33}b_{44} - s_9 s_{11} \rfloor = 0$$

Therefore, the eigenvalus of J_1 satisfy the following relations:

$\lambda_x + \lambda_y = b_{11} + b_{22} < 0$ (17a)

$\lambda_x.\lambda_y = b_{11}b_{22} - b_{12}b_{21}$ (17b)

$\lambda_z + \lambda_w = b_{33} + b_{44}$ (17c)

$\lambda_z.\lambda_w = b_{33}b_{44} - s_9 s_{11}$ (17d)

where $\lambda_x, \lambda_y, \lambda_z$ and λ_w denote to the eigenvalues in the x- direction y-direction, z-direction and w-direction, respectively. So it easy to verify that, all the eigenvalues have negative real part and then the equilibrium point E_1 is locally asymptotically stable in R_+^4 if and only if the following conditions hold.

$$\left.\begin{array}{l} b_{12} < 0 \text{ and } b_{21} > 0 \\ \text{OR} \\ b_{12} > 0 \text{ and } b_{21} < 0 \end{array}\right\} \quad (18a)$$

$b_{33} < 0$ and $b_{44} < 0$ (18b)

$b_{33}b_{44} > s_9 s_{11}$ (18c)

However it is unstable point otherwise.

In the following theorem, the local stability conditions of the positive equilibrium point E_2 are established

Theorem (2) Assume that $E_2 = (x^*, y^*, z^* w^*)$ exists in the $Int\, R_+^4$ and the following condition are satisfied

$$x^* + \frac{s_4 y^*}{x^*} > \left|(s_1 - s_5)y^* + s_2\right| + s_3(e_1 z^* + e_3 w^*) \quad (19a)$$

$$s_5 y^* + \frac{s_2 x^*}{y^*} > \left|s_4 - s_1 x^*\right| + s_6(e_2 z^* + e_4 w^*) \quad (19b)$$

$$\frac{s_{11} w^*}{z^*} > s_3 x^* + s_6 y^* + s_9 + s_8 w^* \quad (19c)$$

$$\frac{s_9 z^*}{w^*} > s_3 x^* + s_6 y^* + \left|s_{11} - s_8 z^*\right| \quad (19d)$$

Then the positive equilibrium point E_2 is locally asymptotically stable in the $Int\, R_+^4$

Proof: The Jacobian matrix of system (2) at the positive equilibrium point E_2 can be written:

$$J_2 = J(E_2) = \lfloor a_{ij} \rfloor_{4\times 4} \quad (20)$$

where:

$$a_{11} = -\left(x^* + \frac{s_4 y^*}{x^*}\right) < 0;\ a_{12} = s_4 - s_1 x^*;\ a_{13} = a_{14} = -s_3 x^*;$$

$$a_{21} = (s_1 - s_5) y^* + s_2;\ a_{22} = -\left(s_5 y^* + \frac{s_2 x^*}{y^*}\right) < 0;\ a_{23} = a_{24} = -s_6 y^* < 0$$

$$a_{31} = e_1 s_3 z^* > 0;\ a_{32} = e_2 s_6 z^*;\ a_{33} = \frac{-s_{11} w^*}{z^*} < 0;\ a_{34} = s_{11} - s_8 z^*;$$

$$a_{41} = e_3 s_3 w^* > 0;\ a_{42} = e_4 s_6 w^* > 0;\ a_{43} = s_9 + s_8 w^* > 0;\ a_{44} = -\frac{s_9 z^*}{w^*} < 0$$

According to Gersgorin theorem the poof is follows if and only if $\left|a_{ii}\right| > \sum_{\substack{i=1 \\ i\neq j}}^{4} \left|a_{ij}\right|$

Then all the eigenvalues of J_2 exists in the region

$$\xi = \cup \left\{ U^* \in C : \left|U^* - a_{ii}\right| < \sum_{\substack{i=1 \\ i\neq j}}^{4} \left|a_{ij}\right| \right\}$$

Therefore according to the given conditions (19) (a-d) all the eigenvalues of J_2 exists in the left half plane and hence E_2 is locally asymptotically stable.■

5. Global stability analysis of system (2)

In this section the global stability for the equilibrium points of system (2) is investigated by using the Lyapunov method as shown in the following theorems.

Theorem (3) Assume that the vanishing equilibrium point E_0 of system (2) is locally asymptotically stable in the R_+^4 with

$$\text{Max}\left\{\frac{e_4 s_2}{s_2 - 1}, e_1\right\} < e_3 < \text{Min}\left\{\frac{e_4(s_4 + s_7 - s_5)}{s_4}, \frac{e_1 e_4}{e_2}, \frac{e_1(s_{11} + s_{12})}{s_{11}}\right\} \quad (21a)$$

$$\left[s_1 + \frac{e_4}{e_2}(s_5 - s_1)\right]^2 < \frac{4 e_4 s_5}{e_3} \quad (21b)$$

Then it is globally asymptotically stable in the R_+^4.

Proof: Consider the following function:

$$V_0(x,y,z,w) = c_1 x + c_2 y + c_3 z + c_4 w$$

Clearly $V_0 : R_+^4 \to R$ is C^1 positive definite function, where c_i $(i = 1,2,3,4)$ are positive constants to be determined. Now since the derivative of V_0 along the trajectory of the system (2) can be written as

$$\frac{dV_0}{dt} = c_1 \frac{dx}{dt} + c_2 \frac{dy}{dt} + c_3 \frac{dz}{dt} + c_4 \frac{dw}{dt}$$

$$\begin{aligned}\frac{dV_0}{dt} = &-c_1 x^2 - c_2 s_5 y^2 - (c_1(s_2 - 1) - c_2 s_2)x + (-c_1 s_1 - c_2(s_5 - s_1))xy \\ &-(c_1 - c_3 e_1)s_3 xz - (c_1 - c_4 e_3)s_3 xw - (c_2(s_4 + s_7) - (c_1 s_4 + c_2 s_5))y \\ &-(c_2 - c_3 e_2)s_6 yz - (c_2 - c_4 e_4)s_6 yw - (c_3 - c_4)s_8 zw \\ &-(c_3(s_9 + s_{10}) - c_4 s_9)z - (c_4(s_{11} + s_{12}) - c_3 s_{11})w\end{aligned}$$

So by choosing the positive constants as:

$c_1 = 1,\ c_2 = \frac{e_4}{e_3},\ c_3 = \frac{1}{e_1},\ c_4 = \frac{1}{e_3}$

It is obtain that:

$$\begin{aligned}\frac{dV_0}{dt} = &-x^2 - \frac{e_4 s_5}{e_3} y^2 + \left(-s_1 + \frac{e_4}{e_3}(s_1 - s_5)\right)xy - \left(s_2 - \left(1 + \frac{e_4 s_2}{e_3}\right)\right)x \\ &-\left(\frac{e_4}{e_3}(s_4 + s_7) - \left(s_4 + \frac{e_4 s_5}{e_3}\right)\right)y - \left(\frac{e_4}{e_3} - \frac{e_2}{e_1}\right)s_6 yz - \left(\frac{1}{e_1} - \frac{1}{e_3}\right)s_8 zw \\ &-\left(\frac{1}{e_1}(s_9 + s_{10}) - \frac{s_9}{e_3}\right)z - \left(\frac{1}{e_3}(s_{11} + s_{12}) - \frac{s_{11}}{e_1}\right)w\end{aligned}$$

Therefore, according to condition (21a) - (21b) we obtain that:

$$\begin{aligned}\frac{dV_0}{dt} \le &-\left(x - \sqrt{\frac{e_4 s_5}{e_3}} y\right)^2 - \left(s_2 - \left(1 + \frac{e_4 s_2}{e_3}\right)\right)x \\ &-\left(\frac{e_4}{e_3}(s_4 + s_7) - \left(s_4 + \frac{e_4 s_5}{e_3}\right)\right)y \\ &-\left(\frac{1}{e_1}(s_9 + s_{10}) - \frac{s_9}{e_3}\right)z - \left(\frac{1}{e_3}(s_{11} + s_{12}) - \frac{s_{11}}{e_1}\right)w\end{aligned}$$

Then $\frac{dV_0}{dt} < 0$, hence V_0 is strictly Lyapunov function. Therefore E_0 is globally asymptotically stable in the R_+^4 .■

<u>Theorem (4)</u> Assume that the predator free equilibrium point E_1 of system (2) is locally asymptotically stable in the $Int.R_+^4$, and the following conditions are satisfied:

$$\left[\frac{s_4}{x} - s_1 - \frac{e_2}{e_1}\left(s_5 - \left(s_1 + \frac{s_2}{y}\right)\right)\right]^2 < \frac{4e_2}{e_1}\left(1 + \frac{s_4 \hat{y}}{x\hat{x}}\right)\left(s_5 + \frac{s_2 \hat{x}}{y\hat{y}}\right) \qquad (22a)$$

$$s_9 + s_{10} > e_1 s_3 \hat{x} + e_2 s_6 \hat{y} \qquad (22b)$$

$$e_3 > \max\left\{\frac{e_1 e_4}{e_2}, e_1, \frac{e_1 s_{11}}{s_9 + s_{10} - (e_1 s_3 \hat{x} + e_2 s_6 \hat{y})}\right\} \qquad (22c)$$

$$e_3 < \frac{e_1(s_{11}+s_{12})}{e_1 s_3 \hat{x} + e_2 s_6 \hat{y} + s_{11}} \tag{22d}$$

Then E_1 is globally asymptotically stable in the R_+^4.

Proof: Consider the following function:

$$V_1(x,y,z,w) = \hat{c}_1\left(x - \hat{x} - \hat{x}\ln\frac{x}{\hat{x}}\right) + \hat{c}_2\left(y - \hat{y} - \hat{y}\ln\frac{y}{\hat{y}}\right) + \hat{c}_3 z + \hat{c}_4 w$$

Clearly $V_1 : R_+^4 \to R$ is C^1 positive definite function, where $\hat{c}_i$ $(i = 1,2,3,4)$ are positive constants to be determined. Now since the derivative of V_1 along the trajectory of the system (2) can be written as:

$$\begin{aligned}\frac{dV_1}{dt} = &-\hat{c}_1\left(1 + \frac{s_4\hat{y}}{x\hat{x}}\right)(x-\hat{x})^2 - \hat{c}_2\left(s_5 + \frac{s_2\hat{x}}{y\hat{y}}\right)(y-\hat{y})^2 \\ &+ \left(-\hat{c}_1\left(s_1 - \frac{s_4}{x}\right) - \hat{c}_2\left(s_5 - \left(s_1 + \frac{s_2}{y}\right)\right)\right)(x-\hat{x})(y-\hat{y}) \\ &- (\hat{c}_1 - e_1\hat{c}_3)s_3 zx - (\hat{c}_1 - e_3\hat{c}_4)s_3 xw - (\hat{c}_2 - e_2\hat{c}_3)s_6 yz \\ &- (\hat{c}_2 - e_4\hat{c}_4)s_6 yw - (\hat{c}_3 - \hat{c}_4)s_8 zw \\ &- (\hat{c}_3(s_9 + s_{10}) - (\hat{c}_1 s_3\hat{x} + \hat{c}_2 s_6\hat{y} + \hat{c}_4 s_9))z \\ &- (\hat{c}_4(s_{11} + s_{12}) - (\hat{c}_1 s_3\hat{x} + \hat{c}_2 s_6\hat{y} + \hat{c}_3 s_{11}))w\end{aligned}$$

So by choosing the positive constants as: $\hat{c}_1 = 1,\ \hat{c}_2 = \frac{e_2}{e_1}, \hat{c}_3 = \frac{1}{e_1},\ \hat{c}_4 = \frac{1}{e_3}$.

Then we get:

$$\begin{aligned}\frac{dV_1}{dt} = &-d_{11}(x-\hat{x})^2 - d_{22}(y-\hat{y})^2 + d_{12}(x-\hat{x})(y-\hat{y}) \\ &- d_{24}yw - d_{34}zw - \hat{B}_1 z - \hat{B}_2 w\end{aligned}$$

here:

$$d_{11} = \left(1 + \frac{s_4\hat{y}}{x\hat{x}}\right) > 0; d_{22} = \frac{e_2}{e_1}\left(s_5 + \frac{s_2\hat{x}}{y\hat{y}}\right); d_{12} = \frac{s_4}{x} - s_1 - \frac{e_2}{e_1}\left(s_5 - \left(s_1 + \frac{s_2}{y}\right)\right);$$

$d_{24} = s_6\left(\frac{e_2}{e_1} - \frac{e_4}{e_3}\right); d_{34} = s_8\left(\frac{1}{e_1} - \frac{1}{e_3}\right); \hat{B}_1 = \left(\frac{s_9 + s_{10}}{e_1} - \left(s_3\hat{x} + \frac{e_2 s_6\hat{y}}{e_1} + \frac{s_9}{e_3}\right)\right)$; Now, since the

$$\hat{B}_2 = \left(\frac{s_{11} + s_{12}}{e_3} - \left(s_3\hat{x} + \frac{e_2 s_6\hat{y} + s_{11}}{e_1}\right)\right)$$

conditions (22b) - (22c) guarantee that d_{24}, d_{34} *and* $\hat{B}_1$ are positive while condition (22d) guarantees that $\hat{B}_2$ is positive. Therefore, by using the given conditions. We obtain that:

$$\frac{dV_1}{dt} \le -\left[\sqrt{d_{11}}(x-\hat{x}) - \sqrt{d_{22}}(y-\hat{y})\right]^2 - \hat{B}_1 z - \hat{B}_2 w$$

Then $\frac{dV_1}{dt} < 0$ under the given conditions and then V_1 is strictly Lyapunov function. Therefore E_1 is globally asymptotically stable in the R_+^4 .■

In the following theorem, the conditions of the globally asymptotically stable of the positive equilibrium point E_2 are established

Theorem (5) Assume that the positive equilibrium point E_2 of system (2) is locally asymptotically stable in the $Int.R_+^4$ with

$$q_{12}^2 < 2q_{11}q_{22} \quad (23a)$$

$$q_{24}^2 < q_{22}q_{44} \quad (23b)$$

$$q_{34}^2 < 2q_{33}q_{44} \quad (23c)$$

where:

$$q_{11} = 1 + \frac{s_4 y^*}{xx^*}; q_{12} = -s_1 + \frac{s_4}{x} - \frac{e_2}{e_1}\left(s_5 - s_1 - \frac{s_2}{y}\right); q_{22} = \frac{e_2}{e_1}\left(s_5 + \frac{s_2 x^*}{yy^*}\right);$$

$$q_{24} = -s_6\left(\frac{e_4}{e_3} - \frac{e_2}{e_1}\right); q_{34} = -\frac{s_8}{e_1} + \frac{s_{11}}{e_1 z} + \frac{s_8}{e_3} + \frac{s_9}{e_3 w}; q_{33} = \frac{s_{11} w^*}{e_1 zz^*}; q_{44} = \frac{s_9 z^*}{e_3 ww^*}$$

Then E_2 is globally asymptotically stable in the $Int.R_+^4$.

Proof: Consider the following function:

$$V_2(x,y,z,w) = c_1^*\left(x - x^* - x^* \ln\frac{x}{x^*}\right) + c_2^*\left(y - y^* - y^* \ln\frac{y}{y^*}\right)$$
$$+ c_3^*\left(z - z^* - z^* \ln\frac{z}{z^*}\right) + c_4^*\left(w - w^* - w^* \ln\frac{w}{w^*}\right)$$

Clearly $V_2 : R_+{}^4 \to R$ is C^1 positive definite function, where $c_i^* (i = 1,2,3,4)$ are positive constants to be determined. Now since the derivative of V_2 a long the trajectory of the system (2) can be written as:

$$\frac{dV_2}{dt} = -c_1{}^*\left(1 + \frac{s_4 y^*}{xx^*}\right)(x - x^*)^2$$
$$+ \left(-c_1^* s_1 + \frac{c_1^* s_4}{x} - c_2^* s_5 + c_2^* s_1 + \frac{c_2^* s_2}{y}\right)(x - x^*)(y - y^*)$$
$$- c_2^*\left(s_5 + \frac{s_2 x}{yy^*}\right)(y - y^*)^2 + s_6(c_3^* e_2 - c_2^*)(y - y^*)(z - z^*)$$
$$+ s_6(c_4^* e_4 - c_2^*)(y - y^*)(w - w^*) + s_3(c_3^* e_1 - c_1^*)(x - x^*)(z - z^*)$$
$$+ \left(c_4^* s_8 + \frac{c_3^* s_{11}}{z} + \frac{c_4^* s_9}{w} - c_3^* s_8\right)(z - z^*)(w - w^*) - \frac{c_3^* s_{11} w^*}{zz^*}(z - z^*)^2$$
$$+ s_3(c_4^* e_3 - c_1)(x - x^*)(w - w^*) - \frac{c_4^* s_9 z^*}{ww^*}(w - w^*)^2$$

By choosing the positive constants as: $c_1^* = 1; c_2^* = \frac{e_2}{e_1}; c_3^* = \frac{1}{e_1}; c_4^* = \frac{1}{e_3}$, then we get

$$\frac{dV_2}{dt} = -q_{11}(x - x^*)^2 + q_{12}(x - x^*)(y - y^*) - q_{22}(y - y^*)^2 - q_{33}(z - z*)^2$$
$$+ q_{24}(y - y^*)(w - w^*) + q_{34}(z - z^*)(w - w^*) - q_{44}(w - w^*)^2$$

Therefore, according to the conditions (23) (a-c) we obtain that:

$$\frac{dV_2}{dt} \le -\left[\sqrt{q_{11}}(x-x^*) - \sqrt{\frac{q_{22}}{2}}(y-y^*)\right]^2$$
$$-\left[\sqrt{\frac{q_{22}}{2}}(y-y^*) - \sqrt{\frac{q_{44}}{2}}(w-w^*)\right]^2$$
$$-\left[\sqrt{q_{33}}(z-z^*) - \sqrt{\frac{q_{44}}{2}}(w-w^*)\right]^2$$

So, $\frac{dV_2}{dt} < 0$ and then V_2 is strictly Lyapunov function. Therefore is globally asymptotically stable in the $Int\ R_+^4$ ■

5. Numerical analysis of system (2)

In this section the dynamical behavior of system (2) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions (19) (a-d) of the positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point as shown in following figure.

$$\begin{aligned} &s_1 = 0.2,\ s_2 = 0.1,\ s_3 = 0.75,\ s_4 = 0.2,\ s_5 = 1,\ s_6 = 1, s_7 = 0.05,\ s_8 = 0.2, \\ &s_9 = 0.1,\ s_{10} = 0.1, s_{11} = 0.2,\ s_{12} = 0.15,\ e_1 = 0.5, e_2 = 0.7,\ e_3 = 0.4,\ e_4 = 0 \end{aligned} \quad (24)$$

Note that, in Figure-2, we will use that (—) to describe the trajectory of system (2) that started at (0.0, 0.8, 0.7, 0.6, 0.5) and (…..) to describe the trajectory of system (2) that started at (0.0, 0.2, 0.5, 0.9, 0.1).

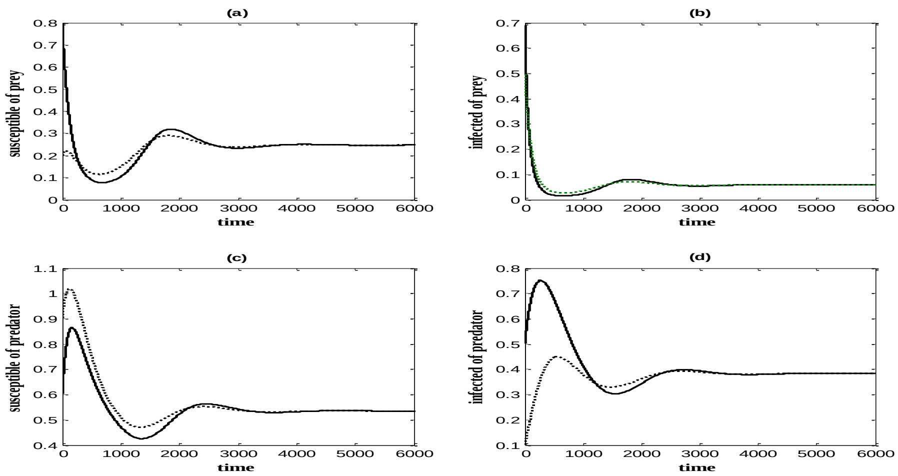

Figure 2- Time series of the solution of system (2) (a) trajectories of x as a function of time, (b) trajectories of y as a function of time, (c) trajectories of z as a function of time,(d) trajectories of w as a function of time.

Clearly, figure-2, shows that the solution of system (2) approaches asymptotically to the positive equilibrium point $E^* = (x^*, y^*, z^*, w^*)$ starting from two different initial points and this is confirming our obtained analytical results.

Now in order to discuss the effect of the parameters values of system (2) on the dynamical behavior of the system, the system is solved numerically for the data given in Eq. (24) with varying one parameter each time. It is observed that varying the parameters values s_i; $i = 2,4,5,8,9,10,11,12$

and $e_i; i=1,2,3,4$ do not have any effect on the dynamical behavior of system (2) and the system still approaches to positive equilibrium point. However, varying the attack rates of susceptible and infected predator s_3 and s_6, respectively keeping other parameters fixed as given in equation (24), leads to extinction in predator species as shown in figure-3, for the parameters given by Eq. (24) with $s_3=0.20, s_6=0.075$, .

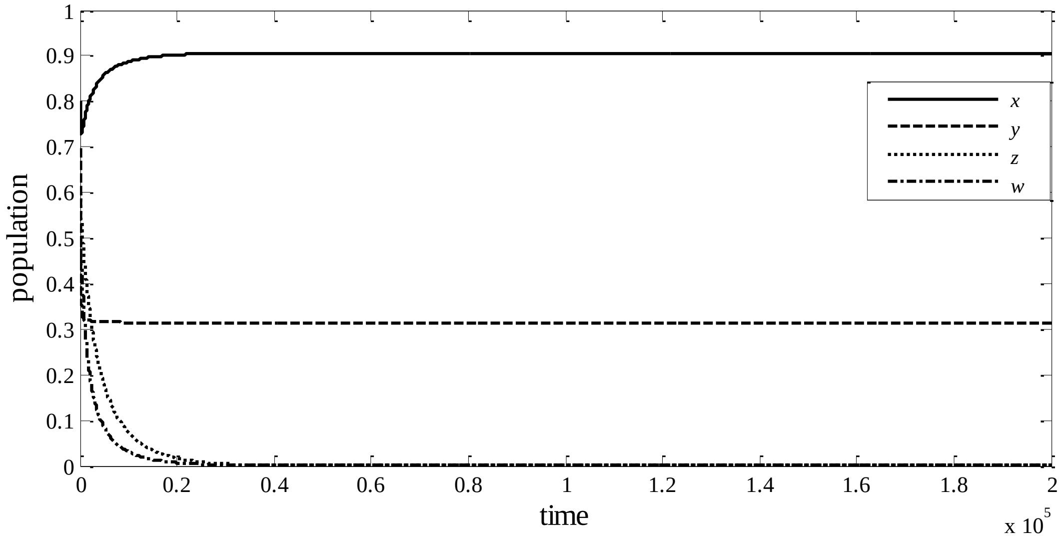

Figure 3- Time series of the solution of system (2)

From the above figure it is clear that as the attack rates of susceptible and infected predator decrease the trajectory of the system (2) approaches asymptotically to the predator free equilibrium point $E_2=(\hat{x},\hat{y}.0,0)$.

Finally, the dynamical behavior at the vanishing equilibrium point E_0 is investigated by choosing $s_2=1$ and $s_7=2$ and keeping other parameters fixed as given in Eq. (24), and then the solution of system (2) is drawn in figure- 4.

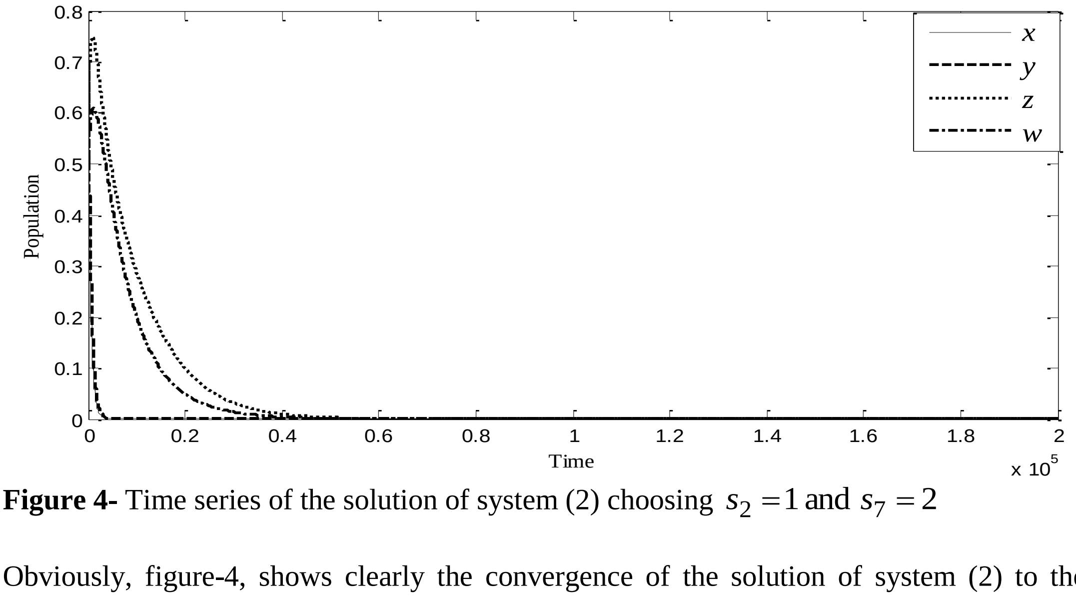

Figure 4- Time series of the solution of system (2) choosing $s_2=1$ and $s_7=2$

Obviously, figure-4, shows clearly the convergence of the solution of system (2) to the vanishing equilibrium point E_0=(0,0,0,0) when the parameters increase up to a specific values. Clearly the used values in figure-4, satisfy the stability conditions of the vanishing equilibrium point.

6. Conclusions and Discussion

In this paper, we proposed and analyzed an eco-epidemiological model that described the dynamical behavior of prey-predator model with Lotka-Volterra type of functional response and linear incidence rate for the disease in prey and predator respectively. It is assumed that the disease doesn't spread outside the specific species (prey or predator), while the disease may be transmitted within the individuals of prey and within the individuals of predator by two ways: from an external source as well as through contact between the individuals of prey and those of predator respectively. The model included four non-linear autonomous differential equations that describe the dynamics of four different population namely susceptible prey x, infected prey y, susceptible predator z, infected predator w. The boundedness of the system (2) has been discussed. The dynamical behavior of system (2) has been investigated locally as well as globally. Further , it is observed that the vanishing equilibrium point (E_0) always exist , and it is locally asymptotically stable point if and only if conditions (15) hold, in addition to that it is globally if the conditions (21a) - (21b) hold. The predator free equilibrium point (E_1) exists under the conditions (5a)-(5b), and it is locally asymptotically stable point if and only if the conditions (18) (a-c) hold as well as it is globally if the conditions (22) (a-d) hold. The positive equilibrium point of system (2) exists provided that the conditions (8) - (11) are hold and the isoclinic $g_1(x, y) = 0$ intersect the *x*-axis at the positive value namely x_1. It is locally asymptotically stable point if and only if conditions (19) (a-d) hold, in addition it is globally if the conditions (23) (a-c) hold. To understand the effect of varying each parameter on the global dynamics of system (2) and to confirm our above analytical results, system (2) has been solved numerically and the following results are obtained:

1. The system (2) dose not have periodic dynamic.
2. For the set of hypothetical parameters values given Eq. (24), system (2) approaches asymptotically to a globally asymptotically stable point $E_2 = (x^*, y^*, z^*, w^*)$.
3. As the attack rates parameters s_3 and s_6 for susceptible and infected predator in system (2) decreases, then the solution of the system (2) still stable and approaches asymptotically to the predator free equilibrium point $E_2 = (\hat{x}, \hat{y}.0,0)$. Clearly decreasing the value of s_3 and s_6 leads to increasing in the value of (susceptible and infected) prey population.

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