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ET-Coessential and ET-Coclosed submodules

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Abstract
Let M be an R-module, where R be a commutative ring with identity. In this paper, we defined a new kind of submodules, namely ET-coessential and ET-Coclosed submodules of M. Let T be a submodule of M. Let $K \leq H \leq M$, K is called ET-Coessential of H in M ($K \subseteq_{\mathbf{ET.ce}} H$), if $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$. A submodule H is called ET- coclosed in M of H has no proper coessential submodule in M, we denote by ($K \subseteq_{\mathbf{ET.cc}} H$) , that is, $K \subseteq_{\mathbf{ET.ce}} H$ implies that $K = H$. In our work, we introduce some properties of ET-coessential and ET-coclosed submodules of M.

Keywords: ET-small submodule, ET-coessential submodule, ET- coclosed submodule, .

حول المقاسات الجزئية الجوهرية الرديف من ET والمقاسات الجزئية المغلقة الرديف من النمط ET النمط

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الخلاصة
ليكن M مقاس احادي محايد وليكن Rحلقة إبداليه ذات عنصر محايد . في هذه الورقة نستعرض نوعين من المقاسات الجزئية تدعى الاولى المقاسات الجزئية الجوهرية الرديفة من النمط ET– والثانية المقاسات الجزئية المغلقة الرديفة من النمط–ET ليكن T مقاس جزئي من المقاس M و $K \leq H \leq M$, المقاس الجزئي H يدعى مقاس جوهري رديف من النمط–ET لـH في M وبرمز له M $K \subseteq_{\mathbf{ET.ce}}$ اذا كان $\frac{M}{K}$ $\ll_{E(\frac{T+K}{K})}$ $\frac{H}{K}$ يدعى مقاس جزئي مغلق رديف H والمقاس الجزئي ل Hفي M.. اذا لم يكن له مقاس جزئي جوهري رديف في M ويرمز لهH $K \subseteq_{\mathbf{ET.cc}}$ في هذا البحث سوف نقوم بدراسة وتطوير خواص هذه المقاسات الجزئية

1. Introduction

Let R is a commutative ring with identity and M is an arbitrary R-module. A proper submodule H of M is called small ($H \ll M$), if for all submodule K of M ($K \leq M$) such that $H+K=M$ implies that $K=M$[1]. A submodule H of M is essential ($H \leq_e M$) if for all $B \leq M$ such that $H \cap B = 0$, then $B = 0$ [2] . A submodule H of M is closed ($H \leq_c M$) if H has no proper essential extensions inside M . that is, if $H \leq_e K \leq_e M$ then H=K [3]. A submodule H of M is called an essential- small ($H \ll_e M$) submodule of M, if for all essential submodule B of M such that $M = H + B$ implies that $B = M$ [4].

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Let T ≤ M, a submodule H of M is said to be "T-small submodule of M", if for all K ≤ M such that ⊆ H+K , then T ⊆ K [5]. In a previous work [6], the authors defined ET-small submodule of M. Let T ≤ M and A submodule H of M is "ET-small submodule of M", if for all K $\leq_e$ M such that T ⊆ H+K, then T ⊆ K, clearly every T-small submodule of M is ET-small submodule of M but the converse is not true. We give in lemma 1 and lemma 2 some properties of ET-small submodule of M.

Lemma1 [6]:

1- Let T, A and B be submodules of M such that T ≤ B and A ≤ B ≤ M and $B \ll_e M$. If $A \ll_{ET} M$, then $A \ll_{ET} B$.

2- Let M be an R-module with submodules A ≤ B ≤ M such that T≤ B. If $A \ll_{ET} B$, then $A \ll_{ET} M$.

3- Let M be an R-module and let T, A and B be submodules of M,then $A \ll_{ET} M$ and $B \ll_{ET} M$ if and only if $A+B \ll_{ET} M$.

Lemma2 [6]:

1- Let M_1 and M_2 be any R-modules and $f : M_1 \to M_2$ be a homomorphism. If T and H are submodules of M such that $H \ll_{ET} M_1$,then $f(H) \ll_{Ef(T)} M_2$.

2- Let M be an R-module and let T, H and N be submodules of M such that H ≤ N≤ M and H ≤ T and $H \leq_c M$, if $\frac{N}{H} \ll_{E\frac{T}{H}} \frac{M}{H}$ then $N \ll_{ET} M$.

T-coessential submodule was given the if K, H submodule of M such that $K \subseteq H$, K is T-coessential of H ($K \subseteq_{T.ce} H$) if $\frac{H}{K} \ll_{\frac{T+K}{K}} \frac{M}{K}$ [7]. In this work, we define the ET-coessential submodule and ET- coclosed submodule of M and we give some properties of this type of submodules. Let T be a submodule of M. Let K ≤ H ≤ M, K is called ET-coessential in M if $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$. Every T-coessential submodule in M is an ET- coessential submodule in M but the converse is not true. We also give other properties. A submodule H is called ET- coclosed in M if H has no proper submodule K for which K⊂ H is a coessential submodule in M, we denote by ($K \subseteq_{\mathbf{ET.cc}} H$), that is, $K \subseteq_{\mathbf{ET.ce}} H$ implies that K = H. Also we give some properties of ET- coclosed submodule of M.

2. ET-coessential submodules.

Definition2.1*:* Let T be a submodule of a module M, let K and H be submodules of M, such that K ⊆ H is called ET-coessential in M if $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$. We denote this by ($K \subseteq_{\mathbf{ET.ce}} H$).

Remarks and Examples 2.2:

1) Consider Z_6 as a Z-module. Let T=$\{\bar{0},\bar{3}\}$, K=$\{\bar{0}\}$ and H=$\{\bar{0},\bar{2},\bar{4}\}$, then $K \subseteq_{\mathbf{ET.ce}} H$ in Z_6 ,where $\frac{\{\bar{0},\bar{2},\bar{4}\}}{\{\bar{0}\}} = \{\bar{0},\bar{2},\bar{4}\} \ll_{ET} Z_6 \cong \frac{Z6}{\{\bar{0}\}}$; see the cited adopted reference [6].

2) Consider Z as a Z-module. Let T=2Z, K={0}, H=3Z , thus $K \not\subseteq_{\mathbf{ET.ce}} H$ in Z since $\frac{H}{K} = \frac{3Z}{\{0\}} = 3Z$ is not E($\frac{2Z+\{0\}}{\{0\}}$)-small submodule of $\frac{Z}{\{0\}}$ = Z, since 2Z⊆3Z+5Z where 5Z $\leq_e$Z, but 2Z⊈5Z, therefore K $\not\subseteq_{ET.ce}$ H.

3) Let M be an R-module and let T, K and H be submodules of M such that K⊆ H. Every T-coessential submodule in M is an ET- coessential submodule in M but the converseis not true.In general, for example, consider Z_4 as a Z-module. If T=$\{\bar{0},\bar{2}\}$, K=$\{\bar{0}\}$ and H=$\{\bar{0},\bar{2}\}$, then K is not a T-coessential submodule of H in Z_4, where $\frac{\{\bar{0},\bar{2}\}}{\{\bar{0}\}} \cong \{\bar{0},\bar{2}\}$ and $Z_4 \cong \frac{Z4}{\{\bar{0}\}}$. But $\{\bar{0},\bar{2}\}$ is not a T-small submodule of Z_4 , by [5], So K is not a T- coessential submodule of H in Z_4. But K is a ET- coessential submodule of H in Z_4, since the $\{\bar{0},\bar{2}\}$and Z_4 are only essential submodules in Z_4 then T⊆$\{\bar{0},\bar{2}\}$and T⊆ Z_4 , so $\{\bar{0},\bar{2}\} \ll_{ET} Z_4$ implies that $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{Z4}{K}$. Hence $K \subseteq_{\mathbf{ET.ce}} H$.

4) Let M be an R - module and let T, K and H be submodules of M such that K⊆ H. If T = 0, then K is a ET- coessential submodule of H in M. Since let $\frac{T+K}{K} \subseteq \frac{H}{K} + \frac{X}{K}$, $\forall \frac{X}{K} \leq_e \frac{M}{K}$, but T = 0, so$\frac{T+K}{K} = \frac{0+K}{K}$ =K ⊆ $\frac{X}{K}$ then $\frac{T+K}{K} \subseteq \frac{X}{K}$. Thus $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$, hence $K \subseteq_{ET.ce} H$.

5) Let M be an uniform R-module then every ET-coessential submodule of M is a T-coessential submodule of M.

Proposition 2.3: Let M be an R-module and let T, K and H be submodules of M such that K⊆ H, then $K \ll_{ET} M$ if $0 \subseteq_{ET.ce} K$.

Proof: Let $K \ll_{ET} M$. Then $\frac{K}{0} \ll_{\frac{T+0}{0}} \frac{M}{0}$. Thus 0 is a T- coessential submodule of H in M.

Conversely, let $0 \subseteq_{ET.ce} K$ in M. To show that $K \ll_{ET} M$. Let $X \leq_e M$ such that $T \subseteq K + X$, then $\frac{T+0}{0} \subseteq \frac{K+X}{0} = \frac{K}{0} + \frac{X}{0}$. Since $0 \subseteq_{ET.ce} K$, then $\frac{K}{0} \ll_{E(\frac{T+0}{0})} \frac{M}{0}$ and hence $\frac{T+0}{0} \subseteq \frac{X}{0}$. Therefore T⊆X . Thus $K \ll_{ET} M$.

The following proposition gives a characteristic of an ET- coessential submodule of M.

Proposition 2.4: Let T be a submodule of a module M and let K and H be submodules of M such that K⊆ H. Then $K \subseteq_{ET.ce} H$ if and only if T⊆H+X implies that T⊆K+X, for every essential submodule X of M.

Proof: Let $K \subseteq_{ET.ce} H$. ∀ $X \leq_e M$ such that T⊆ H+X, then $\frac{T+K}{K} \subseteq \frac{H+X}{K} = \frac{H}{K} + \frac{X+K}{K}$. Since $K \subseteq_{ET.ce} H$, then $\frac{T+K}{K} \subseteq \frac{X+K}{K}$ and hence T⊆T+K⊆X+K.

The converse, to show that $K \subseteq_{ET.ce} H$. Let $\frac{T+K}{K} \subseteq \frac{H}{K} + \frac{X}{K}$, ∀ $\frac{X}{K} \leq_e \frac{M}{K}$, then $\frac{T+K}{K} \subseteq \frac{H}{K} + \frac{X}{K} = \frac{H+X}{K}$, then T⊆T+K⊆H+X. By our assumption, then T⊆X+K. Hence $\frac{T+K}{K} \subseteq \frac{X+K}{K} = \frac{X}{K}$. And $K \subseteq_{ET.ce} H$.

Proposition 2.5: Let T be a submodule of a module M and let K, H and L be submodules of M such that K⊆H⊆L⊆M. Then $H \subseteq_{ET.ce} L$ in M if and only if $\frac{H}{K} \subseteq_{E(\frac{T+K}{K}).ce} \frac{L}{K}$ in $\frac{M}{K}$.

Proof: Let $H \subseteq_{ET.ce} L$ in M, then $\frac{L}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$. Since $\frac{L}{H} \cong \frac{L/K}{H/K}$ $\frac{T+H}{H} \cong \frac{(T+H)/K}{H/K}$ and $\frac{M}{H} \cong \frac{M/K}{H/K}$, by the third isomorphism theorem. Then $\frac{L/K}{H/K} \ll_{E(\frac{(T+H)/K}{H/K})} \frac{M/K}{H/K}$. Thus $\frac{H}{K} \subseteq_{E(\frac{T+K}{K}).ce} \frac{L}{K}$ in $\frac{M}{K}$.

Conversely, suppose that $\frac{H}{K} \subseteq_{E(\frac{T+K}{K}).ce} \frac{L}{K}$ in $\frac{M}{K}$, then $\frac{L/K}{H/K} \ll_{E(\frac{(T+H)/K}{H/K})} \frac{M/K}{H/K}$ .Since $\frac{L}{H} \cong \frac{L/K}{H/K}$, $\frac{T+H}{H} \cong \frac{(T+H)/K}{H/K}$ and $\frac{M}{H} \cong \frac{M/K}{H/K}$, by the third isomorphism theorem . Then $\frac{L}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$.

Thus $H \subseteq_{ET.ce} L$ in M.

Proposition 2.6: Let T be a submodule of a module M, let K , H and L be submodules of M such that K⊆H⊆L⊆M and $H \leq_c M$. Then $K \subseteq_{ET.ce} L$ in M if and only if $K \subseteq_{ET.ce} H$ in M and $H \subseteq_{ET.ce} L$ in M .

Proof: Let $K \subseteq_{ET.ce} L$ in M, then $\frac{L}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$. Since $\frac{H}{K} \subseteq \frac{L}{K} \subseteq \frac{M}{K}$, then $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ [6], hence $K \subseteq_{ET.ce} H$ in M. Now we define $f: \frac{M}{K} \to \frac{M}{H}$ by $f(m+K) = m+H$, ∀m∈M . Since ∀$m+H \in \frac{M}{H}$, ∃ $m+K \in \frac{M}{K}$, suth that $f(m+K) = m+H$ hence f is an epimorphism. Since $\frac{L}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ in M, hence $f(\frac{L}{K}) = \frac{L}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$ [6]. Hence $H \subseteq_{ET.ce} L$ in M.

Conversely, suppose that $K \subseteq_{ET.ce} H$ in M and $H \subseteq_{ET.ce} L$ in M, then $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ and $\frac{L}{H} \ll_{E(\frac{T+H}{H})} \frac{M}{H}$. To prove $K \subseteq_{ET.ce} L$ in M .

Let $\frac{T+K}{K} \subseteq \frac{L}{K} + \frac{X}{K}$, ∀ $\frac{X}{K} \leq_e \frac{M}{K}$ and K⊆X, then $\frac{T+K}{K} \subseteq \frac{L+X}{K}$ and hence T⊆T+K⊆L+X .Therefore $\frac{T+H}{H} \subseteq \frac{L}{H} + \frac{X+H}{H}$. Since X ⊆X+H ⊆M and $X \leq_e M$ then $X+H \leq_e M$ and since $H \leq_c M$ then $\frac{X+H}{H} \leq_e \frac{M}{H}$ [2], since $H \subseteq_{ET.ce} L$ in M, then $\frac{T+H}{H} \subseteq \frac{X+H}{H}$ and hence T⊆T+H⊆X+H .Therefore $\frac{T+K}{K} \subseteq \frac{H}{K} + \frac{X}{K}$. since $K \subseteq_{ET.ce} H$ in M, then $\frac{T+K}{K} \subseteq \frac{X}{K}$.

Thus $K \subseteq_{ET.ce} L$ in M.

Proposition 2.7: Let T be a submodule of a module M. If $K \subseteq_{ET.ce} H$ in M , $K \leq_c M$ and $N \subseteq_{ET.ce} L$ in M, then $K+N \subseteq_{ET.ce} H+L$ in M.

Proof: Let $K \subseteq_{ET.ce} H$ in M and $N \subseteq_{ET.ce} L$ in M, then $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$ and $\frac{L}{N} \ll_{E(\frac{T+N}{N})} \frac{M}{N}$. To show that $K+N \subseteq_{ET.ce} H+L$ in M, let $\frac{T+K+N}{K+N} \subseteq \frac{H+L}{K+N} + \frac{X}{K+N}$, for every $\frac{X}{K+N} \leq_e \frac{M}{K+N}$ and K+N⊆X then $\frac{T+K+N}{K+N} \subseteq$

$\frac{H+L+X}{K+N}$ and hence T⊆T+K+N⊆H+L+X. Therefore $\frac{T+K}{K} \subseteq \frac{H}{K} + \frac{L+X+K}{K}$, since X ⊆L+X+K ⊆M and X$\leq_e$ M then L+X+K $\leq_e$M and since K $\leq_c$M then $\frac{L+X+K}{K} \leq_e \frac{M}{K}$ [2], and K $\subseteq_{ET.ce}$H in M then $\frac{T+K}{K} \subseteq \frac{L+X+K}{K}$, so T⊆T+K⊆L+X+K. Therefore $\frac{T+N}{N} \subseteq \frac{C}{N} + \frac{X+K+N}{N}$. Since X ⊆X+K+N ⊆M and X$\leq_e$ M then X+K+N $\leq_e$M and N $\leq_c$M then $\frac{X+K+N}{N} \leq_e \frac{M}{N}$ [2], since N $\subseteq_{ET.ce}$L in M, then $\frac{T+N}{N} \subseteq \frac{X+K+N}{N}$ and hence T⊆T+N⊆X+K+N=X. Therefore $\frac{T+K+N}{K+N} \subseteq \frac{X}{K+N}$. Thus K+X $\subseteq_{\mathbf{ET.ce}}$ H+L in M .

Corollary 2.8: Let T be a submodule of a module M. If K $\subseteq_{ET.ce}$H in M and N ⊆ M ,then K+N $\subseteq_{ET.ce}$H+N in M. The converse is true if N $\ll_{ET}$ M and K+N $\leq_c$M.

Proof: Suppose that K $\subseteq_{ET.ce}$H in M and N ⊆ M, then $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$. Since N $\subseteq_{ET.ce}$N and K $\subseteq_{ET.ce}$H in M, then K+N $\subseteq_{ET.ce}$H+N in M, by Proposition (2.7).

Proposition 2.9*:* Let T, K and N be submodules of a module M, if K+N $\subseteq_{ET.ce}$H+N and N $\ll_{ET}$ M and K+N $\leq_c$ M then K $\subseteq_{ET.ce}$H.

Proof: Suppose that K+N $\subseteq_{ET.ce}$H+N in M and N $\ll_{ET}$ M . To prove K $\subseteq_{ET.ce}$H in M. Let $\frac{X}{K}$ be an essential submodule of $\frac{M}{K}$ such that $\frac{T+K}{K} \subseteq \frac{H}{K} + \frac{X}{K}$. Then $\frac{T+K}{K} \subseteq \frac{H+X}{K}$ and hence T⊆T+K⊆H+X . Therefore $\frac{T+K+N}{K+N} \subseteq \frac{H+X+N}{K+N}$. Thus $\frac{T+K+N}{K+N} \subseteq \frac{H+N}{K+N} + \frac{X+N}{K+N}$. Since X ⊆X+N⊆M and X $\leq_e$ M then X+N $\leq_e$ M and K+N $\leq_c$M then $\frac{X+N}{K+N} \leq_e \frac{M}{K+N}$ [2], and since K+N $\subseteq_{ET.ce}$H+N in M, then $\frac{T+K+N}{K+N} \subseteq \frac{X+N}{K+N}$ and hence T⊆T+K+N⊆X+N. Since X$\leq_e$ M and N $\ll_{ET}$ M, therefore T ⊆ X . So $\frac{T+K}{K} \subseteq \frac{X}{K}$. Thus K $\subseteq_{ET.ce}$H in M.

Proposition 2.10: Let T be a submodule of a module M and let N $\ll_{ET}$ M .If K $\subseteq_{ET.ce}$H in M and K $\leq_c$ M, then K $\subseteq_{ET.ce}$H+N in M.

Proof: Suppose that N $\ll_{ET}$ M and K $\subseteq_{ET.ce}$H in M, then $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$. To prove K $\subseteq_{ET.ce}$H+N in M. Let $\frac{X}{K}$ be an essential submodule of $\frac{M}{K}$ such that $\frac{T+K}{K} \subseteq \frac{H+N}{K} + \frac{X}{K}$. Then $\frac{T+K}{K} \subseteq \frac{H+N+X}{K}$ and hence $\frac{T+K}{K} \subseteq \frac{H}{K} + \frac{N+X}{K}$. Since X ⊆N+X ⊆M and X$\leq_e$ M then N+X $\leq_e$M and K $\leq_c$M then $\frac{X+K}{K} \leq_e \frac{M}{K}$ [2], since K $\subseteq_{ET.ce}$H in M, then $\frac{T+K}{K} \subseteq \frac{N+X}{K}$ and hence T⊆T+K⊆N+X. But N $\ll_{ET}$ M and X$\leq_e$ M, so T ⊆ X . Therefore $\frac{T+K}{K} \subseteq \frac{X}{K}$. Thus K $\subseteq_{ET.ce}$H+N in M.

Proposition 2.11: Let M and N be R-modules such that T ≤ M and let f : M → N be an epimorphism. If K $\subseteq_{ET.ce}$H in M and K $\leq_c$ M, then f (K)$\subseteq_{\mathbf{Ef(T).ce}}$ f (H) in N.

Proof: Let K$\subseteq_{ET.ce}$H in M , then $\frac{H}{K} \ll_{E(\frac{T+K}{K})} \frac{M}{K}$. To prove f (K) $\subseteq_{ET.ce}f$ (H) in N. Let $\frac{X}{f(K)}$ be an essential submodule of $\frac{N}{f(A)}$ such that $\frac{f(T)+f(K)}{f(K)} \subseteq \frac{f(H)}{f(K)} + \frac{X}{f(K)}$. Then $\frac{f(T)+f(K)}{f(K)} \subseteq \frac{f(H)+X}{f(K)}$ and then $f(T)$ $\subseteq f(T) + f(K) \subseteq f$ (H)+X . so $f^{-1}(f(T)) \subseteq f^{-1}(f$ (H))+ f^{-1}(X), then T +ker(f) ⊆ H+ ker(f)+ f^{-1}(X)[2] .so T ⊆ H+ f^{-1}(X) thus $\frac{T+K}{K} \subseteq \frac{H}{K} + \frac{f^{-1}(X)}{K}$. Since $\frac{X}{K} \leq_e \frac{N}{f(K)}$ then X $\leq_e$ N, therefore f^{-1} (X) $\leq_e$ M and since K $\leq_c$ M implies that $\frac{f^{-1}(X)}{K} \leq_e \frac{M}{K}$ [2], and since K $\subseteq_{\mathbf{ET.ce}}$H in M, then $\frac{T+K}{K} \subseteq \frac{f^{-1}(X)}{K}$ and hence T⊆T+K⊆ f^{-1}(X) then $f(T) \subseteq f(f^{-1}$(X)) = X∩Im(f), since f is an epimorphism then X∩Im(f)=X , so $f(T)$ ⊆ X, therefore $\frac{f(T)+f(K)}{f(K)} \subseteq \frac{X}{f(K)}$. Thus f (K) $\subseteq_{\mathbf{ET.ce}} f$ (H) in N.

3. ET- coclosed submodules

Definition 3.1: Let T be a submodule of a module M. A submodule H is called an ET- coclosed in M if H has no proper coessential submodule in M.

Remarks and Examples 3.1:

1- If $T = M$ and $A \subseteq B$ be submodule of M, then A is ET-coessential of B if and only if A is e-coessential of B. So A is ET-coclosed if and only if A is e-coclosed in M.

2- Consider Z_6 as Z-modules. Let $T = \{\bar{0}, \bar{3}\}$, $A = \{0\}$ and $B = \{\bar{0}, \bar{2}, \bar{4}\}$.

$A \subseteq_{ET.ce} B$ since $\frac{B}{A} \ll_{ET} Z_6$ [3] but $\{\bar{0}\} \neq \{\bar{0}, \bar{2}, \bar{4}\}$, thus B is not ET- coclosed in Z_6, but A is coclosed in B.

3- Consider Z_4 as Z-module. Let $T = \{\overline{0}, \overline{2}\}$, $B = \{\overline{0}, \overline{2}\}$. Now, if $A = \{0\}$, then $\frac{B}{\{0\}} \cong \{\overline{0}, \overline{2}\}$ if $A = \{\overline{0}, \overline{2}\} = \frac{B}{B} = \{0\} \ll_{ET} Z_4$, therefore $B \subseteq_{ET.ce} Z_4$.

Proposition 3.2: Let $A \subseteq B \subseteq M$ and $\subseteq B$, then A is ET-coclosed in M iff A is ET-coclosed in B.

Proof: Suppose that A is ET-coclosed in M and X ⊆ A such that $\frac{A}{X} \ll_{(\frac{T+X}{X})} \frac{B}{X}$ then $\frac{A}{X} \ll_{(\frac{T+X}{X})} \frac{M}{X}$, by[6, Proposition 2.4]. Since A is ET-coclosed in M then A=X.

Conversely, let X ≤ A such that $\frac{A}{X} \ll_{(\frac{T+X}{X})} \frac{B}{X}$, since $A \subseteq B$, then $\frac{A}{X} \subseteq \frac{B}{X}$, thus $\frac{A}{X} \ll_{(\frac{T+X}{X})} \frac{B}{X}$, by [6, Proposition 2.3] since A is ET-coclosed in B.

Proposition 3.3: Let T be a submodule of a module M and let A ⊂ B ⊂ M be submodules.

(1) If B is ET-coclosed in M, then B/A is ET-coclosed in M/A.

(2) If A ≪ B and B/A is ET-coclosed in M/A, then B is ET-coclosed in M.

Proof:

(1) Suppose that N ⊂ B such that N/A ⊂ B/A such that N/A is ET- coessential of B/A in M/A. Then $\frac{B/A}{N/A} \ll_{E(\frac{(T+N)/A}{N/A})} \frac{M/A}{N/A}$ and so $\frac{B}{N} \ll_{E(\frac{(T+N)}{N})} \frac{M}{N}$, then N is ET-coessential of B. But B is ET-coclosed in M, then N = B. implies that $\frac{N}{A} = \frac{B}{A}$, then $\frac{B}{A}$ is ET-coclosed in $\frac{M}{A}$.

(2) Let N⊂B such that N is ET-coessential of L, then $\frac{B}{N} \ll_{E(\frac{(T+N)}{N})} \frac{M}{N}$, implies that $\frac{B/A}{N/A} \ll_{E(\frac{(T+N)/A}{N/A})} \frac{M/A}{N/A}$, therefore $\frac{N}{A} \subseteq_{ET.ce} \frac{\mathrm{B}}{\mathrm{A}}$, but $\frac{B}{N}$ is ET-coclosed in $\frac{\mathrm{M}}{\mathrm{N}}$, then $\frac{N}{A} = \frac{\mathrm{B}}{\mathrm{A}}$, so N=A, then B is ET-coclosed of M.

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