Cubic Trigonometric Spline for Solving Nonlinear Volterra Integral Equations

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Abstract

In this paper, cubic trigonometric spline is used to solve nonlinear Volterra integral equations of second kind. Examples are illustrated to show the presented method's efficiency and convenience.

Keywords: Nonlinear integral equation, Volterra second kind , Cubic trigonometric spline.

شريحة مثلثية مكعبية لحل معادلات فولتيرا التكاملية غير الخطية

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الخلاصه

في هذا البحث ، يتم استخدام شريحة مثلثية مكعب لحل معادلات فولتيرا التكاملية غير الخطية من النوع الثاني. تم توضيح أمثلة لإظهار كفاءة الطريقة المعروضة وراحتها.

1.   Introduction

An integral equation is defined as an equation in which the unknown function $u(x)$ to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations.

The importance of trigonometric functions in different areas, such as electronics or medicine is well known.Recently,trigonometric splines and polynomials gained very highinterest with computer-aided geometric desikgn.In particular,curve designs [1, 2, 3]construct a cubic trigonometric Bezier curve with two shape parameters, on the basis of cubic trigonometric Bernstein functions to generate a curve interpolation scheme.

The solution of integral equation can be approximated by using the non-polynomial spline functions and the collection method [4].Combination of affixed point method and cubic B-spline functions was used [5] to solve the integral equation numerically. The error analysis for the method shows that the approximate solution converges to the exact solution. Non-polynomial spline of functions was used [6] to develop numerical methods to approximate the solution of second kind Volterra integral equations.

There is another method to solve these problems, which is called the Laplace transform series decomposition method (LTSDM), butthe cubic trigonometric spline method is the best method

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because it gives approximate solution which is very close to the exact solution, as shown in the tables that will be presented in this paper.

In this paper, the presentation of the algorithm is achieved in a very simplified way so that the reader can apply it to other types of equations such as Fredholm integral equations.

Previous studies provided more details on trigonometric spline [7, 8, 9, 10].

2.Trigonometric cubic spline method:-[7]

In a simple way, we take [a,b] as interval, in order to improve the numerical method for approximation solution of the following kind:

$$u(x) = f(x) + \int_0^x k\big(x,t,u(t)\big)\, dt \qquad \qquad \ldots (1)$$

For this purpose define $x_j = a + j\,h$, $j = 0,1,2,\cdots,L$

For all jthsegment, the cubic non-polynomial spline $S_j(x)$ has the form:-

$$S_j(x) = a_j \sin k(x - x_j) + b_j \cos k(x - x_j) + c_j\ (x - x_j) + d_j \qquad \qquad \ldots (2)$$

Where a_j , b_j , c_j and d_j are real finite constants and k is the frequency of the trigonometric function which will be used to develop the accuracy of this method.In this paper we consider $k\epsilon(\,0\,,\,1\,]$, where the value of k can be determined by mathematical programs such as Mathlab14. or Mathcad15,by repeating values for k until we reach the random value that gives the best approximate solution.

Now,we will explain this approximate method by considering the following relation:

$S_j'(x_j) = a_j\,k + c_j \approx u'(x_j)$
$S_j''(x_j) = -\,b_j k^2 \approx u''(x_j)$
$S_j'''(x_j) = -a_j k^3 \approx u'''(x_j)$

Now,we can obtain the values of a_j , b_j , c_j and d_j as follows:

$$a_j = -\tfrac{1}{k^3}u'''(x_j). \ . \ . \qquad \qquad (3)$$

$$b_j = -\tfrac{1}{k^2}u''(x_j)\quad \ldots \qquad \qquad (4)$$

$$c_j = u'(x_j) + \tfrac{1}{k^2}u'''(x_j)\ \ldots \qquad \qquad (5)$$

$$d_j = u(x_j) + \tfrac{1}{k}u''(x_j)\ \ldots \qquad \qquad (6)$$

Now,we differentiate equation (1) three times with respect to x and then put:
$x = x_j$, $j = 0,1,2,\cdots,L-1$ and for clarification we will compensate $x_j = 0$
andin the same way for other values:

$$u(0) = f(0) \qquad \qquad \ldots (7)$$

$u'(x) = f'(x) + \int_0^x \frac{\partial k(x,t,u(t))}{\partial x}\, dt + k\big(x,x,u(x)\big)$

$$u'(0) = f'(0) + k\big(0,0,u(0)\big). \ . \ . \qquad \qquad (8)$$

Let $E\big(x,t,u(t)\big) = \frac{\partial k(x,t,u(t))}{\partial x}$

$$u''(x) = f''(x) + \int_0^x \frac{\partial E\big(x,t,u(t)\big)}{\partial x}\, dt + 2E\big(x,x,u(x)\big)$$

$$u''(0) = f''(0) + 2E\big(0,0,u(0)\big) \qquad \qquad \ldots (9)$$

Let $F\big(x,t,u(t)\big) = \frac{\partial E(x,t,u(t))}{\partial x}$

$$u'''(x) = f'''(x) + \int_0^x \frac{\partial F\big(x,t,u(t)\big)}{\partial x}\, dt + 3F\big(x,x,u(x)\big)$$

$$u'''(0) = f'''(0) + 3F\big(0,0,u(0)\big) \qquad \qquad \ldots (10)$$

Thus,we can approximate the solution of nonlinear Volterra integral equations of second kind (1) by using equation (2).

Algorithm:

Step1:-Input $h = \frac{b-a}{L}$, $x_j = x_0 + j\ h, j = 0,1,2,\cdots,L$, $x_0 = a$, $x_L = b$

Step2:- Compute a_j , b_j , c_j and d_j by substituting the equations 7 – 10 in equations 3 – 6

Step3:- Evaluate $S_0(x)$ by using Step2 and equation (2) for $j = 0$

Step4:- Approximate $u_1 = u(x_1) \approx S_0(x_1)$

Step5:- Do the following steps for $j = 1$ to $L - 1$

Step6:- Compute a_j , b_j , c_j and d_j by using equation (3 – 6) and replacing $u'(x_j)$, $u''(x_j)$ and $u'''(x_j)$ in $S_j'(x_j)$, $S_j''(x_j)$ and $S_j'''(x_j)$

Step7:- Approximate $u_{j+1} = S_j (x_{j+1})$

To show that the presented method is efficient and convenient:

Example (1):- Consider the following non-linear Volterra integral equation of second kind:-

$$u(x) = \sin(x) + \frac{2}{3}\cos(x) - \frac{1}{3}\cos^2(x) - \frac{1}{3} + \int_0^x \sin(x - t)\, u^2(t)\, dt$$

Which has the exact solution $u(x) = \sin(x)$,[8]

If $h = 0.1$

$$S(x) = \begin{cases} 0 & if & -1 \leq x \leq 0.0 \\ S10(x) & if & 0.0 \leq x \leq 0.1 \\ S11(x) & if & 0.1 \leq x \leq 0.2 \\ S12(x) & if & 0.2 \leq x \leq 0.3 \\ S13(x) & if & 0.3 \leq x \leq 0.4 \\ S14(x) & if & 0.4 \leq x \leq 0.5 \\ S15(x) & if & 0.5 \leq x \leq 0.6 \\ S16(x) & if & 0.6 \leq x \leq 0.7 \\ S17(x) & if & 0.7 \leq x \leq 0.8 \\ S18(x) & if & 0.8 \leq x \leq 0.9 \\ S19(x) & if & 0.9 \leq x \leq 1.0 \end{cases}$$

Table 1-Approximate Results for Example 1 if h =0.1

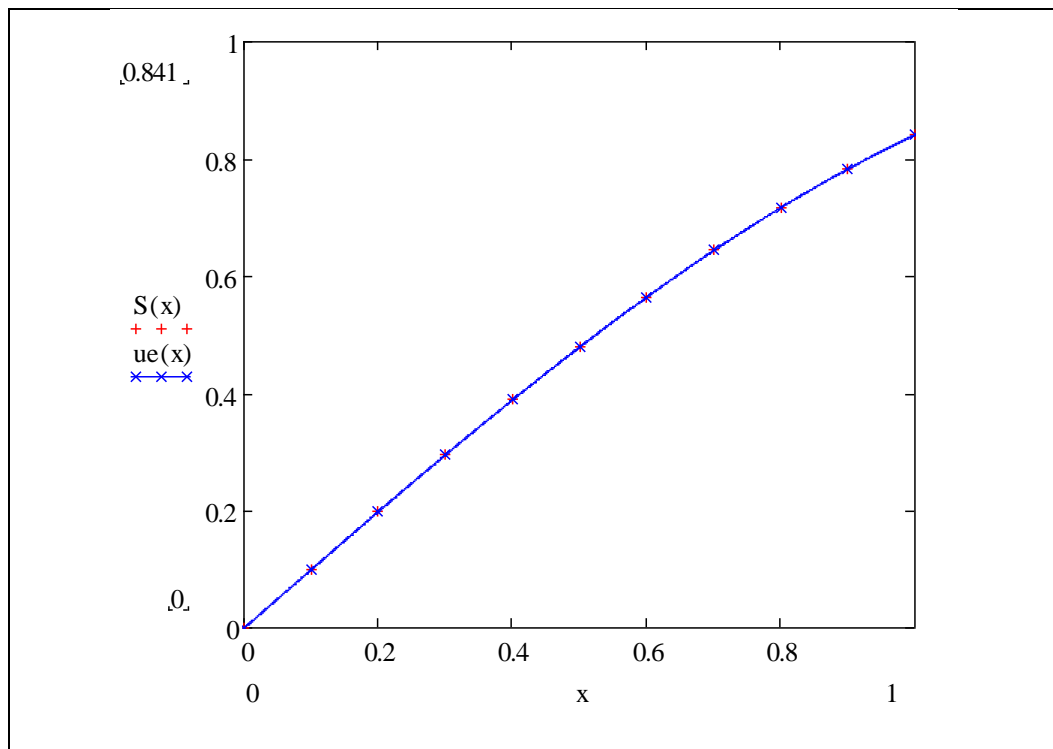| X_j	Exact solution	Approximate solution	Absolute error
0.0	0.0	0.0	0.0
0.1	0.099833416646828	0.099833416646828	0.0
0.2	0.19866933079506	0.19866933079506	0.0
0.3	0.29552020666134	0.29552020666134	0.0
0.4	0.38941834230865	0.38941834230865	0.0
0.5	0.4794255386042	0.4794255386042	0.0
0.6	0.56464247339504	0.56464247339503	0.00000000000001
0.7	0.64421768723769	0.64421768723768	0.00000000000001
0.8	0.71735609089952	0.71735609089952	0.0
0.9	0.78332690962748	0.78332690962748	0.0
1.0	0.8414709848079	0.84147098480789	0.00000000000001
L.S.E.		2.2325351775545e-28	

Figure 1-Comparison of Exact Solution and Approximate Solution if h =0.1

If$h = \frac{1}{3}$

$$S(x) = \begin{cases} 0 & if \; -1 \leq x \leq 0 \\ S10(x) \; if \; 0 \leq x \leq \dfrac{1}{3} \\ S11(x) \; if \; \dfrac{1}{3} \leq x \leq \dfrac{2}{3} \\ S12(x) \; if \; \dfrac{2}{3} \leq x \leq 1 \end{cases}$$

Table 2-Approximate Results for Example 1 if h =(1)/3

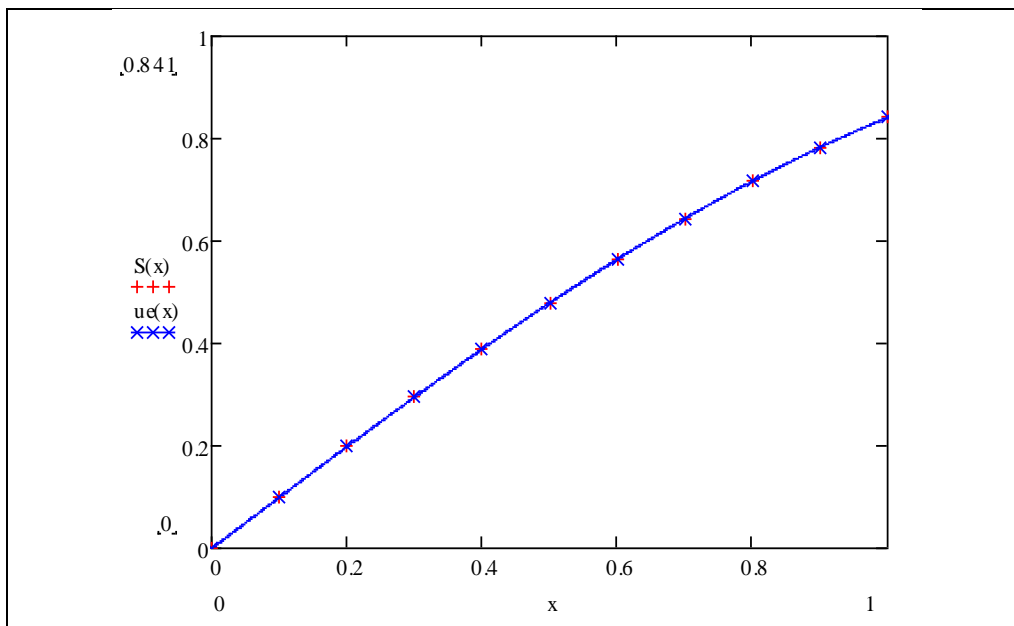| x_j	Exact solution	Approximate solution	Absolute error
0.0	0.0	0.0	0.0
$\dfrac{1}{3}$	0.32719469679615	0.32719469679615	0.0
$\dfrac{2}{3}$	0.61836980306974	0.61836980306974	0.0
1.0	0.8414709848079	0.8414709848079	0.0
L.S.E.		9.6481363554251e-29	

Figure 2-Comparison of Exact Solution and Approximate Solution if h =(1)/3

Example(2):-Consider the following non-linear Volterra integral equation of second kind:-

$$u(x) = x - x\,e^x + \int_0^x e^{x-t}\,e^{u(t)}\,dt$$

Which has the exact solution $u(x) = x$,[8]

If $h = 0.1$

$$S(x) = \begin{cases} 0 & if & -1 \le x \le 0.0 \\ S10(x) & if & 0.0 \le x \le 0.1 \\ S11(x) & if & 0.1 \le x \le 0.2 \\ S12(x) & if & 0.2 \le x \le 0.3 \\ S13(x) & if & 0.3 \le x \le 0.4 \\ S14(x) & if & 0.4 \le x \le 0.5 \\ S15(x) & if & 0.5 \le x \le 0.6 \\ S16(x) & if & 0.6 \le x \le 0.7 \\ S17(x) & if & 0.7 \le x \le 0.8 \\ S18(x) & if & 0.8 \le x \le 0.9 \\ S19(x) & if & 0.9 \le x \le 1.0 \end{cases}$$

Table 3-Approximate Results for Example 2 if h = 0.1

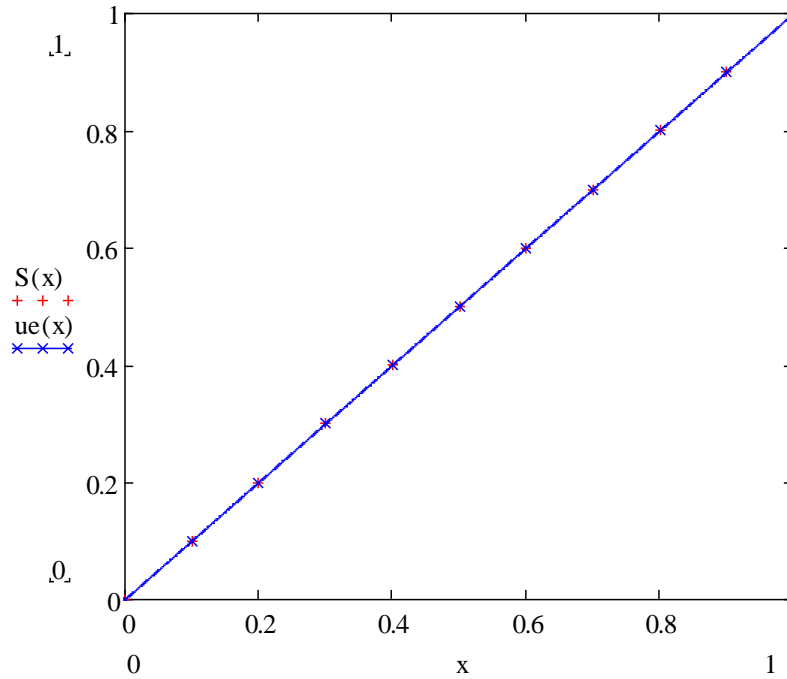| X_j	Exact solution	Approximate solution	Absolute error
0.0	0.0	0.0	0.0
0.1	0.1	0.1	0.0
0.2	0.2	0.2	0.0
0.3	0.3	0.3	0.0
0.4	0.4	0.4	0.0
0.5	0.5	0.5	0.0
0.6	0.6	0.6	0.0
0.7	0.7	0.7	0.0
0.8	0.8	0.8	0.0
0.9	0.9	0.9	0.0
1.0	1.0	1.0	0.0
L.S.E.		0.0	

Figure 3-Comparison of Exact Solution and Approximate Solution if h $= \mathbf{0.1}$

If$h = \frac{1}{3}$

$$S(x) = \begin{cases} 0 & if -1 \leq x \leq 0 \\ S10(x) \ if \ 0 \leq x \leq \frac{1}{3} \\ S11(x) \ if \ \frac{1}{3} \leq x \leq \frac{2}{3} \\ S12(x) \ if \ \frac{2}{3} \leq x \leq 1 \end{cases}$$

Table 4-Approximate Results for Example 2 if h =(1)/3

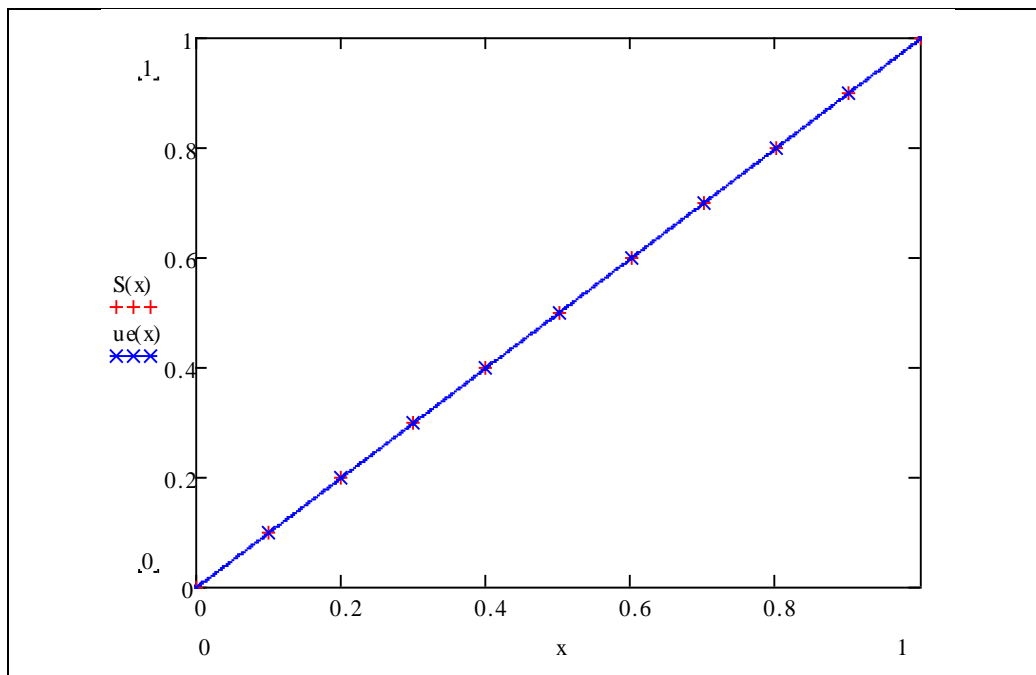| X_j	Exact solution	Approximate solution	Absolute error
0.0	0.0	0.0	0.0
$\frac{1}{3}$	$\frac{1}{3}$	0.33333333333333	3.3333333333333e-15
$\frac{2}{3}$	$\frac{2}{3}$	0.66666666666666	6.6666666666667e-15
1	1	0.99999999999999	1.0e-14
L.S.E.		1.5555555555556e-28	

Figure 4-Comparison of Exact Solution and Approximate Solution if h =(1)/3

Example(3):-Consider the following non-linear Volterra integral equation of second kind:- $u(x) = \cos(x) - \frac{1}{2}x^2 + \int_0^x (x-t)\big(u^2(t) + \sin^2(t)\big)\, dt$

Which has the exact solution $u(x) = \cos\ (x)$, [7]

$$h = \frac{1}{3}$$

$$S(x) = \begin{cases} 0 & if\ -1 \ \le\ x\ \le\ 0 \\ S10(x)\ if\ 0\ \le\ x\ \le\ \dfrac{1}{3} \\ S11(x)\ if\ \dfrac{1}{3}\ \le\ x\ \le\ \dfrac{2}{3} \\ S12(x)\ if\ \dfrac{2}{3}\ \le\ x\ \le\ 1 \end{cases}$$

Table 5-Approximate Results for Example 3 if h $= \frac{1}{3}$

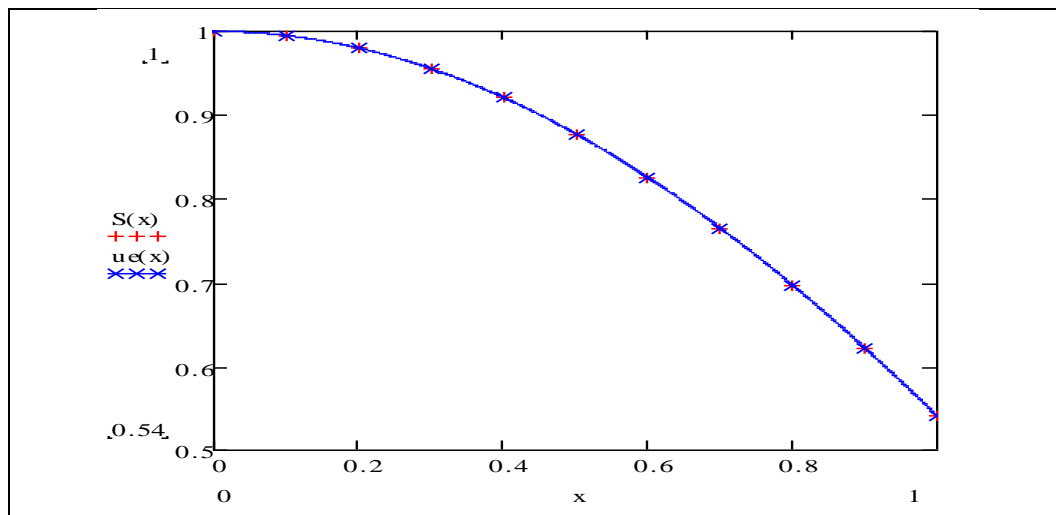| X_j	Exact solution	Approximate solution	Absolute error
0.0	1.0	1.0	0.0
$\frac{1}{3}$	0.94495694631474	0.94495694631474	0.0
$\frac{2}{3}$	0.78588726077695	0.78588726077695	0.0
1	0.54030230586814	0.54030230586814	0.0
L.S.E.		9.5314704530475e-30	

Figure 5 -Comparison of Exact Solution and Approximate Solution if h =(1)/3

$$h = 0.1$$

$$S(x) = \begin{cases} 0 & if & -1 \leq x \leq 0.0 \\ S10(x) & if & 0.0 \leq x \leq 0.1 \\ S11(x) & if & 0.1 \leq x \leq 0.2 \\ S12(x) & if & 0.2 \leq x \leq 0.3 \\ S13(x) & if & 0.3 \leq x \leq 0.4 \\ S14(x) & if & 0.4 \leq x \leq 0.5 \\ S15(x) & if & 0.5 \leq x \leq 0.6 \\ S16(x) & if & 0.6 \leq x \leq 0.7 \\ S17(x) & if & 0.7 \leq x \leq 0.8 \\ S18(x) & if & 0.8 \leq x \leq 0.9 \\ S19(x) & if & 0.9 \leq x \leq 1.0 \end{cases}$$

Table 6-Approximate Results for Example 3 if h =0.1

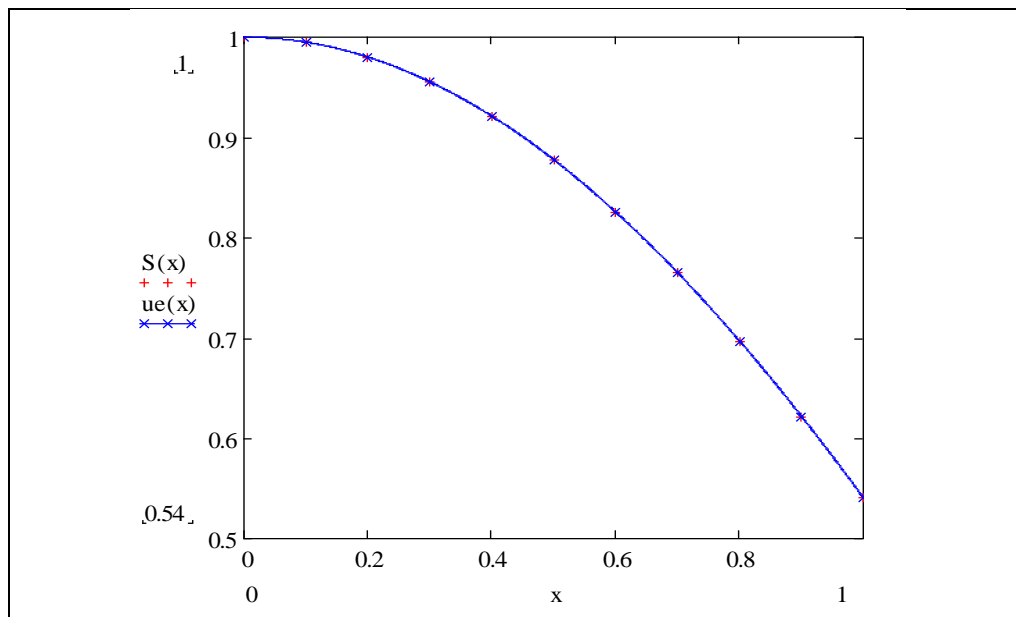| X_j	Exact solution	Approximate solution	Absolute error
0.0	1.0	1.0	0.0
0.1	0.99500416527803	0.99500416527803	0.0
0.2	0.98006657784124	0.98006657784125	0.00000000000001
0.3	0.95533648912561	0.95533648912562	0.00000000000001
0.4	0.92106099400289	0.92106099400289	0.0
0.5	0.87758256189037	0.87758256189038	0.00000000000001
0.6	0.82533561490968	0.82533561490969	0.00000000000001
0.7	0.76484218728449	0.7648421872845	0.00000000000001
0.8	0.69670670934717	0.69670670934718	0.00000000000001
0.9	0.62160996827066	0.62160996827068	0.00000000000002
1.0	0.54030230586814	0.54030230586816	0.00000000000002
L.S.E.		1.4970818729916e-27	

Figure 6 -Comparison of Exact Solution and Approximate Solution if h =0.1

3. Conclusion

The present work is an effort to obtain the approximate solution of Volterraintegral equation of the second kind by using trigonometric cubic spline method. Three test examples were considered with the exact solution,for which the results are given in Tables-(1 , 2 , 3 , 4 , and 5) and Figures-(1 , 2 , 3 , 4 , and 5) to show the accuracy and efficiency of the method. All the computationswereperformed by Mathcad 15

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