DOI: 10.24996/ijs.2025.66.2.10

ISSN: 0067-2904

New Differential Subordination and Superordination Results for a subclass of Meromorphic Univalent Functions Defined by a New Operator

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Received: 5/11/2023 Accepted: 31/1/2024 Published: xx

Abstract

In this paper, we explore and introduce a novel study employing a new operator denoted as $\mathcal{W}^{\eta,s}_{v,\gamma,\delta,\tau,n,r,d}$ within the field of geometric function theory, particularly in the context of sandwich theorems. We derive results pertaining to differential subordination and superordination for this new formula operator. Additionally, we establish specific sandwich theorems.

Keywords: Meromorphic univalent function, Analytic function, Subordinate, differential subordination, dominant, Hadamard product, sandwich theorem,

نتائج جديدة للتبعية التفاضلية والتبعية التفاضلية العليا لفئات من الدوال احادية التكافؤ الميرومورفية المعرفة بواسطة مؤثر جديد

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الخلاصة

في هذا البحث، نستكشف ونقدم دراسة جديدة تستخدم عاملًا جديدًا يُشار إليه بـ $\mathcal{W}^{\eta,s}_{v,\gamma,\delta,\tau,n,r,d}$ في مجال نظرية الدالة الهندسية، خاصة في سياق نظريات الساندويتش. نحن نستخلص النتائج المتعلقة بالتبعية التفاضلية والتبعية التفاضلية العليا لمشغل الصيغة الجديد هذا. بالإضافة إلى ذلك، قمنا بتأسيس نظريات ساندويتش محددة.

1. Introduction

The notation $\mathcal{A}^*$ represents the class of functions characterized as following:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n. \tag{1.1}$$

This class comprises functions that are meromorphic and univalent within the punctured open unit disk U^* defined as $\{z: z \in \mathbb{C}: 0 < |z| < 1\}$. Many researchers have investigated meromorphic functions within various classes and under different circumstances, as discussed in references [1, 2].Consider $\mathcal{H}$ as the linear space encompassing all analytic functions within the open unit disk U. For a positive integer n and a complex number a, we define the following:

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$$\mathcal{H}[a,n] = \{f \in \mathcal{H}: f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

In the context of analytic functions in the $\mathcal{H}$, we say that f is subordinate to g in the open unit disk U, denoted as $f(z) \prec g(z)$, if there exists a Schwarz function w, which is analytic in U, has $w(0) = 0$, and $|w(z)| < 1$ for $z \in U$, such that $f(z) = g(w(z))$ for $z \in U$.
In addition, when the function g is univalent in U, we establish these particular equivalence relationship, as can be found in references (cf., e.g. [3, 4] and [5]):
$f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and the image of f over U is contained within the image of g over U, for $z \in U$.

Definition 1.1: [3] In this scenario, $\psi(r,s,t;z): \mathbb{C}^3 \times U \to \mathbb{C}$, and the analytic function $h(z)$ is defined within U. We require p and $\psi(p(z), zp'(z), z^2p''(z); z)$ to be univalent in U, and p should satisfy the second-order differential superordination condition:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), (z \in U). \quad (1.2)$$

We categorize p as a solution to the differential superordination (1.2). A subordinate of p, is an analytic function $q(z)$ for which $p \prec q$ holds true for all p that satisfy (1.2). A univalent subordinate $\tilde{q}(z)$ that meets the condition $q \prec \tilde{q}$ for all subordinates q related to (1.2) is referred to as the best subordinant.

Definition 1.2: [5] In this situation, consider $\psi(r,s,t;z): \mathbb{C}^3 \times U \to \mathbb{C}$, and let $h(z)$ be a univalent function in U. Also, let p be an analytic function in U, satisfying the differential subordination of second-order:

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), (z \in U). \quad (1.3)$$

In this context, we classify p as a solution to (1.3). The univalent function q is referred to as a dominant of p, if $p \prec q$ holds true for all. A dominant $\tilde{q}(z)$ that adheres to the condition $q \prec \tilde{q}$ for every dominant q associated with (1.3) is termed the best dominant.
Many authors have established conditions that are sufficient for the functions h, q and ψ that fulfill the following conditions. You can refer to references [1, 2,] and [6-11] for more details on these requirements.

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z), (z \in U^*). \quad (1.4)$$

If we have an $f \in \mathcal{A}^*$ represented as (1.1), and a $g \in \mathcal{A}^*$ given by:

$$g(z) = \frac{1}{z} + \sum_{n=2}^{\infty} b_n z^n.$$

The Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n = (g * f)(z).$$

We can utilize the results from references [2], [6], [8-14] to derive suitable criteria for the fulfillment of normalized analytic functions

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z).$$

Here, q_1 and q_2 with $q_1(0) = q_2(0) = 1$, represent univalent functions in U. Recently, Shanmugam et al. [10, 11] and Goyal et al. [8] have obtained Sandwich results for classes of analytic functions. You can also refer to [15] for additional information.

Recent studies in the literature have also explored specific univalent and multivalent functions, along with differential subordination outcomes for higher-order functions. These findings are discussed in references [7], and [16-23].
Atshan et al. [7] defined an integral operator $\Re_*^{\eta}(\nu,\gamma,\delta,\tau,r)$ on $\mathcal{A}^*$ as follows:

$$\Re_*^{0} f(z) = f(z),$$

$$\Re_*^{1} f(z) = \left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right) z^{-1-\left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right)} \int_0^z t^{\left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right)} f(t)\,dt,$$

$$(\nu>1,\gamma>1,\delta>0,\tau>0,0<r<1,z\in U^*).$$

For a $f(z)$ belonging to $\mathcal{A}^*$ and given by (1.1), we have:

$$\Re_*^{\eta} f(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{\nu+\tau-1}{\nu+\tau-1+(n+1)(\gamma+\delta-r)}\right)^{\eta} a_n z^n.$$

Now, the general Hurwitz – lerch zeta function

$$\Phi(z,s,d) = \sum_{n=0}^{\infty}\frac{z^n}{(d+n)^s}, d\in\mathbb{C}\setminus Z_0^-, s\in\mathbb{C},$$

when $0<|z|<1$.

Definition 1.3: For any $f\in\mathcal{A}^*$ and z within U^*, we introduce a new operator defined as follows $\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f:\mathcal{A}^*\to\mathcal{A}^*$, where

$$\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z) = \frac{\Phi(z,s,d)}{zd^{-s}} * \Re_*^{\eta} f(z)$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty}\left(\frac{\nu+\tau-1}{\nu+\tau-1+(n+1)(\gamma+\delta-r)}\right)^{\eta}\left(\frac{d}{n+d+1}\right)^{s} a_n z^n. \quad (1.5)$$

We note from (1.5) that

$$z\left(\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z)\right)' + \left(\frac{\nu+\tau-1}{\gamma+\delta-r}+1\right)\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z) = \left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right)\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta-1,s} f(z). \quad (1.6)$$

The primary objective of this research is to determine adequate conditions under which specific normalized analytic functions f fulfill the following:

$$q_1(z) \prec \left(z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z)\right)^{\mu} \prec q_2(z),$$

and

$$q_1(z) \prec \left((1-\rho)z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z) + \rho z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta+1,s} f(z)\right)^{\mu} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ denote univalent functions within the open unit disk U, and they both satisfy $q_1(0) = q_2(0) = 1$.
In our research, we have established multiple results in the sandwich-type theorem category, utilizing the operator $\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s}$ when applied to the function $f(z)$.

2. Preliminaries

To lay the foundation for our subordination and superordination findings, we introduce these concepts

Definition 2.1: According to reference [4], the set Q consists of all functions q that are both analytic and one-to-one within the region $\overline{U}\backslash E(q)$, where $\overline{U}$ represents U union its boundary (∂U). Here, $E(q)$ is defined as $E(q) = \{\zeta \in \partial U: \lim_{z\to\zeta} q(z) = \infty\}$ and these functions also satisfy the condition $q'(\zeta) \neq 0$ for $\zeta \in \partial U\backslash E(q)$. Additionally, we define the subclasses of Q as follows: The subclass for which $q(0) = a$ is denoted as $Q(a)$, $Q(0)$ is referred to as $Q(0)$, and $Q(1)$ is represented by $Q(1)$, which comprises functions $q \in Q$ with $q(0) = 1$

Lemma 2.1: As per reference [6], consider a convex univalent function $q(z)$ within the region U. Let $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}\backslash\{0\}$, and assume that $Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > max\left\{0, -Re\left(\frac{\alpha}{\beta}\right)\right\}$. If $p(z)$ is an analytic function in U and $\alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z)$, then it follows that $p(z) \prec q(z)$, and q is identified as the best dominant.
Lemma 2.2: According to reference [5], let q be a univalent function within U, and consider $\emptyset$ and θ as analytic functions in the domain D that contains $q(U)$. It is essential that $\emptyset(w) \neq 0$ when w belongs to $q(U)$. Define $Q(z) = zq'(z)\emptyset(q(z))$ and $h(z) = \theta\big(q(z)\big) + Q(z)$. Now, assume the following conditions:
1) Q is starlike univalent in U.
2) $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for z in U.
If p is an analytic function in U with $q(0) = p(0), p(U) \subseteq D$ and $\emptyset\big(p(z)\big) + zp'(z)\emptyset(p(z)) \prec \emptyset\big(q(z)\big) + zq'(z)\emptyset(q(z))$, then it follows that $p(z) \prec q(z)$, and q is identified as the best dominant

Lemma 2.3: In accordance with reference [4], consider a convex univalent function $q(z)$ within the unit disk U. Furthermore, let θ and $\emptyset$ be analytic functions within a domain D that contains $q(U)$. The following conditions are vital:
1) $Re\left\{\frac{\theta'(q(z))}{\emptyset(q(z))}\right\} > 0$ for z within U.
2) $Q(z) = zq'(z)\emptyset(q(z))$ is starlike univalent for z in U.
Assume that p belongs to $\mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\theta\big(p(z)\big) + zp'(z)\emptyset\big(p(z)\big)$ is univalent in U. Additionally, if $\theta\big(q(z)\big) + zq'(z)\emptyset\big(q(z)\big) \prec \theta\big(p(z)\big) + zp'(z)\emptyset\big(p(z)\big)$, then it can be concluded that $q(z) \prec p(z)$, and q is recognized as the best subordinant.

Lemma 2.4: [4] Let $q(z)$ be a convex univalent in U with $q(0) = 1$. Let $\beta \in \mathbb{C}$, that $Re\{\beta\} > 0$. If $p(z) \in \mathcal{H}[q(0),1] \cap Q$ and $p(z) + \beta z p'(z)$ is univalent in U, then $q(z) + \beta z q'(z) \prec p(z) + \beta z p'(z)$, hence, $p(z) \prec q(z)$ and $p(z)$ is named the best dominant.

Lemma 2.5: According to reference [4], consider a convex univalent function $q(z)$ within U, and ensure that $q(0) = 1$. Let $\beta \in \mathbb{C}$ with the condition $Re\{\beta\} > 0$. If $p(z) \in \mathcal{H}[q(0),1] \cap Q$ and $p(z) + \beta z p'(z)$ is univalent within U, then it follows that $q(z) + \beta z q'(z) \prec p(z) + \beta z p'(z)$, which implies that $q(z) \prec p(z)$, and $p(z)$ is identified as the best subordinant.

3. Results of differential subordination

In this context, we present a variety of differential subordination results, utilizing the operator $\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}$ applied to the function $f(z)$.

Theorem 3.1: Consider $q(z)$ as a convex univalent function within U, satisfying $q(0) = 1$, where α is a non-zero complex number in $\mathbb{C}$, and μ is a positive number. Suppose that $q(z)$ meets the following conditions

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > max\left\{0, -Re\left(\frac{\mu}{\alpha}\right)\right\}. \tag{3.1}$$

If $f \in \mathcal{A}^*$ satisfies the subordination

$$\frac{\mu}{z}\left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right)\left(\frac{\mathcal{W}^{\eta-1,s}_{\nu,\gamma,\delta,\tau,n,r,d}\,f(z)}{\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\,f(z)} - \left(\frac{\gamma+\delta-r}{\nu+\tau-1}\right) - 1\right) + \left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\,f(z)\right)^{\mu}$$

$$\prec q(z) + \frac{\alpha}{\mu} zq'(z), \tag{3.2}$$

then

$$\left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\,f(z)\right)^{\mu} \prec q(z), \tag{3.3}$$

and $q(z)$ is the best dominant.

Proof: Consider

$$p(z) = \left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\,f(z)\right)^{\mu}, \tag{3.4}$$

thus, the function $p(z)$ is analytic within U, and it holds that $p(0) = 1$. Consequently, if we differentiate (3.4) subject to z and utilizing (1.6), it follows that

$$\frac{zp'(z)}{p(z)} = \mu\left(\frac{\left(\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\,f(z)\right)'}{\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\,f(z)}\right),$$

then

$$\frac{zp'(z)}{p(z)} = \frac{\mu}{z}\left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right)\left(\frac{\mathcal{W}^{\eta-1,s}_{\nu,\gamma,\delta,\tau,n,r,d}\,f(z)}{\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\,f(z)} - \left(\frac{\gamma+\delta-r}{\nu+\tau-1}\right) - 1\right).$$

The relation (3.2) follows by the hypothesis give us $p(z) + \frac{\alpha}{\mu}zp'(z) \prec q(z) + \frac{\alpha}{\mu}zq'(z)$. Applying Lemma 2.1 where $\beta = \frac{\alpha}{\mu}$ and $\alpha = 1$, we derive (3.3). □

By taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, in Theorem 3.1, we arrive at we can derive the next result.

Corollary 3.1: Suppose that $\mu > 0, \alpha \in \mathbb{C}\backslash\{0\}$ and $(-1 \le B < A \le 1)$.
Let $Re\left(\frac{1+Az}{1+Bz}\right) > max\left\{0, -Re\left(\frac{\mu}{\alpha}\right)\right\}$. If $f \in \mathcal{A}^*$ and fulfills the following subordination condition:
$p(z) \prec \frac{1+Az}{1+Bz} + \frac{\mu}{\alpha}\frac{(A-B)z}{(1+Bz)^2}$,
where $p(z)$ given by (3.4), then $\left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\,f(z)\right)^{\mu} \prec \frac{1+Az}{1+Bz}$,
furthermore, $\frac{1+Az}{1+Bz}$ is called the best dominant.
If we substitute $A = 1$ and $B = -1$ into Corollary 3.1, we obtain the next corollary:
Corollary 3.2: Suppose that $\mu > 0$ and $\alpha \in \mathbb{C}\backslash\{0\}$, and assume that $Re\left(\frac{1+z}{1-z}\right) > max\left\{0, -Re\left(\frac{\mu}{\alpha}\right)\right\}$. If f belongs to $\mathcal{A}^*$ and holds the subordination criteria:
$p(z) \prec \frac{1+z}{1-z} + \frac{\mu}{\alpha}\frac{2z}{(1-z)^2}$,

where $\mathcal{p}(z)$ is given by (3.4), then $\left(z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z)\right)^{\mu} \prec \frac{1+z}{1-z}$, and $\frac{1+z}{1-z}$ is identified as the best dominant.

Theorem 3.2: Consider $q(z)$ as a convex univalent function within the unit disk U, where $q(0) = 1$ and $q'(z) \neq 0$. Let t be a non-zero complex number in $\mathbb{C}$, ζ be a positive number, and suppose that $q(z)$ satisfies:

$$Re\left\{\frac{1}{t}(\zeta + t) + z\frac{q''(z)}{q'(z)}\right\} > 0, \tag{3.5}$$

where t is non-zero complex number and z is within U.
Assuming $\frac{zq'(z)}{q(z)}$ is starlike univalent in U, if f belongs to $\mathcal{A}^*$ and meets

$$G(z) = \psi_{(\eta,s,\nu,\gamma,\delta,\tau,n,r,d;z)} \prec \zeta q(z) + ztq'(z), \tag{3.6}$$

where

$$G(z) = \left((1-\rho)z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z) + \rho z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta+1,s} f(z)\right)^{\mu}\left[\zeta + t\mu\left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right)\left(\frac{(1-\rho)\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta-1,s} f(z) - (1-2\rho)\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z) - \rho\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta+1,s} f(z)}{(1-\rho)\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z) + \rho\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta+1,s} f(z)}\right)\right], \tag{3.7}$$

then $\left((1-\rho)z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z) + \rho z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta+1,s} f(z)\right)^{\mu} \prec \mathcal{p}(z)$, and $\mathcal{p}(z)$ is identified as the best dominant.

Proof: Let

$$\mathcal{p}(z) = \left((1-\rho)z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z) + \rho z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta+1,s} f(z)\right)^{\mu}. \tag{3.8}$$

In this scenario, the function $\mathcal{p}(z)$ is analytic within U, and it holds that $\mathcal{p}(0) = 1$. Consequently, if we differentiate (3.5) subject to z and applying (1.6), we can deduce that

$$\frac{z\mathcal{p}'(z)}{\mathcal{p}(z)}\mu\left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right)\left[\frac{(1-\rho)\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta-1,s} f(z) - (1-2\rho)\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z) - \rho\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta+1,s} f(z)}{(1-\rho)\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z) + \rho\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta+1,s} f(z)}\right]. \tag{3.9}$$

Set $\theta(m) = \zeta m$ and $\psi(m) = t$.
It follows that $\theta(m)$ and $\psi(m)$ are analytic in $\mathbb{C}$, in addition,
$Q(z) = zq'(z)\,\psi(q(z)) = ztq'(z)$ and $h(z) = \theta(q(z)) + Q(z) = \zeta q(z) + ztq'(z)$.
Since it's evident that $Q(z)$ is starlike univalent in U, we can establish the following

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{1}{t}(\zeta + t) + z\frac{q''(z)}{q'(z)}\right\} > 0.$$

Using a simple calculation, we have

$$G(z) = \zeta\mathcal{p}(z) + zt\mathcal{p}'(z), \tag{3.10}$$

where $G(z)$ is as defined in (3.7). Applying (3.6) and (3.10), we get

$$(z) + zt\mathcal{p}'(z) \prec \zeta q(z) + ztq'(z). \tag{3.11}$$

Therefore, using Lemma 2.2, we deduce that $\mathcal{p}(z) \prec q(z)$.
Finally, applying (3.8), we obtain the required result. □
When we substitute $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ into Theorem 3.2 for every $\mu > 0$ and t in $\mathbb{C} \setminus \{0\}$, the condition (3.5) transforms into

$$Re\left\{\frac{1}{t}(\zeta + t) - \frac{2zB}{1+Bz}\right\} > 0. \tag{3.12}$$

Consequently, we obtain this corollary.

Corollary 3.3: Let $\zeta > 0, t \in \mathbb{C}\backslash\{0\}$, and $(-1 \le B < A \le 1)$. Consider the assumption that (3.12) hold.
If $f \in \mathcal{A}^*$ and $G(z) \prec \zeta\left(\frac{1+Az}{1+Bz}\right) + zt\left(\frac{A}{1+Bz} + \frac{(1+Az)B}{(1+Bz)^2}\right)$, where $G(z)$ is as defined in (3.7), then $\left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)\right)^{\mu} \prec \frac{1+Az}{1+Bz}$.
As well as $\frac{1+Az}{1+Bz}$ is defined as the best dominant.
Set $p(z) = \left(\frac{1+z}{1-z}\right)^{\rho}$ $(0 < \rho \le 1)$, the requirement (3.12) in the proof of Theorem 3.2 has the form

$$Re\left\{\frac{1}{t}(\zeta + t) + \frac{z}{(1+z)^2}\left(1 + 2(\rho - 1)\left(\frac{1-z}{1+z}\right)\right)\right\} > 0. \tag{3.13}$$

Therefore, the following corollary follows.

Corollary 3.4: Suppose that $\zeta > 0, t \in \mathbb{C}\backslash\{0\}, m \in \mathbb{C}$ and $(0 < \rho \le 1)$. Consider the assumption (3.13) satisfy.
If $f \in \mathcal{A}^*$ and $G(z) \prec \zeta\left(\frac{1+z}{1-z}\right) + zt\left(\frac{2\rho}{(1+z)^2}\left(\frac{1+z}{1-z}\right)^{\rho-1}\right)$,
where $G(z)$ given by (3.7), then $\left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)\right)^{\mu} \prec \left(\frac{1+z}{1-z}\right)^{\rho}$,
and $\left(\frac{1+z}{1-z}\right)^{\rho}$ is called the best dominant.

4. Results of differential superordination

Theorem 4.1: Suppose $q(z)$ is a convex univalent function within the unit disk U, where $q(0) = 1$, and α is a complex number while μ is a positive value with $Re(\alpha) > 0$. If f belongs to $\mathcal{A}^*$ such that

$$\left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)\right)^{\mu} \ne 0.$$

Furthermore, assume that f fulfills the condition

$$\left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)\right)^{\mu} \in \mathcal{H}[q(0), 1] \cap Q. \tag{4.1}$$

If $p(z)$, that defined as in (3.4), is univalent and meets the superordination condition

$$q(z) + \frac{\alpha}{\mu} zq'(z) \prec \left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)\right)^{\mu} + \frac{\mu}{z}\left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right)\left(\frac{\mathcal{W}^{\eta-1,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)}{\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)} - \left(\frac{\gamma+\delta-r}{\nu+\tau-1}\right) - 1\right), \tag{4.2}$$

then $q(z) \prec \left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)\right)^{\mu}$, and $q(z)$ is recognized as the best subordinant.

Proof: Consider the following

$$p(z) = \left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)\right)^{\mu}. \tag{4.3}$$

Upon differentiate (3.4) subject to z, we obtain

$$\frac{zp'(z)}{p(z)} = \frac{\mu}{z}\left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right)\left(\frac{\mathcal{W}^{\eta-1,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)}{\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)} - \left(\frac{\gamma+\delta-r}{\nu+\tau-1}\right) - 1\right) \tag{4.4}$$

$$\frac{\mu}{z}\left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right)\left(\frac{\mathcal{W}^{\eta-1,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)}{\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}\, f(z)} - \left(\frac{\gamma+\delta-r}{\nu+\tau-1}\right) - 1\right) + p(z) \prec q(z) + \frac{\alpha}{\mu} zq'(z).$$

By performing a straightforward computation and utilizing (1.6), we can derive from (4.4):

$\frac{\mu}{z}\left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right)\left(\frac{\mathcal{W}^{\eta-1,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)}{\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)}-\left(\frac{\gamma+\delta-r}{\nu+\tau-1}\right)-1\right)=\frac{\alpha}{\mu}z\wp'(z).$

By applying Lemma 2.4, we achieve the desired result.

When we substitute $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ into Theorem 4.1, we get this result:

Corollary 4.1: Assume that $\mu>0,\alpha\in\mathbb{C}\backslash\{0\}$ and $(-1\le B<A\le 1)$, where $\left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)\right)^{\mu}\in\mathcal{H}[q(0),1]\cap Q$.

If $\wp(z)$ described by (3.4) is univalent in U and if f belongs to $\mathcal{A}^*$and adheres to the following condition superordination:

$$\frac{1+Az}{1+Bz}+\frac{\alpha}{\mu}\frac{(A+B)z}{(1+Bz)^2}\prec\wp(z), \tag{4.5}$$

then $\frac{1+Az}{1+Bz}\prec\left(z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)\right)^{\mu}$, and $\frac{1+Az}{1+Bz}$ is recognized as the best subordinant.

Theorem 4.2: Consider $q(z)$ as a convex univalent function within the unit disk U, with $q'(z)\neq 0$. Let t be a complex number in $\mathbb{C}\backslash\{0\}$, ζ be a positive real number, and m belong to $\mathbb{C}$. Suppose that $q(z)$ satisfies

$$Re\left\{\frac{\zeta q'(z)}{t}\right\}>0. \tag{4.6}$$

Assume that $f(z)$ belongs to the set $\mathcal{A}^*$ and satisfies the following condition:

$$\left((1-\rho)z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)+\rho z\mathcal{W}^{\eta+1,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)\right)^{\mu}\in\mathcal{H}[q(0),1]\cap Q,$$

Further,

$$\left((1-\rho)z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)+\rho z\mathcal{W}^{\eta+1,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)\right)^{\mu}\neq 0.$$

If the function $\wp(z)$ given by (3.8) is univalent in U and

$$\zeta q(z)+ztq'(z)\prec G(z), \tag{4.7}$$

then, $q(z)\prec\left((1-\rho)z\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)+\rho z\mathcal{W}^{\eta+1,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)\right)^{\mu}$,

and $q(z)$ is recognized as the best subordinant.

Proof: Consider $\wp(z)$ that given in U by (3.8). A calculation reveals that

$$\frac{zq'(z)}{q(z)}=\mu\left(\frac{\nu+\tau-1}{\gamma+\delta-r}\right)\left[\frac{(1-\rho)\mathcal{W}^{\eta-1,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)-(1-2\rho)\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)-\rho\mathcal{W}^{\eta+1,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)}{(1-\rho)\mathcal{W}^{\eta,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)+\rho\mathcal{W}^{\eta+1,s}_{\nu,\gamma,\delta,\tau,n,r,d}f(z)}\right]. \tag{4.8}$$

Set $\theta(m)=\zeta m$ and $\psi(m)=t$.

It follows that $\theta(m)$ and $\psi(m)$ are analytic in $\mathbb{C}\setminus\{0\}$.

Moreover, $\psi(m)\neq 0, m\in\mathbb{C}\setminus\{0\}$. As well as, $Q(z)=zq'(z)\,\psi\big(q(z)\big)=z\varepsilon\frac{q'(z)}{q(z)}$.

Clearly, $Q(z)$ is starlike univalent in U, we have

$$Re\left\{\frac{\theta'(q(z))}{\psi\big(q(z)\big)}\right\}=Re\left\{\frac{\zeta q'(z)}{t}\right\}>0.$$

Through a simple computation, it can be concluded that:

$$G(z)=\theta\big(\wp(z)\big)+\wp(z)=\zeta\wp(z)+zt\wp'(z), \tag{4.9}$$

where $G(z)$ is defined by (3.7).

It can be deduced from both (4.7) and (4.9) that

$$\zeta q(z)+ztq'(z)\prec\zeta\wp(z)+zt\wp'(z). \tag{4.10}$$

Hence, when we employ Lemma 2.3, we obtain that $q(z)\prec\wp(z)$, and the result is follow. □

5. Sandwich results

The Sandwich Theorem is a consequent of combining Theorem 3.1 with Theorem 4.1.

Theorem 5.1: Suppose $q_1$and q_2 are a convex univalent function in U with $q_1(0) = q_2(0) = 1$ and q_2 meets (3.1). Let $Re\{\alpha\} > 0, \mu > 0, \alpha \in \mathbb{C}\backslash\{0\}$.
Consider $q_1$and q_2 are convex univalent functions within the unit disk U, both having $q_1(0) = q_2(0) = 1$ and q_2 satisfies (3.1). Let $Re\{\alpha\} > 0, \mu > 0$, and α belongs to $\mathbb{C}\backslash\{0\}$.
If $f \in \mathcal{A}^*$, such that

$$\left(z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z)\right)^{\mu} \in \mathcal{H}[q(0),1] \cap Q.$$

Moreover, the function $p(z)$ defined as in (3.4) is univalent and it meets this condition

$$q_1(z) + \frac{\alpha}{\mu} z q_1'(z) \prec p(z) \prec q_2(z) + \frac{\alpha}{\mu} z q_2'(z), \tag{5.1}$$

implies that

$$q_1(z) \prec \left(z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z)\right)^{\mu} \prec q_2(z).$$

Here, q_1 and q_2 represent the best subordinant and the best dominant of (5.1), respectively. Furthermore, when Theorem 3.2 is combined with Theorem 4.2, it results in the Sandwich Theorem:

Theorem 5.2: Suppose that q_i are two convex univalent functions in U, with $q_i(0) = 1, q_i'(z) \neq 0$ $(i = 1,2)$. Let q_1 and q_2 meets (4.6) and (3.5), respectively.
If $f \in \mathcal{A}^*$ and we assume that f fulfills the following conditions:

$$\left(z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z)\right)^{\mu} \neq 0,$$

and

$$\left(z\mathcal{W}_{\nu,\gamma,\delta,\tau,n,r,d}^{\eta,s} f(z)\right)^{\mu} \in \mathcal{H}[q(0),1] \cap Q,$$

and $\psi(z)$ is univalent in U, then

$$\zeta q_1(z) + zt q_1'(z) \prec \psi(z) \prec \zeta q_2(z) + zt q_2'(z). \tag{5.2}$$

This implies that $q_1(z)$ is subordinate to $\psi(z)$, which is subordinate to $q_2(z)$. In this context, $q_1(z)$ and $q_2(z)$serve as the best subordinant and the best dominant, respectively, while $\psi(z)$ follows the form described in (3.7).

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