Al-Hawasy et al. Iraqi Journal of Science, 2025, Vol. 66, No. 3, pp: 1186-1199
DOI: 10.24996/ijs.2025.66.3.18

ISSN: 0067-2904

Mixed Galerkin - Implicit Differences Methods for Solving a Coupled Nonlinear Parabolic System with Variable Coefficients

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Received: 2/11/2023 Accepted: 11/3/2024 Published: 30/3/2025

Abstract

The approximate solution of a coupled nonlinear parabolic system with variable coefficients (CNPSVC) is found by using the mixing Galerkin finite element method (GFEM) in the variable of space with implicit finite difference method (IFDM) in the variable of time, for this reason, the method will be denoted by MGIM. In this method and at any step of time t_j the CNPSVC is transformed into couple Galerkin nonlinear algebraic system (CGNAS), which is solved by applying the predictor and the corrector techniques (PCT), these techniques transform the CGNAS into a coupled Galerkin linear algebraic system (CGLAS). Then the Cholesky decomposition (ChDe) is used to solve it. The existence and uniqueness of the solution are proven. The stability and the convergence of the method are studied. Some Illustrative examples are given to solve the proposed system, the results are given by tables and figures and we show the accuracy and effectiveness of the proposed method.

Keywords: Convergence, Coupled Nonlinear Parabolic System with Variable Coefficients, Cholesky Decomposition Method, Galerkin Finite Element Method, Implicit Difference Method, Predictor - Corrector Techniques, Stability.

مزج طريقتي كاليركن والفروقات المنتهية الضمنية لحل نظام معادلات مقترنة من نوع القطع المكافئ غير خطية ذات معاملات متغيرة

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الخلاصة

في هذا البحث تم إيجاد الحل التقريبي لنظام متكون من زوج من معادلات مقترنة من نوع القطع المكافئ غير خطية ذات معاملات متغيرة باستخدام طريقة كاليركن للعناصر المنتهية بالنسبة لمتغير الازاحة مع طريقة ضمنية للفروقات المنتهية بالنسبة لمتغير الزمن. في هذه الطريقة تم تحويل زوج من مسائل القيم

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الحدودية من نوع القطع المكافئ ذات معاملات متغيرة إلى نظام كاليركن الجبري غير الخطي، والذي تم حله باستخدام تقنية التخمين والتصحيح والتي تحول نظام كاليركن الى الى نظام جبري خطي ومن ثم يثم حل النظام باستخدام طريقة جولسكي. تمت برهنة الوجود ووحدانية الحل لهذه المسألة . تم دراسة الاستقرارية والتقارب للطريقة. تم اعطاء مثالين توضيحين لحل زوج من مسائل القيم الحدودية من نوع القطع المكافئ ذات معاملات متغيرة، أعطيت النتائج على شكل جداول ورسومات حيث بينا الكفاءة والدقة للطريقة.

1. Introduction

A wide range of applications in natural science, engineering, and technology, are described generally by mathematical models which lead to nonlinear ODEs [1,2] or PDEs [3-8]. Usually, such of these problems are solved in general by using numerical methods [9,10]. In particular, different numerical methods can be used for solving the parabolic type of these problems with variable coefficients, such as, the Crank-Nicolson scheme with the GFEM [11] in 2019, the method of simple algorithm [12] in 2020, the GFEM [13] in2021. Later, in 2022 the Galerkin method with the Bivariate Bernstein [14], the piecewise constant arguments [15], recently and in 2023 the spline collocation method [16], the orthogonal cubic splines [17] were used to solve the considered problem. Therefore, great attention is given in this paper to investigate the solution of the CNPSVC by using the MGIM.

This paper starts with describing the considered problem and its weak form (WF). The approximation problem (App) obtained from the discretization of the CNPSVC by using the GFEM for the space-variable and the IFDM for the time-variable, then at any step of time (t_j) the problem is reduced to solving CGNLAS which is transformed to CGLAS after using the PCT. Then the ChDe is applied to solve it. The existence and uniqueness of the solution are proven. The stability and the convergence of the method are studied. Finally, illustrative examples are given to solve different problems using the MGIM, the results show the efficiency and the accuracy of this method.

2. Description of the CNPSVC

Let $\Omega = \{\vec{x} = (x_1, x_2) \in R^2\colon\ a < x_1, x_2 < b\} \subset R^2$, be the region with boundary $\partial\Omega$, and let $I = [0, T]$, $Q = \Omega \times I$, $0 < T < \infty$, then the CNPSVC are given by:

$$U_{1t} - \sum_{r,s=1}^{2} \frac{\partial}{\partial x_s}\left[a_{rs}(\vec{x},t)\frac{\partial U_1}{\partial x_r}\right] + h_1(\vec{x},t)U_1 - g(\vec{x},t)U_2 = w_1(\vec{x},t), \text{in } Q \quad (1)$$

$$U_{2t} - \sum_{r,s=1}^{2} \frac{\partial}{\partial x_s}\left[b_{rs}(\vec{x},t)\frac{\partial U_2}{\partial x_r}\right] + h_2(\vec{x},t)U_2 + g(\vec{x},t)U_1 = w_2(\vec{x},t), \text{in } Q \quad (2)$$

with the initial conditions (ICs)

$$U_i(\vec{x},0) = U_i^0(\vec{x}), \text{in } \Omega, (i = 1,2) \quad (3)$$

and the boundary conditions (BCs)

$$U_i(\vec{x},t) = 0, \text{on } \partial\Omega \times I, (i = 1,2) \quad (4)$$

The classical vector solution of system (Eq.(1)-Eq.(4)) is $\vec{U}(\vec{x},t) = (U_1(\vec{x},t), U_2(\vec{x},t)) \in (H^2(Q))^2$, $a_{rs}(\vec{x},t), b_{rs}(\vec{x},t)$ $(\forall r,s = 1,2), h_r(\vec{x},t)$ and $g(\vec{x},t)$ are positive nonzero arbitrary functions in $L^\infty(Q)$, $w_r(\vec{x},t)$ are given functions in $L^2(Q)$ for all $\vec{x} \in \Omega$ $(\forall r = 1,2)$. Let $(.,.)$ &$\|.\|_0$ $((.,.)_1$& $\|.\|_1)$ be symbolized to the inner product and the norm in $L^2(\Omega)$(in $V = H_0^1(\Omega)$).

3. The WF of the Problem

Let $H_0^1(\Omega) = \{v\colon v = v(\vec{x}) \in H^1(\Omega), \forall\vec{x} \in \Omega, with\ v = 0\ on\ \partial\Omega\}$, then the WF of the CNPSVC (Eq.(1)-Eq.(4)) is given ($\forall\ \vec{U} \in (H_0^1(Q))^2$, and $\vec{V} \in (H_0^1(\Omega))^2$) by

$$\langle U_{1t}, v_1\rangle + a_1(t, U_1, v_1) - (g(t)U_2, v_1) = (w_1(U_1), v_1)\ , \text{in } Q \quad (5)$$

$(U_1(0), v_1) = (U_1^0, v_1)$, in Ω (6)
$\langle U_{2t}, v_2\rangle + a_2(t, U_2, v_2) + (g(t)U_1, v_2) = (w_2(U_2), v_2)$, in Q (7)
$(U_2(0), v_2) = (U_2^0, v_2)$, in Ω (8)

where the following are bilinear form
$a_p(t, U_p, v_p) = \sum_{r,s=1}^{2} a_{rs}(\vec{x}, t)\left(\frac{\partial U_p}{\partial x_s}, \frac{\partial v_p}{\partial x_r}\right) + h_p(\vec{x}, t)(U_p, v_p)$, for $p = 1,2$.
The following hypotheses are useful in the study of the problem.

3.1 Assumptions

i-Let $B, \bar{B} \in R^+$ s.t the following are held:
a) $|a(t, \vec{U}, \vec{v})| \le B\|\vec{U}\|_1 \|\vec{v}\|_1$, b) $(t, \vec{U}, \vec{v}) \ge \bar{B}\|\vec{U}\|_1^2$, with $a(t, \vec{U}, \vec{U}) = \sum_{r=1}^{2} a_r(t, U_r, U_r)$.
ii- $w_r(\vec{x}, t, U_r)$ $(\forall r = 1,2)$ is of the type of Caratheodory on $Q \times R$,satisfies the following
a) $|w_r(\vec{x}, t, U_r)| \le \eta_r(\vec{x}, t) + d_r|U_r|$ where $d_r > 0$, $(\vec{x}, t) \in Q, U_r \in R$ and $\eta_r \in L^2(Q, R)$.
b) $|w_r(\vec{x}, t, U_r) - w_r(\vec{x}, t, \bar{U}_r)| \le L_r|U_r - \bar{U}_r|$, where $(\vec{x}, t) \in Q, U_r, \bar{U}_r \in R$, L_r is a Lipischitz constant, $\forall r = 1,2$.
In the following section the discretization for the WF(Eq.(5)-Eq.(8)) is obtained.

4. Discretization of the WF

The WF of (Eq.(5)-Eq.(8)) is discretized through applying the GFEM[18], let $V_N \subset H_0^1(\Omega)$ be a finite dimensional space, and assume $M_1 \in Z^+$, $\bar{\Omega} = \bigcup_{k=1}^{N} O_k$ with $O = O_k^n, k = 1,2, \ldots, N$, $N = M^2$ where $M = M_1 - 1$ be an "admissible" triangulation of $\bar{\Omega}$, let $h = {}^{1}/_{M_1}$, $x_{i1} = ih$ and $x_{i2} = ih$, $(i = 0,1,2, \ldots, M_1)$ be points in $\bar{\Omega}$ s.t
$0 = x_{0p} < x_{1p} < \cdots < x_{ip} < \cdots < x_{(M_1)p} = 1$, with $p = 1,2$.
The interval of the variable of time is divided into the subintervals $I_j = I_j^n \coloneqq [t_j^n, t_{j+1}^n]$with $t_j = j\Delta t$, $j = 0,1, \ldots, NT - 1$ and $\Delta t = \frac{T}{NT}$, $NT \in Z^+$. The discrete WF (DWF) of ((5)-(8)) are written ($\forall$ $v_r \in V_N, r = 1,2$) as follows:
$\left({U_1^n}_{j+1} - {U_1^n}_j, v_1\right) + \Delta t[a_1\left({U_1^n}_{j+1}, v_1\right) - \left(g(t_{j+1}){U_2^n}_{j+1}, v_1\right)] = \Delta t\left(w_1\left({U_1^n}_{j+1}\right), v_1\right)$,
(9)
$(U_1^n(0), v_1) = (U_1^0, v_1)$in Ω (10)
$\left({U_2^n}_{j+1} - {U_2^n}_j, v_2\right) + \Delta t[a_2\left({U_2^n}_{j+1}, v_2\right) + \left(g(t_{j+1}){U_1^n}_{j+1}, v_2\right)] = \Delta t\left(w_2\left({U_2^n}_{j+1}\right), v_2\right)$,(11)
$(U_2^n(0), v_2) = (U_2^0, v_2)$ in Ω (12)

Now, to find the APPS of the CNPSVC, the following algorithm is utilized:

5. Algorithm

To find the APPS $\vec{U}^n = (U_1^n, U_2^n)$ of (Eq.(9)-Eq.(12)), using the GFEM, let $V_N \subset H_0^1(\Omega)$ (be a piecewise affine functions) of dimension N, s.t $V_N = \{v_m, m = 1,2, \ldots, N, with\ v_m(\vec{x}) = 0\ on\ \partial\Omega\}$, let $\vec{V}_N = V_N \times V_N$ then the following proceedings can be applied:
Step1: The WF (Eq.(9)-(Eq.12)) can be written (for $\vec{U}^n \in \vec{V}_N, v_m \in V_N, \forall\ m = 1,2, \ldots, N$) as:
$\left({U_1^n}_{j+1} - {U_1^n}_j, v_i\right) + \Delta t a_1\left({U_1^n}_{j+1}, v_i\right) - \Delta t\left(g(t_{j+1}){U_2^n}_{j+1}, v_m\right) =$
$\Delta t\left(w_1\left(t_{j+1}, {U_1^n}_{j+1}\right), v_m\right)$, (13)
$(U_1^n(0), v_m) = (U_1^0, v_m)$ (14)

$$\left(U^n_{2\,j+1}-U^n_{2\,j},v_m\right)+\Delta t a_2\left(U^n_{2\,j+1},v_m\right)+\Delta t\left(g(t_{j+1})U^n_{1\,j+1},v_m\right)=\Delta t\left(w_2\left(t_{j+1},U^n_{2\,j+1}\right),v_m\right) \quad (15)$$

$$(U^n_2(0),v_m)=(U^0_2,v_m) \quad (16)$$

Step2: Utilizing the GFEM, the APPS $U^n_r(\forall\, r=1,2)$ is approximated using the basis $(v_1,v_2,\dots,v_N)$ of V_N, i.e.
$U^n_1(\vec{x},t_j)=\sum_{k=1}^{N}a_k(t_j)v_k(\vec{x})$, $U^n_2(\vec{x},t_j)=\sum_{k=1}^{N}a_{k+N}(t_j)v_k(\vec{x})$, $\forall j=0,1,\dots,NT-1$,
where $a_k(t_j)$ and $a_{k+N}(t_j)$ are unknown coefficients to be found

Step3: Substitute $\vec{U}^n$ in (Eq.(13)-Eq.(16)), the following CGNAS with their ICs are obtained,

$$(C+\Delta tD)A^{j+1}_k-\Delta tEA^{j+1}_{k+N}=CA^j_k+\Delta t\vec{b}_1(t_{j+1},A^{j+1}_k) \quad (17)$$

$$(C+\Delta tF)A^{j+1}_{k+N}+\Delta tEA^{j+1}_k=CA^j_{k+N}+\Delta t\vec{b}_2(t_{j+1},A^{j+1}_{k+N}) \quad (18)$$

$$CA_k(0)=\vec{b}^0_1 \quad (19)$$

$$CA_{k+N}(0)=\vec{b}^0_2 \quad (20)$$

Where$C=(c_{mk})_{N\times N},c_{mk}=(v_k,v_m),D=(d_{mk})_{N\times N},d_{mk}=a_1(v_k,v_m),F=(f_{mk})_{N\times N}$, $f_{mk}=a_2(v_k,v_m)$,$a_1(v_k,v_m)=\sum_{r,s=1}^{2}a_{rs}(\vec{x})\left(\frac{\partial v_k}{\partial x_s},\frac{\partial v_m}{\partial x_r}\right)+(h_1(\vec{x})v_k,v_m)$, $E=(e_{mk})_{N\times N}$, $e_{mk}=(g(\vec{x})v_k,v_m)$,$a_2(v_k,v_m)=\sum_{r,s=1}^{2}b_{rs}(\vec{x})\left(\frac{\partial v_k}{\partial x_s},\frac{\partial v_m}{\partial x_r}\right)+(h_2(\vec{x})v_k,v_m)$,$A_k(t_j)=\left(a_k(t_j)\right)_{N\times 1}$,$A_{k+N}(t_j)=\left(a_{k+N}(t_j)\right)_{N\times 1}$,$\vec{b}_1=(b_{1m})_{N\times 1},b_{1m}=\left(w_1(\vec{x},t_{j+1},\vec{r}^{\,T}_1A^{j+1}_k),v_m\right)$, $\vec{b}_2=(b_{2i})_{N\times 1}$,$b_{2i}=\left(w_2(\vec{x},t_{j+1},\vec{r}^{\,T}_1A^{j+1}_{k+N}),v_m\right)$, $\vec{b}^0_r=(b^0_{rm})_{N\times 1}$, $b^0_{rm}=(U^0_r,v_m)$, $\forall\, m,k=1,2,\dots,N$, and $r=1,2$.

Step 4: The uniqueness of (Eq.(17)-Eq.(20)) is obtained, since C,D,E and F are positive definite and symmetric. The CGLAS (Eq.(19)-Eq.(20)) are solved (by the ChDe) at first to get A^0_k *and* A^0_{k+N} respectivaly, then to solve the CGNAS (Eq.(17)-Eq.(18)), the PCT is utilized here [18]. In the PT, set $A^{j+1}_{k+(l^*-1)N}=A^{j}_{k+(l^*-1)N}$, $l^*=1,2(\forall j=0,1,2,\dots,NT-1)$ in the R.H.S of (Eq.(17)-Eq.(18)), which turn to CGLAS, then solving them (by the ChDe) to get the predictor solution (PS) $A^{j+1}_{k+(l^*-N)}$. Next in the corrector technique (CT) substitute $\bar{A}^{j+1}_{k+(l^*-1)N}=A^{j+1}_{k+(l^*-1)N}$ in the R.H.S of (Eq.(17)-Eq.(18)) (i.e in $\vec{b}_1$ *and* $\vec{b}_2$), and then solving them (by the ChDe) to get the corrector solution (CS) $A^{j+1}_{k+(l^*-1)N}$ (this PCT can be repeated for more than one time by solving (Eq.(17)-Eq.(18)) after setting the CS $\bar{A}^{j+1}_{k+(l^*-1)N}=A^{j+1}_{k+(l^*-1)N}$ in the R.H.S of them, to get a new CS).
In the next subsection, the ChDe method which was indicated in the above algorithm is illustrated.

5.1 The ChDe Method

To solve the GLAS, the ChDe is utilized if the matrix A is symmetric and positive definite, and hence it can be decomposed into a product of a unique lower triangular matrix K and its transpose [19]. The ChDe can be represented in the following steps:

Step1: $K_{pp}=\left(a_{pp}-\sum_{z=1}^{p-1}K^2_{pz}\right)^{1/2}$ $for\ p=1,2,\dots,N$

Step2: $K_{pq}=\frac{a_{pq}-\sum_{z=1}^{q-1}K_{qz}.K_{pz}}{K_{qq}}$ $for\ q=p+1,\dots,N.$

The CT which was mentioned in the above algorithm can be expressed as follows:

$$\left(\vec{U}_{j+1}^{(l+1)} - \vec{U}_j, \vec{v}_i\right) + \Delta t a\left(\vec{U}_{j+1}^{(l+1)}, \vec{v}_i\right) - \Delta t\left(U_{2}{}_{j+1}^{(l+1)}, v_i\right) + \Delta t\left(U_{1}{}_{j+1}^{(l+1)}, v_i\right) = \Delta t\left[\left(w_1\left(U_1^{(l)}{}_{j+1}\right), v_i\right) + \left(w_2\left(U_2^{(l)}{}_{j+1}\right), v_i\right)\right], i = 1,2, \dots, N \quad (21)$$

where $\vec{U}_{j+1}^{(l)} = \vec{U}_{j+1}^{n}$ represents the (l) iteration of the PS, $\vec{U}_{j+1}^{(l+1)} = \vec{U}_{j+1}^{n}$ represents the $(l+1)$ iteration of the CS and $\vec{U}_j = \vec{U}_j^n$represents the CS for the previous step j, hence Eq.(21) can be written as:

$$\vec{U}^{(l+1)} = H(\vec{U}^{(l)}) \quad (22)$$

6. Existence and Uniqueness of Solution

Before we give the illustrative examples of the proposed method, it is necessary to prove the uniqueness for solutions of the DWF(Eq.(9) – Eq.(12)). Moreover, we show that the sequence of CS is convergent.

Theorem 1 (Existence and Uniqueness of Solution)

The DWF (Eq.(9) – Eq.(12)) with fixed point j $(0 \le j \le NT - 1)$ and for Δt "sufficiently" small, the problem has a unique solution $\vec{U}^n = (U_1^n, U_2^n) = (U_{10}^n, U_{11}^n, \dots, U_{1N}^n, U_{20}^n, U_{21}^n, \dots, U_{2N}^n)$ and the sequence of CS is convergent in R^2.

Proof: Let $\vec{U}^{(l+1)} = \left(U_1^{(l+1)}, U_2^{(l+1)}\right) = (U_{10}^{(l+1)}, U_{11}^{(l+1)}, \dots, U_{1N}^{(l+1)}, U_{20}^{(l+1)}, U_{21}^{(l+1)}, \dots, U_{2N}^{(l+1)})$ and $\vec{\overline{U}}^{(l+1)} = \left(\overline{U}_1^{(l+1)}, \overline{U}_2^{(l+1)}\right) = (\overline{U}_{10}^{(l+1)}, \overline{U}_{11}^{(l+1)}, \dots, \overline{U}_{1N}^{(l+1)}, \overline{U}_{20}^{(l+1)}, \overline{U}_{21}^{(l+1)}, \dots, \overline{U}_{2N}^{(l+1)})$

Are two CS of Eq. (21),

i.e

$$\left(\vec{U}_{j+1}^{(l+1)} - \vec{U}_j, \vec{v}_i\right) + \Delta t a\left(\vec{U}_{j+1}^{(l+1)}, \vec{v}_i\right) - \Delta t\left(U_{2}{}_{j+1}^{(l+1)}, v_i\right) + \Delta t\left(U_{1}{}_{j+1}^{(l+1)}, v_i\right) = \Delta t\left[\left(w_1\left(U_1^{(l)}{}_{j+1}\right), v_i\right) + \left(w_2\left(U_2^{(l)}{}_{j+1}\right), v_i\right)\right] \quad (23)$$

$$\left(\vec{\overline{U}}_{j+1}^{(l+1)} - \vec{\overline{U}}_j, \vec{v}_i\right) + \Delta t a\left(\vec{\overline{U}}_{j+1}^{(l+1)}, \vec{v}_i\right) - \Delta t\left(\overline{U}_{2}{}_{j+1}^{(l+1)}, v_i\right) + \Delta t\left(\overline{U}_{1}{}_{j+1}^{(l+1)}, v_i\right) = \Delta t\left[\left(w_1\left(\overline{U}_1^{(l)}{}_{j+1}\right), v_i\right) + \left(w_2\left(\overline{U}_2^{(l)}{}_{j+1}\right), v_i\right)\right], i = 1,2, \dots, N \quad (24)$$

By subtracting Eq.(24) from Eq.(23), and setting $\vec{v}_i = \vec{U}_{j+1}^{(l+1)} - \vec{\overline{U}}_{j+1}^{(l+1)}$ in the obtained equation and using assumption (ii-b), it yields

$$\left\|\vec{U}_{j+1}^{(l+1)} - \vec{\overline{U}}_{j+1}^{(l+1)}\right\|_0^2 + \Delta t a\left(\vec{U}_{j+1}^{(l+1)} - \vec{\overline{U}}_{j+1}^{(l+1)}, \vec{U}_{j+1}^{(l+1)} - \vec{\overline{U}}_{j+1}^{(l+1)}\right) \le$$
$$\Delta t L_1\left(\left|U_{1j+1}^{(l)} - \overline{U}_{1j+1}^{(l)}\right|, \left|U_{1j+1}^{(l+1)} - \overline{U}_{1j+1}^{(l+1)}\right|\right) + \Delta t L_2\left(\left|U_{2j+1}^{(l)} - \overline{U}_{2j+1}^{(l)}\right|, \left|U_{2j+1}^{(l+1)} - \overline{U}_{2j+1}^{(l+1)}\right|\right)$$
$$\le \Delta t L\left\|\vec{U}_{j+1}^{(l)} - \vec{\overline{U}}_{j+1}^{(l)}\right\|_0\left\|\vec{U}_{j+1}^{(l+1)} - \vec{\overline{U}}_{j+1}^{(l+1)}\right\|_0 \quad , \quad \text{where } L = \max(L_1, L_2)$$

The 2^{nd} term in the LHS is nonnegative (assumption (i-b)) and then utilizing inequality of the Caushy Schwarz on the RHS of above inequality, it yields that

$$\left\|H(\vec{U}_{j+1}^{(l+1)}) - H(\vec{\overline{U}}_{j+1}^{(l+1)})\right\| = \left\|\vec{U}_{j+1}^{(l+1)} - \vec{\overline{U}}_{j+1}^{(l+1)}\right\|_0 \le \alpha\left\|\vec{U}_{j+1}^{(l)} - \vec{\overline{U}}_{j+1}^{(l)}\right\|_0, \text{ where } \alpha = \Delta t L .$$

The contractive of H (i.e. $\alpha < 1$) is obtained from the sufficiently small value of Δt. Hence the DWF (Eq.(9) – Eq.(12)) has a unique solution. On the other hand, $\vec{U}^{(l)} \in R^2, \forall\, l$ and $g\left(\vec{U}^{(l)}\right) = \vec{U}^{(l+1)}, \forall\, l$, then by theorem 3 in [20], the $\{\vec{U}^{(l)}\}$ is convergent to a point $\vec{U}$ in R^2.

7. Stability and the Convergence of the Method

In this section, to study the convergence of the APPS for the DWF (Eq.(9) – Eq.(12)), to solution of the WF (Eq.(5)-Eq.(8)), firstly the stability of the discrete solution must be studied.

7.1 Stability

Lemma 2: For sufficiently small Δt, the following are satisfied

$\|\vec{U}_j^n\|_0^2 \le C_1(\tilde{d}_5), \forall\, j = 0,1,2,\dots,NT-1,NT$

$\Delta t \sum_{j=0}^{NT-1}\|\vec{U}_j^n\|_0^2 \le C(\tilde{d}_5)$, $\sum_{j=0}^{NT-1}\|\vec{U}_{j+1}^n - \vec{U}_j^n\|_0^2 \le C(\tilde{d}_5)$ and $\Delta t \sum_{j=0}^{NT-1}\|\vec{U}_j^n\|_1^2 \le \tilde{d}_7$.

Proof: By substituting $v_1 = 2\Delta t U_{1\,j+1}^n$, $v_2 = 2\Delta t U_{2\,j+1}^n$ in (Eq. (9)- Eq. (11)), rewriting the terms in the L.H.S of the obtained equation by the norms, then adding the resulting equation together, it follows that

$$\left\|U_{1\,j+1}^n\right\|_0^2 - \left\|U_{1\,j}^n\right\|_0^2 + \left\|U_{1\,j+1}^n - U_{1\,j}^n\right\|_0^2 + \left\|U_{2\,j+1}^n\right\|_0^2 - \left\|U_{2\,j}^n\right\|_0^2 + \left\|U_{2\,j+1}^n - U_{2\,j}^n\right\|_0^2 + 2\Delta t\left[a_1\left(U_{1\,j+1}^n, U_{1\,j+1}^n\right) + a_2\left(U_{2\,j+1}^n, U_{2\,j+1}^n\right)\right]$$
$$= 2\Delta t\left[w_1\left(U_{1\,j+1}^n, U_{1\,j+1}^n\right) + w_2\left(U_{2\,j+1}^n, U_{2\,j+1}^n\right)\right] \tag{25}$$

Since

$$\|\vec{U}_{j+1}^n\|_0^2 = \left\|U_{1\,j+1}^n\right\|_0^2 + \left\|U_{2\,j+1}^n\right\|_0^2, \forall j = 0,1,\dots NT-1 \tag{26}$$

$$\|\vec{U}_j^n\|_0^2 = \left\|U_{1\,j}^n\right\|_0^2 + \left\|U_{2\,j}^n\right\|_0^2, \forall j = 0,1,\dots NT-1,NT \tag{27}$$

$$a(\vec{U}_{j+1}^n, \vec{U}_{j+1}^n) = a_1\left(U_{1\,j+1}^n, U_{1\,j+1}^n\right) + a_2\left(U_{2\,j+1}^n, U_{2\,j+1}^n\right) \tag{28}$$

Substituting (Eq.(26)- Eq. (28)) in Eq. (25), we get

$$\|\vec{U}_{j+1}^n\|_0^2 - \|\vec{U}_j^n\|_0^2 + \|\vec{U}_{j+1}^n - \vec{U}_j^n\|_0^2 + 2\Delta t a(\vec{U}_{j+1}^n, \vec{U}_{j+1}^n)$$
$$= 2\Delta t\left[w_1\left(U_{1\,j+1}^n, U_{1\,j+1}^n\right) + w_2\left(U_{2\,j+1}^n, U_{2\,j+1}^n\right)\right] \tag{29}$$

Now, from the assumptions on w_1 and w_2 , one can write the term in R.H.S of Eq. (29) as

$$\left|2\Delta t\left[w_1\left(U_{1\,j+1}^n, U_{1\,j+1}^n\right) + w_2\left(U_{2\,j+1}^n, U_{2\,j+1}^n\right)\right]\right| \le \Delta t\left[\int_\Omega \eta_{1j}^2 dx + \int_\Omega \left|U_{1\,j+1}^n\right|^2 dx\right] +$$
$$\Delta t\left[d_1\int_\Omega \left|U_{1\,j+1}^n\right|^2 dx + d_1\int_\Omega \left|U_{1\,j+1}^n\right|^2 dx\right] + \Delta t\left[\int_\Omega \eta_{2j}^2 dx + \int_\Omega \left|U_{2\,j+1}^n\right|^2 dx\right] +$$
$$\Delta t\left[d_2\int_\Omega \left|U_{2\,j+1}^n\right|^2 dx + d_2\int_\Omega \left|U_{2\,j+1}^n\right|^2 dx\right] \le \Delta t\left[\tilde{d}_1 + d_3\left\|U_{1\,j+1}^n\right\|_0^2 + \tilde{d}_2 + d_4\left\|U_{2\,j+1}^n\right\|_0^2\right] \tag{30}$$

where $d_3 = 1 + 2d_1, d_4 = 1 + 2d_2$.

Since $\left\|U_{r\,j+1}^n\right\|_0^2 \le 2\left\|U_{r\,j+1}^n - U_{r\,j}^n\right\|_0^2 + 2\left\|U_{r\,j}^n\right\|_0^2$, for $r = 1,2$

$$\|\vec{U}_{j+1}^n\|_0^2 \le 2\|\vec{U}_{j+1}^n - \vec{U}_j^n\|_0^2 + 2\|\vec{U}_j^n\|_0^2$$

Substituting these inequalities in Eq. (30), we obtain

$$2\Delta t\left[w_1\left(U_{1\,j+1}^n, U_{1\,j+1}^n\right) + w_2\left(U_{2\,j+1}^n, U_{2\,j+1}^n\right)\right]$$
$$\le \Delta t\left[\tilde{d}_1 + 2d_3\left\|U_{1\,j+1}^n - U_{1\,j}^n\right\|_0^2 + 2d_3\left\|U_{1\,j}^n\right\|_0^2\right]$$

$$+\Delta t\left[\tilde{d}_2+2d_4\left\|U_{2\ j+1}^{n}-U_{2\ j}^{n}\right\|_0^2+2d_4\left\|U_{2\ j}^{n}\right\|_0^2\right] \quad (31)$$

Then using Eq.(31) in Eq.(29), it follows that

$$\left[\left\|\vec{U}_{j+1}^{n}\right\|_0^2-\left\|\vec{U}_{j}^{n}\right\|_0^2+\left\|\vec{U}_{j+1}^{n}-\vec{U}_{j}^{n}\right\|_0^2\right]+2\Delta t a\left(\vec{U}_{j+1}^{n},\vec{U}_{j+1}^{n}\right)\le$$
$$\Delta t\left[\tilde{d}_3+d_5\left\|\vec{U}_{j+1}^{n}-\vec{U}_{j}^{n}\right\|_0^2\right]+\Delta t d_5\left\|\vec{U}_{j}^{n}\right\|_0^2,\ \tilde{d}_3=\left(\tilde{d}_1+\tilde{d}_2\right),\ d_5=2\max\left(d_3,d_4\right) \quad (32)$$

Taking the summation of Eq.(32) for $j=0\ \ to\ j=l-1$, drooping the common terms, and using the assumptions on $a(.,.)$, it becomes

$$\left\|\vec{U}_{l}^{n}\right\|_0^2-\left\|\vec{U}_{0}^{n}\right\|_0^2+\sum_{j=0}^{l-1}\left\|\vec{U}_{j+1}^{n}-\vec{U}_{j}^{n}\right\|_0^2+2\Delta t\sum_{j=0}^{l-1}a\left(\vec{U}_{j+1}^{n},\vec{U}_{j+1}^{n}\right)\le$$
$$\Delta t\left[\sum_{j=0}^{l-1}\tilde{d}_3+d_5\sum_{j=0}^{l-1}\left\|\vec{U}_{j+1}^{n}-\vec{U}_{j}^{n}\right\|_0^2+\Delta t\sum_{j=0}^{l-1}d_5\left\|\vec{U}_{j}^{n}\right\|_0^2\right] \quad (33)$$

Since $\left[\left(\sum_{j=0}^{l}a\left(\vec{U}_{j}^{n},\vec{U}_{j}^{n}\right)\right)-a\left(\vec{U}_{0}^{n},\vec{U}_{0}^{n}\right)\right]=\sum_{j=0}^{l-1}\left[a\left(\vec{U}_{j+1}^{n},\vec{U}_{j+1}^{n}\right)\right]$

By substituting this equality in Eq.(33), with $\tilde{d}_4=\tilde{d}_3 l\,\Delta t\le\tilde{d}_3 NT\Delta t=\tilde{d}_3 T\ge 0$, and adding the common terms, one can conclude that

$$\left\|\vec{U}_{l}^{n}\right\|_0^2+(1-d_5\Delta t)\sum_{j=0}^{l-1}\left\|\vec{U}_{j+1}^{n}-\vec{U}_{j}^{n}\right\|_0^2+2\Delta t\sum_{j=0}^{l-1}a\left(\vec{U}_{j}^{n},\vec{U}_{j}^{n}\right)-2\Delta t a\left(\vec{U}_{0}^{n},\vec{U}_{0}^{n}\right)\le$$
$$\tilde{d}_4+d_5\Delta t\sum_{j=0}^{l-1}\left\|\vec{U}_{j}^{n}\right\|_0^2+\left\|\vec{U}_{0}^{n}\right\|_0^2 \quad (34)$$

Since $\left\|\vec{U}_{0}^{n}\right\|_1\le\tilde{d}\rightarrow\left\|\vec{U}_{0}^{n}\right\|_0\le\tilde{d}$, then from the assumptions $a(.,.)$, it follows that

$2\Delta t a\left(\vec{U}_{0}^{n},\vec{U}_{0}^{n}\right)\le 2\Delta t\left|a\left(\vec{U}_{0}^{n},\vec{U}_{0}^{n}\right)\right|\le 2\Delta t\alpha\left\|\vec{U}_{0}^{n}\right\|_1^2\le 2\Delta t\alpha\tilde{d}=d_6$ and $\left\|\vec{U}_{0}^{n}\right\|_0^2=\tilde{d}_6$

Choosing Δt small such that $d_5\Delta t<1$, gives that $1-d_5\Delta t>0$, now substituting these results in Eq.(34), to obtain

$$\left\|\vec{U}_{l}^{n}\right\|_0^2+(1-d_5\Delta t)\sum_{j=0}^{l-1}\left\|\vec{U}_{j+1}^{n}-\vec{U}_{j}^{n}\right\|_0^2+2\Delta t\sum_{j=0}^{l-1}a\left(\vec{U}_{j}^{n},\vec{U}_{j}^{n}\right)$$
$$\le\tilde{d}_5++d_5\Delta t\sum_{j=0}^{l-1}\left\|\vec{U}_{j}^{n}\right\|_0^2,\ \text{where, } \tilde{d}_5=\tilde{d}_4+d_6+\tilde{d}_6\ge 0 \quad (35)$$

Now, since the 2^{nd} and 3^{nd} terms in L.H.S of Eq. (35) are positive, hence it can be written it in the following form:

$$\left\|\vec{U}_{l}^{n}\right\|_0^2\le\tilde{d}_5++d_5\Delta t\sum_{j=0}^{l-1}\left\|\vec{U}_{j}^{n}\right\|_0^2$$

Now, by applying the discrete Bellman - Gronwalls inequality [19], the above equality gives

$\left\|\vec{U}_{l}^{n}\right\|_0^2\le C_1^2\left(\tilde{d}_5\right)$, for every $l=0,1,\dots,NT$ so it is true for each $j, j=0,1,\dots NT-1,NT$, i.e

$$\left\|\vec{U}_{j}^{n}\right\|_0^2\le C_1\left(\tilde{d}_5\right)\forall j=0,1,\dots,NT-1,NT \quad (36)$$

Then,

$$\Delta t\sum_{j=0}^{NT-1}\left\|\vec{U}_{j}^{n}\right\|_0^2\le C\left(\tilde{d}_5\right),\ \text{where } C\left(\tilde{d}_5\right)=\Delta\text{tNT}C_1\left(\tilde{d}_5\right) \quad (37)$$

Using this result in R.H.S of Eq.(35) with the properties that the 1^{nd} and 3^{nd} terms in L.H.S of Eq.(35) are positive, it becomes (for $l=NT$)

$$\sum_{j=0}^{NT-1}\left\|\vec{U}_{j+1}^{n}-\vec{U}_{j}^{n}\right\|_0^2\le C\left(\tilde{d}_5\right),\ \text{where } C\left(\tilde{d}_5\right)=\left(\tilde{d}_5+d_5 C(d)\right)/\left(1-d_5\Delta t\right) \quad (38)$$

Finally the 1^{st} and 2^{nd} terms in Eq.(35), it becomes (for $l=NT$)

$2\Delta t\gamma\sum_{j=0}^{NT}\left\|\vec{U}_{j}^{n}\right\|_1^2\le\tilde{d}_5+d_5 C(d)=d_7$, or

$$\Delta t\sum_{j=0}^{NT}\left\|\vec{U}_{j}^{n}\right\|_1^2\le\tilde{d}_7\ where\ \tilde{d}_7=\frac{d_7}{2\gamma} \quad (39)$$

7.2 Convergence

In this section, we prove that the APPS $\vec{U}^n = (U_1^n, U_2^n)$for the DWF (Eq.(9) – Eq.(12)), is convergent to the solution $\vec{U} = (U_1, U_{12})$ of the WF (Eq.(5)-Eq.(8)).

The following definitions for the functions " almost everywhere on I " are useful in the proof of next theorem, so let

$\vec{U}_-^n(t) \coloneqq \vec{U}_j^n, t \in I_j^n, \forall\, j = 0,1, \dots, NT$

$\vec{U}_+^n(t) \coloneqq \vec{U}_{j+1}^n, t \in I_j^n, \forall\, j = 0,1, \dots, NT - 1$

Let $\vec{U}_\wedge^n(t)$ be an affine function on each I_j^n, such that

$\vec{U}_\wedge^n(t) \coloneqq \vec{U}_j^n, \forall\, j = 0,1, \dots, NT$.

Theorem 3 : The discrete APPS $\vec{U}_-^n(t), \vec{U}_+^n(t)$ and $\vec{U}_\wedge^n(t)$ converge strongly in $(L^2(Q))^2$, as $n \to \infty$

Proof: From lemma 2 $\left\|\vec{U}_j^n\right\|_0^2 \le C_1(\tilde{d}_5)$ and $\left\|\vec{U}_j^n\right\|_1^2 \le \tilde{d}_7, \forall\, j = 0,1, \dots, NT - 1$.

Then $\left\|\vec{U}_-^n\right\|_{L^2(Q)}^2, \left\|\vec{U}_+^n\right\|_{L^2(Q)}^2, \left\|\vec{U}_\wedge^n\right\|_{L^2(Q)}^2, \left\|\vec{U}_-^n\right\|_{L^2(I,V)}^2, \left\|\vec{U}_+^n\right\|_{L^2(I,V)}^2, \left\|\vec{U}_\wedge^n\right\|_{L^2(I,V)}^2$ are bounded inequality Eq.(41) gives

$\Delta t \sum_{j=0}^{NT-1} \left\|\vec{U}_{j+1}^n - \vec{U}_j^n\right\|_0^2 \le \Delta t C(\tilde{d}_5) \to 0\,, as\ \Delta t \to 0$, hence

$\vec{U}_+^n \to \vec{U}_-^n$ is strongly (ST) in $(L^2(Q))^2$ (40)

By using Alaoglus theorem in [19], there are subsequences of $(\{\vec{U}_-^n\},\{\vec{U}_+^n\},\{\vec{U}_\wedge^n\})$. Using the same notations again, s.t $\vec{U}_-^n \to \vec{U}$, $\vec{U}_+^n \to \vec{U}$, $\vec{U}_\wedge^n \to \vec{U}$ which are weakly convergent in $(L^2(Q))^2$ and in $(L^2(I,V))^2$.

Then by using the first compactness theorem [21], to get $\vec{U}_\wedge^n \to \vec{U}$ ST in $(L^2(Q))^2$. Also $\vec{U}_-^n \to \vec{U}$ and $\vec{U}_+^n \to \vec{U}$ ST in $(L^2(Q))^2$.

Now, let $\{\vec{V}_N\}_{N=1}$ be a sequence of subsequence of $\vec{V}$, such that $\forall\, \vec{v} = (v_1, v_2) \in \vec{V}$, then by using the Galerkin approach, for each $\vec{v} \in \vec{V}$, there is a $\{\vec{v}_N\}$ with $\vec{v}_N = (v_{1N}, v_{2N}) \in \vec{V}_N, \forall N, s.t\ \vec{v}_N \to \vec{v}$ ST in $\vec{V}$ then $\vec{v}_N \to \vec{v}$ in $(L^2(Q))^2$.

Consider that $\vec{\eta}(t) \in (C^2[0,T])^2$ for which $\vec{\eta}(T) = \acute{\vec{\eta}}(T) = 0$ and $\vec{\eta}(0) \ne 0, \acute{\vec{\eta}}(0) \ne 0$

Let $\vec{\eta}^n(t)$ be a piecewise continuous interpolation (PCI) of $\eta(t)$ w.r.t I_j^n, and let

$\vec{\xi} = \vec{v}\vec{\eta}(t)$& $\vec{\xi}^n = \vec{v}_N\vec{\eta}^n(t)$, with

$\vec{\xi}_-^n \coloneqq \vec{v}_N\vec{\eta}_-^n(t), t \in I_j^n, j = 0,1, \dots, NT - 1, \vec{v}_N \in \vec{V}_N$

$\vec{\xi}_+^n \coloneqq \vec{v}_N\vec{\eta}_+^n(t), t \in I_j^n, j = 0,1, \dots, NT - 1, \vec{v}_N \in \vec{V}_N$

$\vec{\xi}_\wedge^n \coloneqq \vec{v}_N\vec{\eta}^n(t), t \in I, \vec{v}_N \in \vec{V}_N$.

By substituting $\vec{v} = \vec{\xi}_{j+1}^n$ in (Eq.(9)-Eq.(11)), then summing both sides of the obtained equations for $j = 0$ to $j = NT - 1$, and using the discrete integration by parts (DIBP) for the 1^{st} term in the L.H.S of (Eq.(9)-Eq.(11))

$$-\int_0^T (U_{1+}^n, (\acute{\xi_{1\wedge}^n}))dt + \int_0^T [a_1(U_{1+}^n, \xi_{1+}^n) - (g(t_+^n)U_{2+}^n, \xi_{1+}^n)]dt = \int_0^T (w_1(t_+^n, U_{1+}^n), \xi_{1+}^n)\,dt - (U_{1T}^n, \xi_1^n(T)) + (U_{10}^n, \xi_1^n(0)) \quad (41)$$

$$-\int_0^T (U_{2+}^n, (\acute{\xi_{2\wedge}^n}))dt + \int_0^T [a_2(U_{2+}^n, \xi_{2+}^n) + (g(t_+^n)U_{1+}^n, \xi_{2+}^n)]dt = \int_0^T (w_2(t_+^n, U_{2+}^n), \xi_{2+}^n)\,dt - (U_{2T}^n, \xi_2^n(T)) + (U_{20}^n, \xi_2^n(0)) \quad (42)$$

Also since

$\vec{\eta}^n(t) \to \vec{\eta}(t)\ in\ (C(I))^2 \subset (L^2(I))^2, \vec{V}_N \to \vec{v}$ ST in $(L^2(I,V))^2$ and in $(L^2(\Omega))^2$, then $\vec{\xi}_+^n = \vec{v}_N\vec{\eta}_+^n \to \vec{v}\vec{\eta} = \vec{\xi}$ ST in $(L^2(I,V))^2$ (and in $(L^2(Q))^2$), $\vec{v}_N\vec{\eta}^n(0) \to \vec{v}\,\vec{\eta}(0)$ ST in $(L^2(Q))^2$, and then $(\acute{\vec{\xi}_\wedge^n}) = \vec{v}_N\acute{\vec{\eta}^n} \to \vec{v}\,\acute{\vec{\eta}} = \vec{v}\,\acute{\vec{\eta}}$ ST in $(L^2(I,V))^2$.

and since $t_{-}^{n} \to t\ ST\ in\ (L^{\infty}(I), \vec{U}_{+}^{n}$, $\vec{U}_{-}^{n}$, $\vec{U}_{\wedge}^{n} \to \vec{U}$ ST in $(L^2(Q))^2$, $\vec{U}_0^n \to \vec{U}^0$ ST in $\vec{V}, \vec{U}_T^n \to \vec{U}^T$ ST in $\vec{V}$ .Then from these convergence, one can get that

$$-\int_0^T (U_1, v_1)\,\acute{\eta_1}(t)dt + \int_0^T [a_1(U_1, v_1) - (g(t)U_2, v_1)]\eta_1(t)dt = \int_0^T (w_1(t, U_1), v_1)\,\eta_1(t)dt - (U_1^T, v_1)\eta_1(t) + (U_1^0, v_1)\eta_1(0) \tag{43}$$

$$-\int_0^T (U_2, v_2)\,\acute{\eta_2}(t)dt + \int_0^T [a_2(U_2, v_2) + (g(t)U_1, v_2)]\eta_2(t)dt = \int_0^T (w_2(t, U_2), v_2)\,\eta_2(t)dt - (U_2^T, v_2)\eta_2(t) + (U_2^0, v_2)\eta_2(0) \tag{44}$$

Then the following cases appear:

Case 1: Choose $\eta_r(t) \in D[0,T], i.e\ \eta_r(0) = \eta_r(T) = 0, \forall r = 1,2$, substituting (Eq.(46)-Eq.(47)), using integration by part (IBP) for the 1^{st} terms in the L.H.S

$$\int_0^T (U_{1t}, v_1)\,\eta_1(t)dt + \int_0^T [a_1(U_1, v_1) - (g(t)U_2, v_1)]\eta_1(t)dt = \int_0^T (w_1(t, U_1), v_1)\,\eta_1(t)dt \tag{45}$$

$$\int_0^T (U_{2t}, v_2)\,\eta_2(t)dt + \int_0^T [a_2(U_2, v_2) + (g(t)U_1, v_2)]\eta_2(t)dt = \int_0^T (w_2(t, U_2), v_2)\,\eta_2(t)dt \tag{46}$$

which gives that

$$(U_{1t}, v_1) + a_1(U_1, v_1) - (g(t)U_2, v_1) = (w_1(t, U_1), v_1)$$
$$(U_{2t}, v_2) + a_2(U_2, v_2) - (g(t)U_1, v_2) = (w_2(t, U_2), v_2)$$

which gives that $\vec{U} = (U_1, U_2)$ is a solution of (Eq.(5)-Eq.(7)), a.e. on I.

Case 2: Choose $\eta_r(t) \in C^1[0,T]$, such that $\eta_r(T) = 0\ and\ \eta_r(0) \neq 0, \forall r = 1,2$, using IBP for the 1^{st} terms in the L.H.S of (Eq.(45)-Eq.(46)), we obtain

$$-\int_0^T (U_1, v_1)\,\acute{\eta_1}(t)dt + \int_0^T [a_1(U_1, v_1) - (g(t)U_2, v_1)]\eta_1(t)dt = \int_0^T (w_1(t, U_1), v_1)\,\eta_1(t)dt + (U_{10}, v_1)\eta_1(0) \tag{47}$$

$$-\int_0^T (U_2, v_2)\,\acute{\eta_2}(t)dt + \int_0^T [a_2(U_2, v_2) + (g(t)U_1, v_2)]\eta_2(t)dt = \int_0^T (w_2(t, U_2), v_2)\,\eta_2(t)dt + (U_{20}, v_2)\eta_2(0) \tag{48}$$

Now, by subtracting (46) from (50) and (47) from (51) we get

$$(U_{r0}, v_r)\eta_r(0) = (U_r^0, v_r)\eta_r(0)\ , \forall\ r = 1,2$$

Thus

$$(U_{r0}, v_r) = (U_r^0, v_r), \forall\ r = 1,2$$

which give the ICs (Eq.(6)-Eq.(8)) are held, then the limit point $\vec{U}$ is a solution to the WF (Eq.(5)-Eq.(8)).

8. Numerical Examples

In this section, two numerical examples are carried out to show the accuracy of the proposed method.

Example 8.1:

Let $Q = \Omega \times I, \Omega = (0,1) \times (0,1), I = [0,1]$, the CNPSVC are given as

$$U_{1t} - \frac{\partial}{\partial x_1}\left[\left(1 + x_1 + e^{x_1/2}\right)\frac{\partial U_1}{\partial x_1}\right] - \frac{\partial}{\partial x_2}\left[\left(1 + x_2 + e^{x_2^2}\right)\frac{\partial U_1}{\partial x_2}\right] + (1 + e^{0.7x_1x_2})U_1 - (x_1^2 + 2x_1x_2 + 31x_2^2 + 11)U_2 = w_1(\vec{x}, t, U_1)$$

$$U_{2t} - \frac{\partial}{\partial x_1}\left[\left(1 + e^{x_1^2}\right)\frac{\partial U_2}{\partial x_1}\right] - \frac{\partial}{\partial x_2}\left[\left(e^{1+x_2^2}\right)\frac{\partial U_2}{\partial x_2}\right] + (e^{1+x_1+x_2})U_2 + (x_1^2 + 2x_1x_2 + 31x_2^2 + 11)U_1 = w_2(\vec{x}, t, U_2),$$

with the ICs

$$U_1(\vec{x}, 0) = U_1^0(\vec{x}) = 0.1x_1x_2(1 - x_1)(1 - x_2)\sqrt{\cos e^{-t}}\ , \text{in } \Omega$$

$U_2(\vec{x},0) = U_2^0(\vec{x}) = x_1^2 x_2^2 (1-x_1)^2(1-x_2)^2 e^{-0.3t}$

And the BCs,

$U_1(\vec{x},t) = 0, on\ \partial\Omega \times I$ and $U_2(\vec{x},t) = 0, on\ \partial\Omega \times I$

Such that the right hand term $w_1(\vec{x},t,U_1)$ and $w_2(\vec{x},t,U_2)$ are given as

$w_1(\vec{x},t,U_1) = x_1x_2(x_1-1)(x_2-1)\Big[0.1\sqrt{\cos(e^{-t})}\Big(e^{0.7x_1x_2} - sin\Big(0.1x_1x_2(x_1-1)(x_2-1)\sqrt{\cos(e^{-t})}\Big)+1\Big) - x_1x_2(x_1-1)(x_2-1)(x_1^2+2x_1x_2+31x_2^2+11)e^{-0.3t} + 0.05e^{-t}\Big(\frac{sin(e^{-t})}{\sqrt{\cos(e^{-t})}}\Big)\Big] - 0.1\{\big(2x_2e^{x_2^2}+1\big)\Big[x_1(x_1-1)\sqrt{\cos(e^{-t})}\,(2x_2-1)\Big] + \Big(0.5e^{x_1/2}+1\Big)\Big[x_2(x_2-1)\sqrt{\cos(e^{-t})}\,(2x_1-1)\Big]\} - 0.2\sqrt{\cos(e^{-t})}\,[x_2(x_2-1)\Big(x_1+e^{x_1/2}+1\Big) + x_1(x_1-1)\big(x_2+e^{x_2^2}+1\big)]$

and

$w_2(\vec{x},t,U_2) = x_1^2x_2^2(x_1-1)^2(x_2-1)^2e^{-0.3t}(e^{x_1+x_2+1}-0.1) - e^{x_1^2+1}[2x_2^2(x_2-1)^2e^{-0.3t}(6x_1(x_1-1)+1)] - 2x_1e^{x_1^2+1}\big[x_1x_2^2(x_2-1)^2e^{-0.3t}\big(2(2x_1^2-3x_1+1)\big)\big] - 2x_2e^{x_2^2+1}\big[x_2x_1^2(x_1-1)^2e^{-0.3t}\big(2(2x_2^2-3x_2+1)\big)\big] - e^{x_2^2+1}[2x_1^2(x_1-1)^2e^{-0.3t}(6x_2(x_2-1)+1)] - x_1x_2(x_1-1)(x_2-1)[x_1x_2(x_1-1)(x_2-1)e^{-0.3t}cos(x_1^2x_2^2(x_1-1)^2(x_2-1)^2e^{-0.3t})] - x_1x_2(x_1-1)(x_2-1)\big[0.1\big((x_1^2+2x_1x_2+31x_2^2+11)\big)\sqrt{\cos(e^{-t})}\,\big]$

The exact solutions of the above system are

$U_1(\vec{x},t) = 0.1x_1x_2(1-x_1)(1-x_2)\sqrt{\cos(e^{-t})}\,,\ U_2(\vec{x},t) = x_1^2x_2^2(1-x_1)^2(1-x_2)^2e^{-0.3t}$

This problem is solved by the MGIM with M=9, NT=20 and T=1, then the APPS $\vec{U}^n$ and the EXS $\vec{U}$ at x_1 and x_2 at the time $\hat{t} = 0.5$ are shown in Table 1 and Figure (1), the absolute maximum error is 0.0023.

Table 1: The Exact, the Approximation solutions and the absolute errors

x_1	x_2	EXS	APPS	Absolute error	x_1	x_2	EXS	APPS	Absolute error
0.1	0.1	0.0007	0.0005	0.0002	0.1	0.1	0.0001	0.0001	0.0000
0.3	0.1	0.0017	0.0012	0.0005	0.3	0.1	0.0003	0.0005	0.0002
0.5	0.1	0.0020	0.0014	0.0006	0.5	0.1	0.0004	0.0007	0.0003
0.7	0.1	0.0017	0.0012	0.0005	0.7	0.1	0.0003	0.0005	0.0002
0.9	0.1	0.0007	0.0006	0.0001	0.9	0.1	0.0001	0.0001	0.0000
0.1	0.3	0.0017	0.0011	0.0006	0.1	0.3	0.0003	0.0005	0.0002
0.3	0.3	0.0040	0.0025	0.0015	0.3	0.3	0.0017	0.0022	0.0005
0.5	0.3	0.0048	0.0030	0.0018	0.5	0.3	0.0024	0.0030	0.0006
0.7	0.3	0.0040	0.0027	0.0013	0.7	0.3	0.0017	0.0021	0.0004
0.9	0.3	0.0017	0.0013	0.0004	0.9	0.3	0.0003	0.0004	0.0001
0.1	0.5	0.0020	0.0013	0.0007	0.1	0.5	0.0004	0.0007	0.0003
0.3	0.5	0.0048	0.0028	0.0020	0.3	0.5	0.0024	0.0030	0.0006
0.5	0.5	0.0057	0.0033	0.0024	0.5	0.5	0.0034	0.0041	0.0007
0.7	0.5	0.0048	0.0030	0.0018	0.7	0.5	0.0024	0.0029	0.0005
0.9	0.5	0.0020	0.0014	0.0006	0.9	0.5	0.0004	0.0006	0.0002
0.1	0.7	0.0017	0.0011	0.0006	0.1	0.7	0.0003	0.0005	0.0002
0.3	0.7	0.0040	0.0024	0.0016	0.3	0.7	0.0017	0.0022	0.0005
0.5	0.7	0.0048	0.0028	0.0020	0.5	0.7	0.0024	0.0029	0.0005
0.7	0.7	0.0040	0.0026	0.0014	0.7	0.7	0.0017	0.0020	0.0003
0.9	0.7	0.0017	0.0012	0.0005	0.9	0.7	0.0003	0.0004	0.0001
0.1	0.9	0.0007	0.0005	0.0002	0.1	0.9	0.0001	0.0001	0.0000
0.3	0.9	0.0017	0.0012	0.0005	0.3	0.9	0.0003	0.0005	0.0002
0.5	0.9	0.0020	0.0014	0.0006	0.5	0.9	0.0004	0.0006	0.0002
0.7	0.9	0.0017	0.0012	0.0005	0.7	0.9	0.0003	0.0004	0.0001
0.9	0.9	0.0007	0.0006	0.0001	0.9	0.9	0.0001	0.0001	0.0000

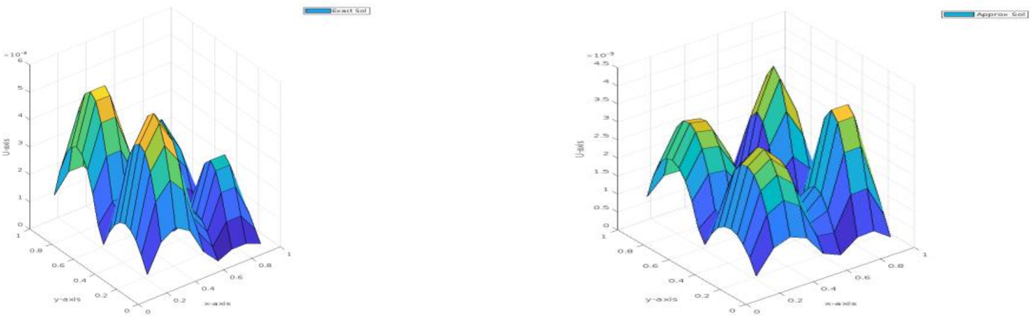

Figure 1: Shows the Exact and the Approximation Solutions

Example 8.2:

Let $Q = \Omega \times I, \Omega = (0,1) \times (0,1), I = [0,1]$, the CNPSVC are given as

$$U_{1_t} - \frac{\partial}{\partial x_1}\left[x_1 \frac{\partial U_1}{\partial x_1}\right] - \frac{\partial}{\partial x_2}\left[x_2 \frac{\partial U_1}{\partial x_2}\right] + (x_1 x_2)U_1 - (x_1 + x_2)^2 U_2 = w_1(\vec{x}, t, U_1),$$

$$U_{2_t} - \frac{\partial}{\partial x_1}\left[x_1^2 \frac{\partial U_2}{\partial x_1}\right] - \frac{\partial}{\partial x_2}\left[x_2^2 \frac{\partial U_2}{\partial x_2}\right] + (x_1^2 x_2^2)U_2 + (x_1 + x_2)^2 U_1 = w_2(\vec{x}, t, U_2),$$

where the ICs are

$$U_1(\vec{x}, 0) = U_1^0(\vec{x}) = sin(x_1 x_2)\big(1 - cos(1 - x_1)(1 - x_2)\big)$$

$$U_2(\vec{x}, 0) = U_2^0(\vec{x}) = x_2^5 \sqrt{x_1}(1 - x_1^2)(1 - \sqrt{x_2})$$

and the BCs are,

$$U_1(\vec{x}, t) = 0, on\ \partial\Omega \times I, \quad U_2(\vec{x}, t) = 0, on\ \partial\Omega \times I$$

such that the right hand term $w_1(\vec{x}, t, U_1)$ and $w_2(\vec{x}, t, U_2)$ are given as

$$w_1(\vec{x}, t, U_1) = e^{\pi t} cos(x_1 x_2)\big[0.5(x_1 + x_2)\big(cos\big((x_1 - 1)(x_2 - 1)\big) - 1\big)$$
$$-(x_1 x_2^2 + x_2 x_1^2 - 2x_1 x_2) sin\big((x_1 - 1)(x_2 - 1)\big)\big] - 0.5[(x_1 + x_2 -$$
$$2)e^{\pi t} sin(x_1 x_2) sin\big((x_1 - 1)(x_2 - 1)\big) + e^{-t^2}\sqrt{x_1}x_2^5(x_1 + x_2)^2(x_1^3 - 1)(\sqrt{x_2} - 1)\big]$$
$$-0.5e^{\pi t} sin(x_1 x_2)[(x_1(x_2^2 + x_1 x_2 - 2x_2 + 1) + x_2) cos\big((x_1 - 1)(x_2 - 1)\big)$$
$$+\big((x_1(x_2^2 + x_1 x_2 + 1) + \pi)\big)\big(cos\big((x_1 - 1)(x_2 - 1)\big) - 1\big) \quad sin\big(e^{\pi t} sin(x_1 x_2)\big(cos\big((x_1 -$$
$$1)(x_2 - 1)\big) - 1\big)/2\big)\big]$$

&

$$w_2(\vec{x}, t, U_2) = 0.5e^{-t^2} x_2^7 (x_1^3 - 1)(\sqrt{x_2} - 1)\sqrt{x_1^5} - x_2^2(x_1^3 - 1)e^{-t^2}\sqrt{x_1}\left[2.375\sqrt{x_2^7} +\right.$$
$$\left.10(\sqrt{x_2} - 1)\right] \ -x_1 x_2^5(\sqrt{x_2} - 1)e^{-t^2}\left[3\sqrt{x_1^5} + 0.5\frac{(x_1^3 - 1)}{\sqrt{x_1}}\right] - 5x_2(x_1^3 - 1)e^{-t^2}\sqrt{x_1}\left[0.1\sqrt{x_2^9} +\right.$$
$$\left.x_2^4(\sqrt{x_2} - 1)\right] - 0.5e^{\pi t} sin(x_1 x_2)(x_1 + x_2)^2\big[\big(cos\big((x_1 - 1)(x_2 - 1)\big) - 1\big)\big] -$$
$$0.5x_1^2 x_2^5(\sqrt{x_2} - 1)e^{-t^2}[9\sqrt{x_1^3} - \frac{0.25(\sqrt{x_2} - 1)}{\sqrt{x_1^3}}] - 0.5x_1^2 x_2^5(\sqrt{x_2} - 1)e^{-t^2}\left[9\sqrt{x_1^3} -\right.$$
$$\left.0.25(\sqrt{x_2} - 1)/\sqrt{x_1^3}\right] - x_2^5(x_1^3 - 1)(\sqrt{x_2} - 1)e^{-t^2}\sqrt{x_1}\left[t + 0.5cos\left(\left(x_2^5(x_1^3 - 1)(\sqrt{x_2} -\right.\right.\right.$$
$$\left.\left.\left.1)e^{-t^2}\sqrt{x_1}\right)/2\right)\right]$$

The exact solution of the above system is

$$U_1(\vec{x}, t) = sin(x_1 x_2)\Big(1 - cos\big((1 - x_1)(1 - x_2)\big)\Big)\, e^{\pi t}/2$$

$$U_2(\vec{x}, t) = x_2^5 \sqrt{x_1}(1 - x_1^2)(1 - \sqrt{x_2})\, e^{-t^2}/2.$$

This problem is solved by the MGIM with M=9, NT=20 and T=1, then the APPS $\vec{U}^n$ and the EXS $\vec{U}$ at x_1 and x_2 at the time $\hat{t} = 0.5$ are shown in Table 2 and Figure (2), the absolute maximum error is 0.0022.

Table 2: The Exact, the Approximation solutions and the absolute errors

x_1	x_2	EXS	APPS	Absolute error	x_1	x_2	EXS	APPS	Absolute error
0.1	0.1	0.0075	0.0061	0.0014	0.1	0.1	0.0000	0.0008	0.0008
0.3	0.1	0.0139	0.0129	0.0010	0.3	0.1	0.0000	0.0011	0.0011
0.5	0.1	0.0120	0.0115	0.0005	0.5	0.1	0.0000	0.0008	0.0008
0.7	0.1	0.0061	0.0058	0.0003	0.7	0.1	0.0000	0.0004	0.0004
0.9	0.1	0.0009	0.0007	0.0002	0.9	0.1	0.0000	0.0001	0.0001
0.1	0.3	0.0139	0.0128	0.0011	0.1	0.3	0.0001	0.0003	0.0002
0.3	0.3	0.0254	0.0253	0.0001	0.3	0.3	0.0002	0.0006	0.0004
0.5	0.3	0.0218	0.0219	0.0001	0.5	0.3	0.0003	0.0006	0.0003
0.7	0.3	0.0110	0.0108	0.0002	0.7	0.3	0.0002	0.0004	0.0002
0.9	0.3	0.0016	0.0012	0.0004	0.9	0.3	0.0001	0.0001	0.0000
0.1	0.5	0.0120	0.0113	0.0007	0.1	0.5	0.0011	0.0003	0.0008
0.3	0.5	0.0218	0.0216	0.0002	0.3	0.5	0.0019	0.0008	0.0011
0.5	0.5	0.0185	0.0183	0.0002	0.5	0.5	0.0022	0.0011	0.0011
0.7	0.5	0.0093	0.0088	0.0005	0.7	0.5	0.0020	0.0011	0.0009
0.9	0.5	0.0013	0.0009	0.0004	0.9	0.5	0.0009	0.0006	0.0003
0.1	0.7	0.0061	0.0056	0.0005	0.1	0.7	0.0034	0.0019	0.0015
0.3	0.7	0.0110	0.0104	0.0006	0.3	0.7	0.0057	0.0036	0.0021
0.5	0.7	0.0093	0.0084	0.0009	0.5	0.7	0.0066	0.0045	0.0021
0.7	0.7	0.0046	0.0036	0.0010	0.7	0.7	0.0059	0.0043	0.0016
0.9	0.7	0.0006	0.0001	0.0005	0.9	0.7	0.0027	0.0022	0.0005
0.1	0.9	0.0009	0.0006	0.0003	0.1	0.9	0.0037	0.0028	0.0009
0.2	0.9	0.0014	0.0009	0.0005	0.2	0.9	0.0052	0.0041	0.0011
0.3	0.9	0.0016	0.0010	0.0006	0.3	0.9	0.0063	0.0051	0.0012
0.5	0.9	0.0013	0.0006	0.0007	0.5	0.9	0.0073	0.0060	0.0013
0.7	0.9	0.0006	-0.0000	0.0006	0.7	0.9	0.0065	0.0055	0.0010
0.9	0.9	0.0001	-0.0002	0.0003	0.9	0.9	0.0030	0.0027	0.0003

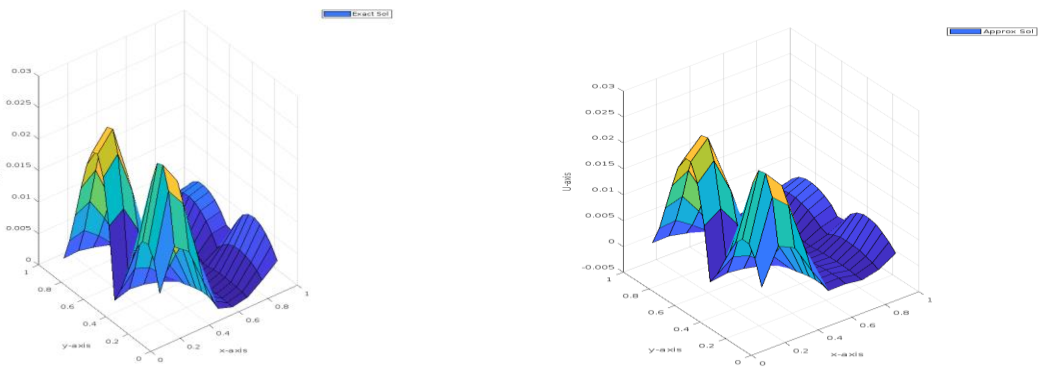

Figure 2 Shows the Exact and the Approximation Solutions

9. Conclusions

The approximate method "MGIM "has been proposed for solving CNPBVPVC. Two numerical examples have been given to examine the efficiency and the accuracy of the MGIM. The results in figures 1 and 2 show the maximum errors that arise from the differences between the approximate and the exact vector solutions for the considered problems. It is observed that the proposed method is efficient and accurate. The transformed system of equations (the CGLAS) was solved by the ChDe. This method is faster than the other methods like gauss elimination and the Haar wavelets methods, because it saves a lot of number of calculations. The GFEM was applied easily and the elements in the CGNAS are in analytic form (exact) comparing with other methods that the elements are in approximate or in a full discrete form. The uniqueness of the APPS of the DWF was proved. Furthermore the stability and the convergence of the method was demonstrated. It is important to mention here that the approximate vector solution are shown in the given figures at the value of $\hat{t} = 0.5$, in fact the same results with same accuracy were obtained at any value of $\hat{t}$ provided this value belongs to I.

10. Disclosure and conflict of interest

"Conflict of Interest: The authors declare that they have no conflicts of interest."

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