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# Closed-coessential and Closed-coclosed Submodules.

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**Abstract**

The aim of this study is to present the concept of closed-coessential submodule and closed-coclosed submodule, the consideration of certain properties. Allow $R$ be a ring with identity and define $F$ as a left $R$-module on the left, with $H$ and $E$ acting as its submodules, in that sense , $E \leq H \leq F$ then $E$ is called closed- coessential submodule in $E$ of $H$ $(E \leq_{c.ce} H)$ , if $\frac{H}{E} \ll_c \frac{F}{E}$.
On the other hand, a submodule $H$ of $F$ is known as closed-coclosed submodule, if $E$ is closed-coessential submodule of $H$ in $F$. Finally, in this article we introduce some properties of these types of submodules under some conditions which are in analogy with the known properties for coessential and coclosed submodules properties. And we discuss the relation between them with the examples and remarks are needed in our work.

**Keywords**: closed-small submodule, closed-coessential submodule, closed-coclosed submodule.

# المقاسات الجزئيه الضد الجوهريه الاساسيه من النمط المغلق والمقاسات الجزئيه الضد المغلقه الاساسيه من النمط المغلق

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**الخلاصة**

الهدف من هذا البحث هو تقديم مفاهيم المقاسات الجزئيه الضد الجوهريه الاساسيه من النمط المغلقوالمقاسات الجزئيه الضد المغلقه الاساسيه من النمط المغلق , وسوف نقوم بالنظرفي بعض الخواص لهذه المفاهيم.ولتكن بحيث $E \leq H \leq F$ فان $E$ $F$ مقاسات جزئيه في $E, H$ و $R$ مقاسا الى اليسار من النمط $F$ وليكن $F$ حلقه مع العنصر المحايد $R$ يدعى بانه مقاس جزئي ضد المغلق الاساسي $F$ للمقاس $H$ المقاس الجزئي $\frac{H}{E} \ll_c \frac{F}{E}$. يدعى بانه مقاس جزئي اذا كان $F$ . في $H$ مقاس جزئي ضد الجوهري الاساسي من النمط المغلق من $E$ مقاس جزئي من النمط المغلق اذا كان واخيرا, في هذه الدراسه قدمنا بعض الخصائص لهذه الانواع من المقاسات الجزئيه تحت عدة شروط والتي هيه قياسا على الخصائص المعروفه للمقاسات الجزئيه الضد جوهريه الاساسيه والضد المغلقه الاساسيه . وأيضا نناقش العلاقه بينهما مع الامثله والملاحظات التي نحتاجها في عملنا.

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**Introduction**

In this paper, every ring has a unique identity, and all modules will be until left $R$-modules.Assume that $H$ is a submodule of $F$, and let $F$ be an $R$-module. A submodule $H$ is called a small submodule (notation $H \ll F$), if for any submodule $E$ of $F$ such that $H + E = F$, implies that $E = F$ [1]. A proper submodule $H$ of an $R$-module $F$ is called essential submodule in $F(H \leq_c F)$, if for every non-zero submodule $E$ of $F$, $H \cap E \neq 0$ [1]. In $F$, a submodule $H$ is referred to a closed, if it has no proper essential extension in $F$ [2]. For $E \leq H \leq F$ , $E$ is called coessential submodule of $H$ in $F(E \leq_{ce} H)$ if $\frac{H}{E} \ll \frac{F}{E}$ [3], and $H$ is said to be coclosed submodule in $F$ and its denoted by $(H \leq_{cc} F)$, if $H$ has no proper coessential submodule in $F$ [2], [4].In an earlier study [5], the concept of closed-small submodule was presented, such that a proper submodule $H$of $F$ is called closed-small ($c$-small) submodule of $F$ denoted by $(H \ll_c F)$. If $H + E = F$ , where $E \leq F$, then $E$ is closed submodule in $F$ $(E \leq_c F)$. It is evident that every small submodule of $F$ is c-small submodule of $F$, but still, the opposite is not true, in the study of many subjects has aroused the interest many authors of generalizations of small submodules see [6-14]. We will use closed submodules to introduce a new generalization of small submodules namely closed-small submodule. Several authors have expressed interest in studying various generalizations of coessential and coclosed submodules [15-20]. For this work, we provide the concept of closed-coessential submodule as a coessential submodule generalization, such that a submodule $E$ of an $R$-module $F$ claims to be closed-coessential submodule $(E \leq_{cce} H)$ of $H$ a submodule in $F$, if $\frac{H}{E} \ll \frac{F}{E}$ where $E \leq H \leq F$.Several properties of this type of submodule are given in the first portion. The concept of closed-coclosed submodule has been given in section two, so that a submodule $H$ which is belong to an $R$-module $F$ has been named as closed-coclosed submodule of $F$ ( $H \ll_{c.cc} F$ ), if $E$ is a closed-coessential submodule of $H$ in $F$. Furthermore, as we provide a few fundamental properties of this type of submodules. Some properties of closed-small (c-small) submodule of $F$ are necessary in this work have been provided in Lemma 1.1.

**Lemma 1.1** [5]:

1- Let $f: F \to F`$ be an isomorphism where $F$ and $F`$ be an $R$-modules, such that $H \ll_c F`$, then $f^{-1}(H) \ll_c F$ .

2- Let $H$ and $E$ are submodules of a module $F$ such that $E \leq H \leq F$ , if $H \ll_c F$, then $E \ll_c F$.

3- If $E, H$ are the submodules of an $R$-module $F$, then, such that $E \leq H \leq F$ and $H \ll_c F$ , then $\frac{H}{E} \ll_c \frac{F}{E}$.

We now demonstrate the lemma that was employed in this work.

**2. Closed-coessential submodule.**

Within this part we present the idea of the closed-coessential submodule and many of its properties.

**Definition 2.1:** Let $F$ be an $R$-module, and let $E$ and $H$ be their submodules such that $E \leq H \leq F$, then $E$ is called a closed-coessential (c-coessential ) submodule of $H$ in $F$.It will denoted by ( $E \leq_{c.ce} H$) if $\frac{H}{E} \ll_c \frac{F}{E}$.

**Remarks and Examples 2.2:**

1. As each small submodule is $c$-small. Then every coessential submodule is $c$-coessential submodule.

**Proof:** Let $H, E$ be submodules of $F$ such that $E \leq H \leq F$. As $E \leq_{ce} H$, then $\frac{H}{E} \ll \frac{F}{E}$. Which implies that $\frac{H}{E} \ll_c \frac{F}{E}$, and hence $E \leq_{c.ce} H$.

The converse, however, is untrue. For example, $Z_6$ as the $Z$-module: $\{\bar{0}\}$ is c-coessential. submodule of $\{\bar{0}, \bar{3}\}$ in $Z_6$, since $\frac{\{\bar{0},\bar{3}\}}{\{\bar{0}\}} \simeq \{\bar{0}, \bar{3}\} \ll_c \frac{Z_6}{\{\bar{0}\}} \simeq Z_6$ and hence $\{\bar{0}, \bar{3}\}$ is $c$-small in $Z_6$, see [5]. So $\{\bar{0}\}$ is not coessential in $Z_6$.

2. In $Z_4$ as the $Z$-module: $\{\bar{0}\}$ is c-coessential submodule of $\{\bar{0}, \bar{2}\}$ in $Z_4$ given that. $\frac{\{\bar{0},\bar{2}\}}{\{\bar{0}\}} \simeq \{\bar{0}, \bar{2}\} \ll_c \frac{Z_4}{\{\bar{0}\}} \simeq Z_4$, see [5].
3. Consider $Z$ as $Z$-module. Since there is no closed-small in $Z$, hence there is no closed-coessential.
4. Consider $Z_8$ as $Z$-module: $\{\bar{0}, \bar{4}\}$ c-coessential submodule of $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ in $Z_8$, since $\frac{\{\bar{0},\bar{2},\bar{4},\bar{6}\}}{\{\bar{0},\bar{4}\}} \simeq \{\bar{0}, \bar{4}\} \ll_c \frac{Z_8}{\{\bar{0},\bar{4}\}} \simeq \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$.
5. Assume that $F$ has two submodules of $F$, if $E \leq H \leq F$, and $\frac{F}{E}$ is a semi simple module, then $E$ is a coessential submodule of $H$ if $E \leq_{c.ce} H$.

We need to prove the following.

**Proposition 2.3:** Assume that $F$ is an $R$-module and $H$ is a submodule of $F$, then $H \ll_c F$ if and only if $\{\bar{0}\} \leq_{c.ce} H$ in $F$.

**Proof:** ⇒) Let us say $H \ll_c F$. By Lemma 1.1 we have $\frac{H}{\{\bar{0}\}} \ll_c \frac{F}{\{\bar{0}\}}$, so $\{\bar{0}\} \leq_{c.ce} H$ in $F$.

⇐) Let $\{\bar{0}\} \leq_{c.ce} H$ in $F$ and let $H + E = F$ where $E$ be submodule of $F$, so $\frac{H+E}{\{\bar{0}\}} = \frac{F}{\{\bar{0}\}}$ hence $\frac{H}{\{\bar{0}\}} + \frac{E}{\{\bar{0}\}} = \frac{F}{\{\bar{0}\}}$. Since $\{\bar{0}\} \leq_{c.ce} H$ in $F$, then $\frac{H}{\{\bar{0}\}} \ll_c \frac{F}{\{\bar{0}\}}$, so $\frac{E}{\{\bar{0}\}} \leq_c \frac{F}{\{\bar{0}\}}$ and then $H \ll_c F$.

**Proposition 2.4:** Let $F$ be an $R$-module, and let $E, H,$ and $U$ be its submodules such that $E \leq H \leq U \leq F$, then $H \leq_{c.ce} U$ in $F$ iff, $\frac{H}{E} \leq_{c.ce} \frac{U}{E}$ in $\frac{F}{E}$.

**Proof:** ⇒) Suppose $H \leq_{c.ce} U$ in $F$, hence $\frac{U}{H} \ll_c \frac{F}{H}$. As $\frac{U}{H} \simeq \frac{U/E}{H/E}$ and $\frac{F}{H} \simeq \frac{F/E}{H/E}$ by Third Isomorphism Theorem, then $\frac{U/E}{H/E} \ll_c \frac{F/E}{H/E}$ and hence $\frac{H}{E} \leq_{c.ce} \frac{U}{E}$ in $\frac{F}{E}$.

⇐) Suppose $\frac{H}{F} \leq_{c.ce} \frac{U}{F}$ in $\frac{F}{E}$, hence $\frac{U/E}{H/E} \ll_c \frac{F/E}{H/E}$ and by using Third Isomorphism Theorem, we get $\frac{U}{H} \simeq \frac{U/E}{H/E} \ll_c \frac{F/E}{H/E} \simeq \frac{F}{H}$ hence $\frac{U}{H} \ll_c \frac{F}{H}$ then $H \leq_{c.ce} U$ in $F$.

**Proposition 2.5:** Let $f: F \to H$ be an isomorphism. If $E \leq L \leq F$ and $E \leq_{c.ce} L$ in $F$, then $f(E) \leq_{c.ce} f(L)$ in $H$.

**Proof:** Assume $\frac{f(L)}{f(E)} + \frac{C}{f(E)} = \frac{H}{f(E)}$ hence $f(L) + C = H$, and so $f^{-1}(f(L) + C) = f^{-1}(H) = F$. This implies $f^{-1}\big(f(L)\big) + f^{-1}(C) = F$. But $f^{-1}(f(L) \geq L$, so $L + f^{-1}(C) \leq F$. Now let $x \epsilon F$, then $f(x) \in H$ and $f(x) = f(l) + c$, for some $l \in L, c \in C$ hence $f(x - l) = c$, that is $x - l \in f^{-1}(c)$.Since $x = l + (x - l) \in L + f^{-1}(C)$ .Thus $F = L + f^{-1}(C)$ and $\frac{F}{E} = \frac{L}{E} + \frac{f^{-1}(C)+E}{E}$. But $\frac{L}{E} \ll_c \frac{F}{E}$, so that $\frac{f^{-1}(C)+E}{E} \leq_c \frac{F}{E}$. Thus $f^{-1}(C) + E$ closed in $F$, and it follows

that $f(f^{-1}(C)+E)$ closed in $H$ and so $C+f(E)$ closed in $H$, thus $C$ closed in $H$ by [5] and $\frac{C}{f(E)} \leq_c \frac{H}{f(E)}$
(Since $f(E) \subseteq C$), therefore $\frac{f(L)}{f(E)} \ll_c \frac{H}{E}$,such that $f(E) \leq_{c.ce} f(L)$.

**Proposition 2.6:** Suppose $F$ be an R-module. If $E \leq_{c.ce} H$, then $E \leq_{c.ce} L$ were $E \leq L \leq H$ and $E, L, H$ are submodules of $F$.
**Proof:** Let $E \leq W \leq F$ with $\frac{L}{E}+\frac{W}{E}=\frac{F}{E}$, thus $L+W=F$. But $L \leq H$, therefore, $F=H+W$ and then $\frac{F}{E}=\frac{H}{E}+\frac{W}{E}$. $E \leq_{c.ce} H$, then $\frac{H}{E} \ll_c \frac{F}{E}$, thus $\frac{W}{E}$ closed in $\frac{F}{E}$, this implies that $E \leq_{c.ce} L$ *in* $F$.
**Proposition 2.7:** Consider the $R$-module $F$ and $E, L$, H are submodules of $F$.
If $E \leq_{c.ce} L$ and $H \ll F$, then $E \leq_{c.ce} L+H$ in $F$.

**Proof:** Suppose that $E \leq W \leq F$ with $\frac{L+H}{E}+\frac{W}{E}=\frac{F}{E}$ then $L+H+W=F$,but $H \ll F$, therefore, $L+W=F$ and hence $\frac{L}{E}+\frac{W}{E}=\frac{F}{E}$. But $E \leq_{c.ce} L$ and $\frac{L}{E} \ll_c \frac{F}{E}$, thus $\frac{W}{E}$ is closed in $\frac{F}{E}$. This mean that $\frac{L+H}{E} \ll_c \frac{F}{E}$ so $E \leq_{c.ce} L+H$ in $F$.

## 3. Closed-coclosed submodule.

In this section we define the closed-coclosed submodule concept and discuss some of its properties

**Definition 3.1:** A submodule $H$ of $F$ is called closed-coclosed in $F$ for short (c-coclosed) submodule if whenever $E \leq_{c.ce} H$ for a submodule $E$ of $H$ in $F$, implies that $H=E$. It will be denoted by ( $H \leq_{c.cc} F$). Similarly, $H$ is referred to as closed-coclosed submodule of $F$, if $H$ has no proper $c$-coessential submodule.

**Remarks and Examples 3.2:**

1. Since every small submodule is c-small, then every c-coclosed submodule is Co-closed submodule.

**Proof:** Suppose $H$ be closed-coclosed submodule of $F$ and $E \leq H$,
such that $\frac{H}{E} \ll \frac{F}{E}$ hence see [5], $\frac{H}{E} \ll_c \frac{F}{E}$, so $E \leq_{c.ce} H$ in $F$ and $E=H$,
hence $H$ is coclosed submodule.

2. the convers is not true. For example, $Z_6$ as Z-module: $\{\bar{0}, \bar{2}, \bar{4}\}$ is coclosed submodule of $Z_6$, given that $\{\bar{0}\}$is the only submodule of $\{\bar{0}, \bar{2}, \bar{4}\}$ such that $\frac{\{\bar{0}, \bar{2}, \bar{4}\}}{\{\bar{0}\}} \backsimeq \{\bar{0}, \bar{2}, \bar{4}\}$ and $\frac{Z_6}{\{\bar{0}\}} \backsimeq Z_6$ so, $\{\bar{0}, \bar{2}, \bar{4}\}$ is not small of $Z_6$ and $\{\bar{0}\} \neq \{\bar{0}, \bar{2}, \bar{4}\}$, but $\{\bar{0}, \bar{2}, \bar{4}\}$ is not c-coclosed since $\{\bar{0}, \bar{2}, \bar{4}\}$ is c-small of $Z_6$ see Remark 2.2, but $\{\bar{0}\} \neq \{\bar{0}, \bar{2}, \bar{4}\}$.
3. Consider$Z_6$ as $Z$-module: $\{\bar{0}, \bar{3}\}$ isn't $c$-coclosed of $Z_6$ since the $c$-coessential submodule of $\{\bar{0}, \bar{3}\}$ is $\{\bar{0}\}$ by [5], and $\{\bar{0}\} \neq \{\bar{0}, \bar{3}\}$,also $\{\bar{0}, \bar{3}\}$ not coclosed submodule in $Z_6$.
4. In $Z_4$ as $Z$-module: $\{\bar{0}, \bar{2}\}$ is not $c$-coclosed of $Z_4$, since $\{\bar{0}\} \leq_{c.ce} \{\bar{0}, \bar{2}\}$, by [5] but $\{\bar{0}, \bar{4}\} \neq \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$.
5. In $Z_8$ as $Z$-module: $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is not $c$-coclosed of $Z_8$ ,since $\{\bar{0}, \bar{4}\} \leq_{c.ce} \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$, by [5] but $\{\bar{0}, \bar{4}\} \neq \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$.

6. $Z$ as the $Z$-module: There is no $c$-coclosed submodule in $Z$, so every submodule is $c$- coclosed.
7. Let $F$ be an $R$-module and let $E$ and $H$ be its submodules, such that $E \leq H \leq F$, if $\frac{F}{E}$ is a semi simple module, then $H$ is coclosed submodule of $F$ if and only if, $H$ is $c$-coclosed submodule.

**Proposition 3.3:** Assume $F$ be an R-module with the submodules $E, U$, and $H$ defined as follows: $U \leq H \leq F$, then $H \leq_{c.cc} F$ if and only if, $\frac{H}{U} \leq_{c.cc} \frac{F}{U}$.

**Proof:** ⇒) Let $\frac{E}{U} \leq \frac{H}{U}$ and $\frac{E}{U} \leq_{c.ce} \frac{H}{U}$ in $\frac{F}{U}$, so by Proposition 2.4 we get $E \leq_{c.ce} H$ in $F$, and since

$H \leq_{c.cc} F$, then $E = H$ and hence $\frac{E}{U} = \frac{H}{U}$.

⇐) Let $E \leq_{c.ce} H$ in $F$, so by Proposition 2.4 we get $\frac{E}{U} \leq_{c.ce} \frac{H}{U}$ in $\frac{F}{U}$, and since $\frac{H}{U} \leq_{c.cc} \frac{F}{U}$ then $\frac{E}{U} = \frac{H}{U}$

and so $E = H$

**Proposition 3.4:** Let $U \leq E \leq L \leq F$ .If $E \leq_{cce} L$ and for any $W \leq F$ , $L + W \leq_c E$ then $E + W \leq_c F$.

**Proof:** Let $L + W \leq_c F$ , so $\frac{L}{E} + \frac{W+E}{E} = \frac{F}{E}$ , but $E \leq_{cce} L$ so $\frac{L}{E} \ll_c \frac{F}{E}$ Implies that $\frac{W+E}{E} \leq_c \frac{F}{E}$ .So $w + E \leq_c F$ .

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