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R^*-Small Submodules and R^*-Hollow Modules

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Abstract
The purpose of this study is to present a new concepts on a module M over a ring, these concepts is called R^*-small sub module, R^*-hollow module which present generalizations of the small submodule and hollow module, respectively. Key characteristics of these concepts such as the image and direct sum of R^*-small sub module, R^*-hollow module are R^*-small submodule, R^*-hollow submodule, respectively.

Keywords: e^*-essential submodule, R^*-small submodule, R^*-hollow module

المقاسات الجزئية الصغيرة من النمط- R^* و المقاسات المجوفة من نمط - R^*

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الخلاصة
الهدف من هذا البحث هو اعطاء مفاهيم جديدة للمقاس M على الحلقة , المقاسات الجزئيه من هذه المفاهيم هي المقاس الجزئي الصغير من النمط - R^* وكذلك المقاس المجوف من النمط - R^* والتي تعتبر تعميمات لمفاهيم سابقه هي المقاس الجزئي الصغير والمقاس المجوف على التوالي. برهنا خواص اساسيه لهذه المفاهيم منها : صوره المقاس الجزئي الصغير من النمط - R^* تكون مقاس جزئي من النمط - R^*

1. Introduction

Let M be a unitary left R-module, and let R be any ring with one. A submodule A of M is referred to as small in M (indicated by $A \ll M$) if whenever $A + B = M$ for some $B \subseteq M$ implies that $B = M$ [1-4]. An R-module with a value greater than zero M is said to be hollow if each suitable submodule of M is small in M [5-9]. For every R-endomorphism f of M a submodule A of M if $f(A) \subseteq A$, then f is called fully invariant [10]. The cosingular submodule of M was released by Oscan in [11] in the following manner: $Z^*(M)$={$m \in M$; $Rm << E(M)$}, where $E(M)$ is an injective hull of M see [12] and [13], where Z^*(M) was called the cosingular submodule. Now, if $Z^*(M) = M$, then M is called cosingular. Many authors present generalizations of small submodule see [14-16]. Baanon and Khalid in [17] was introduced the e^*-essential submodule by using the concept of cosingular submodule. If $A \cap B \neq 0$, for each non-zero cosingular submodule B of M subsequently a submodule A is deemed to be e^*-essential. Also, in [17] it was introduced $Rad_e * (M)$ as being the point where each maximal e^*-essential submodule

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intersects. If M does not include an e^*-essential maximal submodule, then $Rad_{e^*}(M) = M$. In fact it was proved that Rad_{e^*}(M) is the sum of all e^*-small submodule of M, as well as Rad_{e^*}(M) is a fully invariant submodule [17]. We use these concepts to introduce R^*-small submodule and investigate some properties. We also introduce and discuss the R^*-hollow module which is a generalization of the hollow module. We prove main properties of these concepts.

2. R^*-small submodule

In this section R^*-small submodule was presented with some properties.

Definition 2.1:

A submodule A of M is called $Rad_R\,*$-small in M for short R^*-small submodule if whenever $M = A + B$, and $Rad_R * \left(\frac{M}{B}\right) = \frac{M}{B}$ (briefly $R * \left(\frac{M}{B}\right) = \frac{M}{B}$) implies that $B = M$. It will be denoted by $A \ll_{R^*} M$.

Remarks and examples 2.2:

1. It is evident that each small submodule is R^* -small the reverse, however is untrue. For example Z_6 *as* Z-module, since $Z_6 = <\bar{2}> \oplus <\bar{3}>$, and $R^*\left(\frac{Z_6}{<\bar{3}>}\right) \cong R^*(<\bar{2}>) = 0$ i.e., $R^*\left(\frac{Z_6}{<3>}\right) = <\bar{3}> \neq <\bar{2}> \cong \frac{Z_6}{<\bar{3}>}$, but $<\bar{2}> \neq <\bar{3}>$, hence $<\bar{2}>$ and $<\bar{3}>$ are R^*-small submodules, but not small submodules.
2. As the Z-module Z_4. In Z_4, 0 and $\{0,2\}$ are R^*-small submodules according to (1).
3. Consider $M = Z \oplus Z_{p\infty}$ as Z-module. Since $\frac{M}{Z} \cong Z_{p\infty}$, therefore, $R^*(\frac{M}{Z}) = R^*(Z_{p\infty}) = Z_{p\infty}$ i.e., $R^*(\frac{M}{Z}) = \frac{M}{Z}$, but M $\neq Z$ which means that $Z_{p\infty}$ is not R^*-small submodule in M.
4. Let $M = 2Z \oplus Z_2$ as Z-module. $R^*\left(\frac{M}{2Z}\right) = R^*(Z_2) = 0$, i.e., $R^*\left(\frac{M}{2Z}\right) = 2Z \neq Z_2 \cong \frac{M}{2Z}$. Thus Z_2 is R^*-small submodule in M.
5. Consider Z as Z-module. Z$= 2Z + 3Z$ *as* Z-module. Since $\frac{Z}{2Z} \cong Z_2$, hence $R^*\left(\frac{Z}{2Z}\right) = R^*(Z_2) = 0$ i.e., $R^*\left(\frac{M}{2z}\right) = 2Z \neq Z_2 \cong \frac{M}{2Z}$. Thus $3Z$ is R^*-small submodule in Z.

We need to prove the following:

Proposition 2.3:

Assume that M has two submodules of M if $R^*\left(\frac{M}{A}\right) = \frac{M}{A}$ and $A \subseteq B \subseteq M$, then $R^*\left(\frac{M}{B}\right) = \frac{M}{B}$.

Proof.

Define a function $f : \frac{M}{A} \to \frac{M}{B}$ as follow $f(m + A) = m + B \ \forall\, m \in M$, it is clear that f is an epimorphism (proposition 6in [3]) $f\left(R^*\left(\frac{M}{A}\right)\right)$ contained in $R^*\left(\frac{M}{B}\right)$. Hence $f\left(\frac{M}{A}\right) = \frac{M}{B} \subseteq R^*(\frac{M}{B})$. Therefore, $R^*\left(\frac{M}{B}\right) = \frac{M}{B}$.

Corollary 2.4:

Consider A and B to be two submodules of M and let M be any R-module. If $R^*\left(\frac{M}{A}\right) = \frac{M}{A}$, then $R^*\left(\frac{M}{A+B}\right) = \frac{M}{A+B}$.

Proposition 2.5:

Suppose M be an R-module If $R^*(M) = M$ and $A \subseteq M$.Then $A \ll_{R^*} M$ iff $A \ll M$.

Proof.

$\Longrightarrow$) Suppose $A+B=M, B\subseteq M$. To prove $B=M$ we claim that $R^*\left(\frac{M}{B}\right)=\frac{M}{B}$, let $f: M\to \frac{M}{B}$ be the natural epimorphism (Proposition 6 [17]) $f\left(R^*(M)\right)$ contained in $R^*(\frac{M}{B})$ and $R^*(M)=M$,then $f(M)\subseteq R^*(\frac{M}{B})$,therefore $\frac{M}{B}\subseteq R^*(\frac{M}{B})$ and $R^*\left(\frac{M}{B}\right)=\frac{M}{B}$, since $A\ll_{R^*} M$, then $B=M$, hence $A\ll M$.

$\Leftarrow$) Clearly by Remarks and examples 2.2.

These are some characteristics of R^*-small submodules.

Proposition 2.6:

1. If $A\leq B\leq M$, then $B\ll_{R^*} M$ if and only if $A\ll_{R^*} M$ and $\frac{B}{A}\ll_{R^*}\frac{M}{A}$.
2. If A and B are M submodules, then $A+B\ll_{R^*} M$ if and only if both $A\ll_{R} M$ and $B\ll_{R^*} M$.
3. Let $A\leq B\leq M$, if $A\leq_{R^*} B$, then $A\leq_{R^*} M$.
4. Let f:$M\to M^\grave{}$ be R-homomorphism such that $A\ll_{R^*} M$, then $f(A)\ll_{R^*} M^\grave{}$.
5. Let $M=M_1\oplus M_2$, $A_1\leq M_1$, and $A_2\leq M_2$, then $A_1\oplus A_2\ll_{R^*} M_1\oplus M_2$ if and only if $A_1\ll_{R^*} M_1$ and $A_2\ll_{R^*} M_2$.

Proof.

(1) $\Rightarrow$) Assume that L contained in M, in such a way that $L+A=M$ and $R^*\left(\frac{M}{L}\right)=\frac{M}{L}$, then $L+B=M$. Since $B\ll_{R^*} M$,then $L=M$. Now, let $\frac{K}{A}$ be any sub module of $\frac{M}{A}$ such that $\frac{K}{A}+\frac{B}{A}=\frac{M}{A}$ and $R^*\left(\frac{M}{K}\right)=\frac{M}{K}$, then $B+K=M$ since $B\ll_{R^*} M$, so $K=M$. Thus $\frac{K}{A}=\frac{M}{A}$.

$\Leftarrow$)$B+K=M$, K contained in M, such that $R^*\left(\frac{M}{K}\right)=\frac{M}{K}$. Since $\frac{B+K}{A}=\frac{M}{A}$, so $\frac{B}{A}+\frac{A+K}{A}=\frac{M}{A}$ and $R^*\left(\frac{\frac{M}{A}}{\frac{K+A}{A}}\right)=R^*(\frac{M}{K+A})=\frac{M}{K+A}$ by using Corollary 2.4, but $\frac{B}{A}\ll_{R^*}\frac{M}{A}$ which implies that $\frac{M}{A}=\frac{K+A}{A}$, hence $M=A+K$ as $A\ll_{R^*} M$, therefore, $K=M$.

2) Let C be a module under M such that $A+C=M$ and $R^*\left(\frac{M}{C}\right)=\frac{M}{C}$, then $A+B+C=M$ but $R^*\left(\frac{M}{C}\right)=\frac{M}{C}$ and since $A+B\ll_{R^*} M$ which implies that $C=M$, hence $A\ll_{R^*} M$. Also, $B\ll_{R^*} M$ by the same argument.

Conversely, let $A+B+C=M$ and $R^*\left(\frac{M}{C}\right)=\frac{M}{C}$. So, $A+(B+C)=M$ and $R^*\left(\frac{M}{B+C}\right)=\frac{M}{B+C}$ by using Corollary 2.4 and since $A\ll_{R^*} M$, therefore, $M=B+C$, as $B\ll_{R^*} M$ and $R^*\left(\frac{M}{C}\right)=\frac{M}{C}$, then $M=C$.

(3) Suppose that $A+K=M$ and $R^*\left(\frac{M}{K}\right)=\frac{M}{K}$, $B=M\cap B=(A+K)\cap B=(K\cap B)+A=B$ (by Modular Low), to show $R^*(\frac{B}{K\cap B})=\frac{B}{K\cap B}$ (by Second Isomorphic Theorem) $\frac{B}{K\cap B}\cong\frac{K+B}{K}\cong\frac{M}{K}$,but $R^*(\frac{M}{K})=\frac{M}{K}$, hence $R^*(\frac{B}{K\cap B})=\frac{B}{K\cap B}$ and since $A\ll_{R^*} B$, then $K\cap B=B$, so $B\subseteq K$, implies that $A\subseteq K$, then $K=M$.

(4) Suppose that $f(A)+B=f(M)$, for $B\subseteq f(M)$ and $R^*\left(\frac{f(M)}{B}\right)=\frac{f(M)}{B}$. Then $A+f^{-1}(B)=M$, to show $R^*\left(\frac{M}{f^{-1}(B)}\right)=\frac{M}{f^{-1}(B)}$. Define the map $g:\frac{M}{f^{-1}(B)}\to\frac{f(M)}{B}$ by $g(x+f^{-1}(B))=f(x)+B$,$\forall x\in M$ g is well defined , let $x_1+f^{-1}(x_1)=x_2+f^{-1}(x_2)$ iff $x_1-x_2\in f^{-1}(B)$ if and only if $f(x_1)-f(x_2)\in B$ if and only if $f(x_1)+B=f(x_2)+B$, g is

onto, let $w \in \frac{f(M)}{B}$ $so\ w = f(x) + B, x \in M$, then $g(x + f^{-1}(B))= f(x) + B$, therefore, g is onto . g is one to one, let $f(x_1) + B = f(x_2) + B$ if and only if $f(x_1) - f(x_2) \in B$ if and only if $f(x_1 - x_2) \in B$ if and only if $x_1 - x_2 \in f^{-1}(B)$ implies that $x_1 + f^{-1}(B) = x_2 + f^{-1}(B)$, hence g is isomorphism. Now, by [17] $g(R^*(\frac{M}{f^{-1}(B)})) = R^*\left(\frac{f(M)}{B}\right) = \frac{f(M)}{B}$, therefore, $R^*(\frac{M}{f^{-1}(B)})=g^{-1}\left(\frac{f(M)}{B}\right) = \frac{M}{f^{-1}(B)}$ implies $R^*\left(\frac{M}{f^{-1}(B)}\right) = \frac{M}{f^{-1}(B)}$. Since $A \ll_{R^*} M$, therefore, $M = f^{-1}(B)$ implies that $f(M) = B$.
(5) ⇒) Let $P_1: M_1 \oplus M_2 \to M_1$ be a projection map on M_1, since $A_1 \oplus A_2 \ll_{R^*} M_1 \oplus M_2$ by (4), $P_1(A_1 \oplus A_2) \ll_{R^*} M_1$, therefore, $A_1 \ll_{R^*} M_1$. Similarly, $A_2 \ll_{R^*} M_2$.
⇐) Let $J_1: M_1 \to M_1 \oplus M_2$ be the injection map from M_1 .Since $A \ll_{R^*} M_1$, therefore by (4) $j(A_1) = A_1 \oplus 0 \ll_{R^*} M_1 \oplus M_2$ and since $J_2: M_2 \to M_1 \oplus M_2$ by (2) , then $A_1 \oplus 0 + 0 \oplus A_2 = A_1 \oplus A_2 \ll_{R^*} M_1 \oplus M_2$.

Proposition 2.7:
Let M be an R-module and $A \subseteq B \subseteq M$ .If B is direct summand in M and $A \ll_{R^*} M$,then $A \ll_{R^*} B$.
Proof.
Let $A + L = B$, and B be a direct summand of M, then $M = B \oplus B_1$ for $B_1 \subseteq M$, so $M= A + L + B_1$, since $\frac{M}{L+B_1} = \frac{B+L+B_1}{L+B_1} \cong \frac{B}{B\cap(L+B_1)} = \frac{B}{L+(B\cap B_1)} = \frac{B}{L}$ (by using Second Isomorphism Theorem and Modular Law). Consequently, $R^*\left(\frac{B}{L}\right) = \frac{B}{L}$, then by Corollary 2.4, $R^*\left(\frac{M}{L+B_1}\right) = \frac{M}{L+B_1}$, but $A \ll_{R^*} M$, then $M = L + B_1$. Now, $B = B \cap M = B \cap (L + B_1) = L + (B \cap B_1) = L$ (by Modular Law). Thus, $A \ll_{R^*} B$.

Proposition 2.8:
Assume that M is an R-module and $A, B,$ and C are submodules of M with $A \subseteq B \subseteq C \subseteq M$, if $B \ll_{R^*} C$, then $A \ll_{R^*} M$.
Proof.
Suppose that $A + L = M$ and $R^*\left(\frac{M}{L}\right) = \frac{M}{L}$, so $B + L = M$, since $C \subset M$, hence $C = M \cap C = (B + L) \cap C = B + (L \cap C)$ by Modular Low. To demonstrate $R^*\left(\frac{C}{L\cap C}\right) = \frac{C}{L\cap C}$, since $\frac{C}{L\cap C} \cong \frac{L+C}{C} \cong \frac{M}{C}$ by Second Isomorphic Theorem, but $R^*\left(\frac{M}{C}\right) = \frac{M}{C}$, then $R^*\left(\frac{C}{L\cap C}\right) = \frac{C}{L\cap C}$. Since $B \ll_{R^*} C$, then $C = L \cap C$, so $C \subseteq L$, but $A \subseteq C$. Hence, $A \subseteq L$, since $A + L = M$, then $L = M$. Therefore, $A \ll_{R^*} M$.

Remark 2.9:
It is uncommon for the converse of Proposition 2.7 to be true. The example that follows demonstrates. Consider $M = Z \oplus Z_{p\infty}$ as Z- module $0 \oplus Z \subseteq 0 \oplus Z_{p\infty} \subseteq Z \oplus Z_{p\infty}$ it is clear that $Z \ll_{R^*} Z \oplus Z_{p\infty}$, but $Z_{p\infty}$ is not R^*-small in $Z \oplus Z_{p\infty}$(see 2.2 (3)).

3. R^*-hollow module
Definition 3.1:
A module M a is non- zero if for each proper submodule of M is Rad_R $*$-small submodule of M, then M referred to an Rad_R $*$-hollow module (or simply R^*-hollow module).

Remarks and examples 3.2:

1. Every simple module is R^*-hollow module. For example, Z_p as Z-module, (where p is a prime number).
2. It is obvious that any hollow module is R^*-hollow module. However, the reverse is not true. For example, Z_6 as Z-module.
3. Z_4as Z-module is R^*-hollow module.
4. Consider $M = Z \oplus Z_{p\infty}$as Z-module is not R^*-hollow module. Since $Z_{p\infty}$ is proper sub module of M ,but $Z_{p\infty}$is not R^*-small module of M.

The resulting theorem gives an explains of the R^*-hollow module.

Theorem 3.3:

M is R^*-hollow module if and only if for each proper submodule A of M is small in M with $R^*\left(\frac{M}{A}\right) = \frac{M}{A}$ where M is R-module.

Proof.

⇒) Assuming that A is a proper submodule of M such that $R^*\left(\frac{M}{A}\right) = \frac{M}{A}$. We have to show that $A \ll M$. Assume that there exists a proper submodule B of M such that $M = A + B$, since M is R^*-hollow ,then B$\ll_{R^*}$ M ,but $R^*\left(\frac{M}{A}\right) = \frac{M}{A}$, then $M = A$ which is a contradiction . Thus $A \ll M$.

⇐) To demonstrate that M is R^*- hollow module. Take A is a proper submodule of M. Assume that A is not R^*-small, there exists a proper submodule B of M such that $R^*\left(\frac{M}{B}\right) = \frac{M}{B}$ and $M = A + B$. By our assumption $B \ll M$, then $A = M$ which is a contradiction. Thus, M is R^*-hollow.

Proposition 3.4:

A non-zero epimorphic image of R^*-hollow module is R^*-hollow .

Proof.

Given $f: M \to M^{`}$ be an epimorphism and $M$ be $R^*$-hollow module and assume B$\subsetneq M^{`}$, therefore f^{-1}(B) $\subsetneq M$. To show that f^{-1}(B) is a proper in M. If $f^{-1}(B) = M$, then $B = f(M) = M^{`}$ which implies B $= M^{`}$ and that is a contradiction, so $f^{-1}(B)$ is a proper submodule of M. Since M is an R^*-hollow, therefore $f^{-1}(B) \ll_{R^*} M$ and, by Proposition 2.6 we have $f\left(f^{-1}(B)\right) \ll_{R^*} M^{`}$, then $B \ll_{R^*} M^{`}$.

Corollary 3.5:

Suppose M is an R-module and $A \subseteq M$. if M is an R^*-hollow, then $\frac{M}{A}$ is an R^*-hollow.

Proof.

Let $f: M \to \frac{M}{A}$ be a natural epimorphism and M be an R^*-hollow Proposition 3.4, $\frac{M}{A}$ is an R^*-hollow.

Remark 3.6:

It is not required that the direct sum of an R^*-hollow modules be R^*-hollow. For instance $M = Z \oplus Z_{p\infty}$as Z-module as previously demonstrated that every proper submodule of Z is R^*-small so Z as Z-module is R^*-hollow and every proper submodule of $Z_{p\infty}$ is not big, hence is R^*-small so $Z_{p\infty}$as Z-module is R^*-hollow. However, M is not an R^*-hollow.

The direct sum of R^*-hollow modules is an R^*-hollow under the criteria we will now list. Not that a submodule A of M is called completely invariant for each $f \in End(M)$, $f(A) \subseteq A$

. In [11], [18], likewise a module M is said to be a duo module if for each of its submodules is completely invariant.

Proposition 3.7:
Let $M = M_1 \oplus M_2$, M be a duo module, then M is an R^*-hollow if and only if M_1 and M_2 are R^*-hollow. Provided that $A \cap M_1 \neq M_i$, for all $i = 1,2$, $A \subseteq M$.
Proof.
⇒) Let M be an R^*-hollow module and $A_1 \oplus A_2 \subsetneq M_1 \oplus M_2$ with $A_1 \subsetneq M_1$ and $A_2 \subsetneq M_2$. Consider that $\pi_1: M_1 \oplus M_2 \to M_1$ be a projection map, which defined as follows , $\pi_1(m_1 + m_2) = m_1$, for all $m_1 \in M_1, m_2 \in M_2$, since $A_1 \oplus A_2 \ll_{R^*} M_1 \oplus M_2$ .Then by Proposition 2.6, we have $\pi_1(A_1 \oplus A_2) \ll_{R^*} \pi_1(M_1 \oplus M_2)$, so $A \ll_{R^*} M_1$. That means M_1 is an R^*-hollow. By using the same way one can show that M_2 is an R^*-hollow.
⇐) Let M_1 and M_2 be an R^*-hollow, and let $A \subsetneq M$. Since M is a duo module, then $A = A_1 \oplus A_2$, where $A_1 = A \cap M_1$and $A_2 = A \cap M_2$, since $A_1 \ll_{R^*} M_1$ and $A_2 \ll_{R^*} M_2$. Then by Proposition 2.6, $A_1 \oplus A_2 \ll_{R^*} M = M_1 \oplus M_2$.
A module M is said to be distributive if for each of its submodules A, B, and C in M such that $(A + B) \cap C = A \cap C + B \cap C$ [19], [20].
Proposition 3.8:
Assuming that $M = M_1 \oplus M_2$ be a distributive module, then M is an R^*-hollow if and only if M_1 and M_2 are R^*-hollow are provided that $A \cap M_i \neq M_i$ for all $i = 1,2$.
Proof.
By using the same argument in Proposition 3.7.

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