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Proving the Equality of the Spaces $Q_b^r(A)$, $Q_b^l(A)$ and $BL(X)$ where X is a Complex Banach Space

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Abstract
Cabrera and Mohammed proved that the right and left bounded algebras of quotients $Q_b^r(A)$ and $Q_b^l(A)$ of norm ideal A on a Hilbert space H are equal to $BL(H)$ Banach algebra of all bounded linear operators on H. In this paper, we prove that $(Q_b^r(A), \|\cdot\|_r) = (Q_b^l(A), \|\cdot\|_l) = (BL(X), \|\cdot\|_\infty)$ where A is a norm ideal on a complex Banach space X.

Keywords: bounded algebra of quotient, ultraprime algebra, norm ideal.

برهان المساواة للفضاءات $Q_b^r(A)$ و $Q_b^l(A)$ و $BL(X)$ حيث X فضاء باناخ المعقد

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الخلاصة
كابريرا ومحمد برهنا بأن جبر القسومات اليمني واليساري المقيدة $Q_b^r(A)$ و $Q_b^l(A)$ على المثالية المعيارية A على فضاء هلبرت H مساوية الى $BL(H)$ جبر باناخ لكل المؤثرات الخطية المقيدة على H. في هذا البحث ، برهنا $(Q_b^r(A), \|\cdot\|_r) = (Q_b^l(A), \|\cdot\|_l) = (BL(X), \|\cdot\|_\infty)$ على المثالية المعيارية A على فضاء باناخ المعقد X.

1. Introduction

The evolution of the ring theory of quotients of a prime ring was presented in 1969 by Martindale [1]. In 1986, Mathieu used ultraprime algebra to give an analytic form of algebras of quotients of ultraprime algebra [2]. Later in 2003, Cabrera and Mohammed provided a similar analytic form of totally prime algebra. One interesting result is that the left and right bounded algebras of quotients of norm ideal on a Hilbert space H are equal to complex Banach algebra of all bounded linear operators on H see [3].

In this paper, we improve this result using Banach spaces instead of Hilbert spaces. Throughout this paper all algebras are associative.

Recall in [2] a normed algebra $(A, \|\cdot\|)$ is ultraprime if there exists $c > 0$ such that $c\|x\|\|y\| \leq \|M_{x,y}\|$ for all $x, y \in A$. Where $M_{x,y}$ is a linear operator from A into A defined by $M_{x,y}(z) = xzy$ for all $z \in A$. $Q^r(A)$ is denoted to be the right Martindale algebras of quotients [1], L_q^I is a linear operator from I into A, defined by $L_q^I(x) = qx$, for all $x \in I$.

For a prime normed algebra A, the right bounded algebra of quotient of A is given by

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$Q_b^r(A) = \{q \in Q^r(A)/\ I$ is ideal of A, $qI \subseteq A$, L_q^I bounded$\}$, with algebra semi norm defined by $\|q\|_r = \inf\ \{\|L_q^I\|, I$ is ideal of A, $qI \subseteq A$, L_q^I is bounded$\}$.

When A is ultraprime algebra, then $Q_b^r(A)$ is an ultraprime algebra, and the inclusion of A into $Q_b^r(A)$ is topological [2, Theorem 4.1]. Similarly, left bounded algebra of quotient are defined and denoted by $Q_b^l(A)$.

Mohammed and Cabrera [3, Theorem 2] proved for a norm ideal $(A, \|\cdot\|)$ in $BL(H)$ the bounded linear operators in H, then $(Q_b^r(A), \|\cdot\|_r) = (Q_b^l(A), \|\cdot\|_l) = (BL(H), \|\cdot\|_\infty)$.

Our aim in this paper is to improve the above result (see Theorem 2.2 and 2.5).

2. Bounded algebras of quotients of norm ideal.

Recall in [4] that, the norm ideal is an ideal A of $BL(X)$ where X is a Banach space, with norm $\|\cdot\|$ satisfying the following properties:

i. $\|x \otimes f\| = \|x\|\|f\|$, for all $x \in X$ and $f \in X'$.
ii. $\|FTG\| \le \|F\|_\infty \|T\| \|G\|_\infty$ for all $T \in A$ and $F, G \in BL(X)$.

For two vector spaces X, Y and dual specs X', Y', in [5, p. 240] the adjoint of an operator $T: X \longrightarrow Y$ is an operator $T': Y' \longrightarrow X'$ defined by $(T'\varphi)(x) = \varphi(T(x))$ for any $\varphi \in Y'$ and $x \in X$.

In the following proposition we used $L(X)$ to denote the algebra of all linear operators form vector space X to X, $End(X_D)$ is denoted to the algebra of all endomorphism on right $D-$module X. To compute the right bounded algebras of quotients we begin with the following result.

Proposition 2.1

Let X be a complex Banach space and A is a norm ideal of $BL(X)$. Then $Q^r(A) = L(X)$.

Proof

Let X' be the dual space of X. Then $\langle X, X' \rangle$ is a pair of dual space. Using [6, Structure Theorem, p. 75] A is a primitive algebra with non-zero socle. Since X is a left $A-$module, by [7, Theorem 4.3.7 (vii) and (viii), p. 144], $Q^r(A) \cong End(X_D)$, where D is the centralizer of the irreducible left $A-$module X. So D is a complex normed division algebra, by Maizer Theorem [7, Theorem 2, p. 71], $D \cong 1 \cdot \mathbb{C}$, so $Q^r(A) \cong End(X_D) \cong L(X_{1\cdot\mathbb{C}}) \cong L(X)$. Therefore $Q^r(A) = L(X)$. ■

The next theorem is the main result in this paper.

Theorem 2.2

Let X be a complex Banach space and A be a norm ideal of $BL(X)$. Then $(Q_b^r(A), \|\cdot\|_r) = (BL(X), \|\cdot\|_\infty)$

Proof

Since A is a norm ideal of $BL(X)$, it follows that A contains a non-zero socle $FBL(X)$. By proposition 2.1, $Q^r(A) = L(X)$. Using [8, Lemma 2, 12, p. 26], $(Q_b^r(A), \|\cdot\|_r)$ is right bounded algebra of quotient with semi norm $\|\cdot\|_r$. For proving $BL(X) \subseteq Q_b^r(A)$, let $G \in BL(X)$, so $G(FBL(X)) \subseteq A$. We denoted Id_X the identity operator in $BL(X)$, and let $T \in FBL(X)$.

$$\left\|L_G^{FBL(X)}(T)\right\| = \|GT\| = \|GTId_X\|$$
$$\le \|G\|_\infty \|T\| \|Id_X\|_\infty = \|G\|_\infty \|T\|$$

Therefore $L_G^{FBL(X)}$ is bounded, so $G \in Q_b^r(A)$. Thus $BL(X) \subseteq Q_b^r(A)$. Now

$$\|G\|_r = \left\|L_G^{FBL(X)}\right\| = \sup_{T \in FBL(X)} \{\left\|L_G^{FBL(X)}(T)\right\|, \|T\| = 1\}$$
$$\le \sup_{T \in FBL(X)} \{\|G\|_\infty \|T\|, \|T\| = 1\} = \|G\|_\infty$$

Then we get $\|G\|_r \le \|G\|_\infty$-----(1)

Conversely

Let $G \in Q_b^r(A)$, for proving $G \in BL(X)$, let $0 \ne x \in X$ and $f \in X'$ with $\|f\| = 1$, so $G \in L(X)$ by proposition 2.1.

$$\|G(x)\| = \|G(x)\|\|f\| = \|G(x) \otimes f\| = \|G(x \otimes f)\|$$

Since $x \otimes f$ is finite rank operator, so $x \otimes f \in FBL(X)$

$$= \left\|L_G^{FBL(X)}(x \otimes f)\right\|$$

Since $G \in Q_b^r(A)$, so $L_G^{FBL(X)}$ is bounded

$$\le \left\|L_G^{FBL(X)}\right\| \|(x \otimes f)\| = \left\|L_G^{FBL(X)}\right\| \|x\|\|f\|$$
$$= \|G\|_r \|x\|$$

Therefore $\|G(x)\| \leq \|G\|_r \|x\|$ for all $x \in X$, so $G \in BL(X)$. We get that $BL(X) = Q_b^r(A)$.
To prove the converse of (1) consider

$$\|G\|_\infty = \sup_{x\in X}\{\|G(x)\|, \|x\| = 1\}$$
$$\leq \sup_{x\in X}\{\|G\|_r\|x\|, \|x\| = 1\} = \|G\|_r$$

This implies that

$\|G\|_\infty \leq \|G\|_r$-------(2)

From (1) and (2), $\|G\|_\infty = \|G\|_r$. Also, we have $(Q_b^r(A), \|\cdot\|_r)$ is a normed algebra and $(Q_b^r(A), \|\cdot\|_r) = (BL(X), \|\cdot\|_\infty)$.■

The following result is used to compute the left algebras of quotients.

Proposition 2.3

Let X be a complex Banach space and A is a norm ideal of $BL(X)$ containing a non-zero socle. Then $Q^l(A) = L(X)$.

Proof

From the proof of proposition 1.1, A is right primitive algebra with non-zero socle. By using [9, Theorem 11.11, p. 174] $Q^l(A) = L(X)$.■

Corollary 2.4

Let X be a complex Banach space and A is a norm ideal of $BL(X)$ containing a non-zero socle. Then $Q^r(A) = Q^l(A) = L(X)$.

Our second main result in the following Theorem.

Theorem 2.5

Let X be a complex Banach space and A be a norm ideal of $BL(X)$. Then $\left(Q_b^l(A), \|\cdot\|_l\right) = (BL(X), \|\cdot\|_\infty)$

Proof

Since A is a norm ideal of $BL(X)$, it follows thatit contains a non-zero socle $FBL(X)$. By proposition 2.3, $Q^l(A) = L(X)$. Using [8, Lemma 2.12, p. 26], $\left(Q_b^l(A), \|\cdot\|_l\right)$ is left bounded algebra of quotient with semi norm $\|\cdot\|_l$. For proving that $BL(X) \subseteq Q_b^l(A)$, let $G \in BL(X)$, so $(FBL(X))\mathrm{G} \subseteq A$.

We denoted Id_X the identity operator in $BL(X)$, and let $T \in FBL(X)$.

$$\left\|R_G^{FBL(X)}(T)\right\| = \|TG\| = \|Id_X TG\|$$
$$\leq \|Id_X\|_\infty \|T\| \|G\|_\infty = \|T\| \|G\|_\infty$$

Therefore $R_G^{FBL(X)}$ is bounded, so $G \in Q_b^l(A)$. Thus$BL(X) \subseteq Q_b^l(A)$. Now

$$\|G\|_l = \left\|R_G^{FBL(X)}\right\|$$
$$= \sup_{T\in FBL(X)}\{\left\|R_G^{FBL(X)}(T)\right\|, \|T\| = 1\}$$
$$\leq \sup_{T\in FBL(X)}\{\|T\|\|G\|_\infty, \|T\| = 1\} = \|G\|_\infty$$

Then we get $\|G\|_l \leq \|G\|_\infty$------(1)

Conversely

Let $G \in Q_b^l(A)$, for proving $G \in BL(X)$, let $0 \neq x \in X$ and $f \in X'$ with $\|f\| = 1$, so $G \in L(X)$ by proposition 2.1.

$$|f(G(x))| = |(G'f)(x)| = \|x\| \frac{|(G'f)(x)|}{\|x\|}$$
$$\leq \|x\| \sup_{x\in X}\left\{\frac{|(G'f)(x)|}{\|x\|}, \|x\| \neq 0\right\} = \|x\|\|G'f\|$$

For an arbitrary element $u \in X$ with $\|u\| = 1$

$$\left|f\big(G(x)\big)\right| \leq \|x\|\|G'f\|\|u\| = \|x\|\|u \otimes G'f\|_\infty$$

From the properties of operator finite rank by [10, proposition 6.1.5, p. 90]. For $z \in X$, we have

$(u \otimes G'f)(z) = (G'f)(z)u$ definition of finite rank operator

$= f(G(z))u$ adjoint operator

$= (u \otimes f)(G(z))$

Now

$$\|u\otimes G'f\|_\infty = \sup_{z\in X}\{\|(u\otimes G'f)(z)\|, \|z\| = 1\}$$
$$= \sup_{z\in X}\{\|(u\otimes f)(G(z))\|, \|z\| = 1\}$$
$$\le \sup_{z\in X}\{\|(u\otimes f)\|_\infty \|G(z)\|, \|z\| = 1\}$$
$$= \|(u\otimes f)\|_\infty \sup_{z\in X}\{\|G(z)\|, \|z\| = 1\}$$
$$= \|u\|\|f\|\|G\|_\infty$$
$= \|f\|\|G\|_\infty$, then

$$|f(G(x))| \le \|x\|\|u\otimes G'f\|_\infty \le \|x\|\|f\|\|G\|_\infty$$

We have

$\|G(x)\| = \sup_{f\in X'}\{|f(G(x))|, \|f\| = 1\}$ by [5, proposition 11.9,p. 235]

$$\le \sup_{f\in X'}\{\|x\|\|f\|\|G\|_\infty, \|f\| = 1\}$$
$$= \|x\|\|G\|_\infty \sup_{f\in X'}\{\|f\|, \|f\| = 1\} = \|x\|\|G\|_\infty$$

Therefore $\|G(x)\| \le \|G\|_r\|x\|$ for all $x \in X$, so $G \in BL(X)$, then $Q_b^l(A) \subseteq BL(X)$.

From the above, we get that

$$|f(G(x))| \le \|x\|\|u\otimes G'f\|_\infty = \|x\|\|(u\otimes f)G\|_\infty$$

since $G \in BL(X)$ we get $u\otimes G'f = (u\otimes f)G$

$$\le \|x\|\|(u\otimes f)G\|$$
$$= \|x\| \left\|R_G^{FBL(X)}(u\otimes f)\right\|$$
$$\le \|x\| \left\|R_G^{FBL(X)}\right\| \|u\otimes f\|$$
$$\le \|x\| \left\|R_G^{FBL(X)}\right\| \|u\|\|f\|$$
$$= \|x\| \left\|R_G^{FBL(X)}\right\| \|f\|$$
$$|f(G(x))| \le \|x\| \left\|R_G^{FBL(X)}\right\| \|f\|$$

$\|G(x)\| = \sup_{f\in X'}\{|f(G(x))|, \|f\| = 1\}$ by [5, proposition 11.9,p. 235]

$$\le \sup_{f\in X'}\{\|x\| \left\|R_G^{FBL(X)}\right\| \|f\|, \|f\| = 1\}$$
$$= \|x\| \left\|R_G^{FBL(X)}\right\| \sup_{f\in X'}\{\|f\|, \|f\| = 1\}$$
$$= \|x\|\|G\|_l$$
$$\|G\|_\infty = \sup_{x\in X}\{\|G(x)\|, \|x\| = 1\}$$
$$\le \sup_{x\in X}\{\|x\|\|G\|_l, \|x\| = 1\} = \|G\|_l$$

This implies that $\|G\|_\infty \le \|G\|_l$-------(2)

From (1) and (2),$\|G\|_\infty = \|G\|_l$. Also, we get that $(Q_b^l(A), \|\cdot\|_l)$ is a normed algebra, also $(Q_b^l(A), \|\cdot\|_l) = (BL(X), \|\cdot\|_\infty)$.■

Corollary 2.6

For a complex Banach space X and norm ideal A of $BL(X)$. Then $(Q_b^r(A), \|\cdot\|_r) = (Q_b^l(A), \|\cdot\|_l) = (BL(X), \|\cdot\|_\infty)$

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