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Two Involution Clean Rings with Applications in Graph Theory

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Abstract

The definition of invo-clean rings is generalized to two involution clean rings. In this paper, we aimed to identify the structure with determined the basic properties of these rings. A ring is a two-involution clean if all elements are the sum of two involutions and idempotent elements. Additionally, the graph of two involution clean rings has been defined, and some properties of the new graph, such as: connected, the diameter, girth and others, have been proven.

Keywords: Strongly nil-clean, Invo-clean, Invo-t-clean, Hosoya polynomial, Wiener index.

الحلقات النقية الملتفة من النمط-2

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الخلاصة

تم تعميم الحلقات النقية الملتفة الى حلقات نقية ملتفة من النمط-2. وحاولنا ان نجد البنية مع تحديد بعض الخواص الاساسية لهذه الحلقات في هذا البحث. تكون الحلقة نقية ملتفة من النمط-2 اذا استطعنا كتابة كل عنصر في الحلقة كحاصل جمع عنصرين ملتفين مضافاً اليه العنصر المتحايد. وكذلك تم تعريف البيان لهذه الحلقات وبُرهِنَ بعض الخصائص للبيان الجديد كالاتصال والقطر والخصر وغيرها .

1. Introduction

In this paper, R is an associated ring with an identity element, 1, which differs from the zero element. As usual, $invo(R)$ is a set of all involution elements of the ring R, and for any $w \in invo(R)$ we have $w^2 = 1$. We denote $Idmp(R)$ for the set for all idempotent in the ring R, while $Trip(R)$ denoted to the set containing of all tripotent elements such that for any $u \in Trip(R)$, we have $u^3 = u$ of R. As well as $Nil(R)$ is the set of all nilpotent elements of R. A ring R is an involution clean if, for all $x \in R$ can be expressed as $x = u + e$, $u \in invo(R)$ with $e \in Idmp(R)$. This definition dates back to 2017, when it was introduced by Danchev [1]. A new generalized to invo-clean ring is the invo-t-clean ring which is studied in [2]. A

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ring R is called invo-t-clean if, for any $a \in R$, it can be expressed as $a = u + t$, $u \in invo(R)$ and $t \in Trip(R)$. An invo-t-clean is strongly invo-t-clean if $ut = tu$.
Every tripotent element is also an invo-clean element, since $t = t + t^2 - 1 + 1 - t^2$, where $(t + t^2 - 1)^2 = 1$ and $(1 - t^2)^2 = 1 - t^2$, with $t = t^3$. All idempotent elements are invo-clean, see [1]. Moreover, Z_2, Z_3, Z_4 and Z_6 are invo-clean and invo-t-clean; although the opposite is not true, is still any invo-clean ring is invo-t-clean; the example, Z_5. As usual, $M_2(R)$ represents the matrix ring of 2×2. $T_2(R)$ is an upper triangular matrix ring of 2×2, and Z_n is the ring of integer modulo n.
The zero divisor graph is one of the famous concepts connecting the commutative rings with graph theory; this concept started with Beck [3]. Many others, like Habibi, Çelıkel, and Abdıoğlu [4], study the clean graph defined in different types of clean rings. In this paper, we describe a graph that is defined depending on two involution-clean rings.
A graph is an ordered pair of non-empty set of vertices $\mathcal{V}$, and set of edges E, $\mathfrak{G} = (\mathcal{V}, E)$ where $E \subseteq \mathcal{V} \times \mathcal{V}$. The symbol $|\mathfrak{G}|$ denotes total vertices and $\|\mathfrak{G}\|$ indicates the total of edges. The girth is represented as $g(\mathfrak{G})$ which is the shortest of cycle length in $\mathfrak{G}$. The graph we get from the 2-invo clean ring will be simple graph (without loops and multi-edges). The degree of a vertex v is defined as the number of edges incident to it, that is indicated by $deg_{\mathfrak{G}}(\mathcal{v})$ or, $\deg(\mathcal{v})$. The symbols $\delta(\mathfrak{G})$ and $\Delta(\mathfrak{G})$ are represent the minimum and maximum degrees of a graph $\mathfrak{G}$, respectively [5]. The average degree denoted by $ad(\mathfrak{G})$ is defined as :
$ad(\mathfrak{G}) = \frac{1}{|\mathfrak{G}|}\sum_{\forall \mathcal{v} \in \mathcal{V}} \deg(\mathcal{v}) = \frac{2\|\mathfrak{G}\|}{|\mathfrak{G}|}$, and it may be noted that , $\delta(\mathfrak{G}) \leq ad(\mathfrak{G}) \leq \Delta(\mathfrak{G})$.
The symbol of distance $d(\mathcal{v}, \mathcal{u})$ is defined in the connected graph is defined as a positive number of the length of the shortest $(\mathcal{v} - \mathcal{u})$-path in graph $\mathfrak{G}$. The Wiener index [6] it will be sum of the lengths of the shortest $(\mathcal{v} - \mathcal{u})$-path in the graph $\mathfrak{G}$, i.e.,
$\mathcal{W}(\mathfrak{G}) = \frac{1}{2}\sum_{\forall \mathcal{v}, \mathcal{u} \in \mathcal{V}} \mathrm{d}(\mathcal{v}, \mathcal{u})$.
The average distance is calculated as:
$\mathcal{D}(\mathfrak{G}) = \frac{2\mathcal{W}(\mathfrak{G})}{(|\mathfrak{G}|-1)|\mathfrak{G}|}$.
The symbol $d(\mathfrak{G}, \mathcal{K})$ represents the number of pairs (v, u) at distances $\mathcal{K}$ in a graph $\mathfrak{G}$, where $\mathcal{K} = 0,1, \dots, diam(\mathfrak{G})$. We denoted to the diameter of the graph $\mathfrak{G}$ by $diam(\mathfrak{G})$. Notice that, $d(\mathfrak{G}, 0) = |\mathfrak{G}|$, and $d(\mathfrak{G}, 1) = \|\mathfrak{G}\|$. Hosoya polynomial of a connected graph $\mathfrak{G}$, [7] is defined as follows:
$$\mathcal{H}(\mathfrak{G}, x) = \sum_{\mathcal{K}=0}^{diam(\mathfrak{G})} d(\mathfrak{G}, \mathcal{K})\, x^{\mathcal{K}}.$$
Further, M_{vu}-polynomial of $\mathfrak{G}$ in [8] and defined by:
$M_{vu}(\mathfrak{G}; x, y) = \sum_{uv \in E(\mathfrak{G})} m_{\deg(u),\deg(v)}(\mathfrak{G})\, x^{\deg(u)} y^{\deg(v)}$,
where $m_{\deg(u),\deg(v)}(\mathfrak{G})$ is the number of edges uv of $\mathfrak{G}$ such that $\{\deg(u), \deg(v)\} = \{\mathrm{i}, \mathrm{j}\}$.
There are many topological indices that depend on M_{vu}-polynomial in their calculation, and these indices:

- Product connectivity index of $\mathfrak{G}$
$P(\mathfrak{G}) = \sum_{uv \in E(\mathfrak{G})} \frac{1}{\sqrt{deg(u)deg(v)}}$.
- Sum connectivity index of $\mathfrak{G}$
$S(\mathfrak{G}) = \sum_{uv \in E(\mathfrak{G})} \frac{1}{\sqrt{deg(u) + \deg(v)}}$.
- Arithmetic-geometric index of $\mathfrak{G}$
$AG(\mathfrak{G}) = \sum_{uv \in E(\mathfrak{G})} \frac{deg(u) + deg(v)}{2\sqrt{deg(u)deg(v)}}$.
- Atom bond connectivity index of $\mathfrak{G}$

$ABC(\mathfrak{G}) = \sum_{uv\in E(\mathfrak{G})}\sqrt{\frac{deg(u)+deg(v)-2}{deg(u)deg(v)}}$.

You can see more topological indices in [9].

2. Two involution clean rings.

This section is devoted to the study of the algebraic properties of rings of a 2-invo-clean ring.

Definition 2.1: An element $a \in R$ called a two-involution clean element if $a = u_1 + u_2 + e$, where $u_1, u_2 \in invo(R), e \in Idmp(R)$. We denote the set of 2-invo-clean elements in R by $2 - invo_c(R)$. A ring is called two-involution clean (in short, 2-invo-clean) if every element in R is two-involution clean element. A ring R is strongly 2-invo-clean ring if R is 2-invo-clean ring and u_1, u_2, e are commutative.

Interestingly, any $1 + e \in 2 - invo_c(R)$ for any ring R, because $1 + e = (2 \cdot e - 1) + 1 + (1 - e)$ where $(1 - e)^2 = 1 - e$ and $(2e - 1)^2 = 1$.

Examples 2.2:

1- The ring of integer modulo 14, Z_{14} is not 2-invo-clean ring, but it has the elements $2 - invo_c(Z_{14}) = \{0,1,2,3,5,6,7,8,9,10,12,13\}$,

2- The ring Z_n is a 2-invo-clean ring whenever $n \in \{2,3,4,5,6,8,10,12,15,20,24,30,40, 60,120\}$.

3- The upper triangular matrices $T_2(Z_2), T_2(Z_3)$ are 2-invo-clean rings.

Theorem 2.3: Let T be homomorphic from a ring R onto ring S, and R is a 2-invo-clean ring R onto S. Then S is a 2-invo-clean ring.

Proof: Let $y \in S$, then there is $x \in R$ such that $T(x) = y$. Since x is a 2-invo-clean element, we can write x as $x = u_1 + u_2 + e$ where $u_1, u_2 \in invo(R)$, and $e \in Idmp(R)$.

Since T is homomorphic, then, $y = T(x) = T(u_1 + u_2 + e) = T(u_1) + T(u_2) + T(e)$ and $(T(u_1))^2 = T({u_1}^2) = T(1_R) = 1_S$, where 1_R is the identity element in R and 1_S is the identity element in S. Similarly, we get $\left(T(u_2)\right)^2 = 1_S$. Hence $T(u_1), T(u_2) \in invo(S)$ and $\left(T(e)\right)^2 = T(e^2) = T(e) \in Idmp(R)$. Then, y is a 2-invo-clean element in S.■

Corollary 2.4: Assume R is a 2-involution clean ring, and the ideal I in R. Then R/I is 2-invo-clean ring if T is homomorphic from R to R/I.

Corollary 2.5:

1- Any invo-clean element is an invo-t-clean.

2- Any invo-t-clean element is 2-invo-clean.

3- Any invo-clean element is 2-invo-clean.

Proof:

1- In [2], explain that.

2- Let $x \in R$, $x = u_1 + t = u_1 + (t^2 + t - 1) + (1 - t^2)$, because $(t^2 + t - 1)^2 = 1$ and $(1 - t^2)^2 = (1 - t^2)$. Hence, $x \in 2 - invo_c(R)$.

3- It is clear that 1 and 2 implies 3.

The below example shows that the opposite of Corollary 2.5 is not true

Examples 2.6:

1- Let Z_{17} be a ring of integer modulo 17; the idempotent elements of Z_{17} are $\{0,1\}$ and the tripotent elements of Z_{17} are $\{0, 1, 16\}$ and the involution elements of Z_{17} are $\{1,16\}$. The 2-invo-clean elements are $\{0, 1, 2, 3, 15, 16\}$, the invo-clean elements are $\{0,1,2,16\}$, and the invo-t-clean elements are $\{0, 1, 2, 15, 16\}$. Note that element 3 is 2-invo-clean but not invo-clean and it is invo-t-clean. The element 15, it is invo-t-clean, but it is not invo-clean.

2- The ring $T_2(Z_4)$ is 2-invo-clean, which is not invo-clean nor invo-t-clean ring.

$$Idmp\big(T_2(Z_4)\big) = \{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\0&1\end{bmatrix}, \begin{bmatrix}0&2\\0&1\end{bmatrix}, \begin{bmatrix}0&3\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}1&2\\0&0\end{bmatrix}, \begin{bmatrix}1&3\\0&0\end{bmatrix}\}.$$

$$Trip\big(T_2(Z_4)\big) = \{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}0&0\\0&3\end{bmatrix}, \begin{bmatrix}0&1\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\0&3\end{bmatrix}, \begin{bmatrix}0&2\\0&1\end{bmatrix}, \begin{bmatrix}0&2\\0&3\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&3\\0&1\end{bmatrix}, \begin{bmatrix}0&3\\0&3\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&3\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\0&3\end{bmatrix}, \begin{bmatrix}1&2\\0&0\end{bmatrix}, \begin{bmatrix}1&2\\0&1\end{bmatrix}, \begin{bmatrix}1&2\\0&3\end{bmatrix}, \begin{bmatrix}1&3\\0&0\end{bmatrix}, \begin{bmatrix}1&3\\0&3\end{bmatrix} \begin{bmatrix}3&0\\0&0\end{bmatrix}, \begin{bmatrix}3&0\\0&1\end{bmatrix}, \begin{bmatrix}3&0\\0&3\end{bmatrix}, \begin{bmatrix}3&1\\0&0\end{bmatrix}, \begin{bmatrix}3&1\\0&1\end{bmatrix}, \begin{bmatrix}3&2\\0&0\end{bmatrix}, \begin{bmatrix}3&0\\0&0\end{bmatrix}, \begin{bmatrix}3&2\\0&1\end{bmatrix}, \begin{bmatrix}3&2\\0&3\end{bmatrix}, \begin{bmatrix}3&3\\0&0\end{bmatrix}, \begin{bmatrix}3&3\\0&1\end{bmatrix}\}.$$

$$invo\big(T_2(Z_4)\big) = \{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&3\end{bmatrix}, \begin{bmatrix}1&1\\0&3\end{bmatrix}, \begin{bmatrix}1&2\\0&1\end{bmatrix}, \begin{bmatrix}1&2\\0&3\end{bmatrix}, \begin{bmatrix}1&3\\0&3\end{bmatrix}, \begin{bmatrix}3&0\\0&1\end{bmatrix}, \begin{bmatrix}3&0\\0&3\end{bmatrix}, \begin{bmatrix}3&1\\0&1\end{bmatrix}, \begin{bmatrix}3&2\\0&1\end{bmatrix}, \begin{bmatrix}3&2\\0&3\end{bmatrix}, \begin{bmatrix}3&3\\0&1\end{bmatrix}\}.$$

There is 56 elements are invo-clean in $T_2(Z_4)$, and the elements which are 2-invo-clean but not invo-clean are $\{\begin{bmatrix}2&3\\0&2\end{bmatrix}, \begin{bmatrix}3&1\\0&3\end{bmatrix}, \begin{bmatrix}2&1\\0&2\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}3&3\\0&3\end{bmatrix}, \begin{bmatrix}1&3\\0&1\end{bmatrix}, \begin{bmatrix}0&3\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}\}$.

But, there is 60 elements are invo-t-clean in $T_2(Z_4)$, and the elements which are 2-invo-clean but not invo-t-clean are

$$\left\{\begin{bmatrix}3&1\\0&3\end{bmatrix}, \begin{bmatrix}3&3\\0&3\end{bmatrix}, \begin{bmatrix}1&3\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}\right\}.$$

We will now present some properties related to the 2-invo-cleane elements.

Proposition 2.7: An element x in a ring R is a 2-invo-clean element if and only if $1 - x$ is a 2-invo-clean element.

Proof: Assume that x is a 2-invo-clean element, then $x = u_1 + u_2 + e$ where $u_1, u_2 \in invo(R)$, and $e \in Idmp(R)$, then $1 - x = 1 - (u_1 + u_2 + e) = 1 - u_1 - u_2 - e$, where $-u_2, 1 - u_1 \in invo(R)$ and $1 - e \in Idmp(R)$. Hence, $1 - x$ is a 2-invo-clean element.
Conversely, let $1 - x$ be 2-invo-clean element, then $1 - x = u_1 + u_2 + e \Rightarrow x = 1 - u_1 - u_2 - e = -u_1 - u_2 + (1 - e)$, where $-u_2, -u_1 \in invo(R)$ and $1 - e \in Idmp(R)$. Hence, x is a 2-invo-clean element. ■

Corollary 2.8: If element x is an invo-clean in the ring R, the following hold.
1- $1 + x$ is a 2-invo-clean.
2- $1 - x$ is a 2-invo-clean.

Proof:
1- Let $x = u_1 + e$ **,** where $u_1 \in invo(R)$, $e \in Idmp(R)$. $1 + x = 1 + u_1 + e$. Hence, $1 + x$ is a 2-invo-clean element.
2- From Corollary 2.5 (3) and Proposition 2.7, $1 - x$ is a 2-invo-clean element. ■

Examples 2.9:
The converse of (1) and (2) in Corollary 2.8 is not valid. For example,
1- In the ring Z_9, the elements 3 and 7 are 2-invo-clean elements, but the elements 3 and 7 are not invo-clean elements. The element 7 is an invo-t-clean element, but the element 7 not an invo-clean element. The element $1 + 6 = 7$ is a 2-invo-clean element, but 6 is not an invo-clean element, and the element $1 - 7 = -6 = 3$ is a 2-invo-clean element, but 7 is not an invo-clean element.

2- For example, in non-commutative ring, $T_2(Z_3)$. The elements $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 2\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$ are 2-invo-clean elements and invo-t-clean elements, but they are not invo-clean elements, as $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$ is a 2-invo-clean element, but $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ is not an invo-clean element, and $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 2\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$ is a 2-invo-clean element, but $\begin{bmatrix}0 & 2\\0 & 0\end{bmatrix}$ is not invo-clean element.

Theorem 2.10: Let R be a ring, and the sum of two involution elements is involution. Then R is an invo-clean ring if and only if R is a 2-invo-clean ring.
Proof: Let $z \in R$ be a 2-invo-clean element, then $z = u_1 + u_2 + e = q + e$, where $q \in invo(R), e \in Idmp(R)$. Hence, R is an invo-clean ring.
The converse comes from Corollary 2.5 (3). ■

Theorem 2.11: Let R be a tripotent ring. Then R is a 2-invo-clean ring if and only if R is an invo-t-clean ring.
Proof: Let $z \in R$ be a 2-invo-clean element, then $z = u_1 + u_2 + e = u_1 + t_1$, since R is tripotent ring, $u_2 + e \in R$ so $t_1 = u_2 + e \in Tri(R)$. Hence, R it well be invo-t-clean ring. But the converse comes from Corollary 2.5 (2). ■

Proposition 2.12: Let $3 \in 2 - invo_c(R)$ in a ring R, with $3 = u_1 + u_2 + e$, $u_1, u_2 \in invo(R)$, $e \in Idmp(R)$. Then, the following holds:
1- $2u_1e = 2u_2e \text{ or } 2e = 2u_1u_2e$.
2- $15e = 15$.
3- $6u_1 = 6u_2$
4- $60u_1 = 60u_2 = 60e = 60$.
Proof (1): Since $3 = u_1 + u_2 + e$, where $u_1, u_2 \in invo(R)$ and $e \in Idmp(R)$. Thus
$3e = u_1e + u_2e + e$ so $2e = u_1e + u_2e$.
Now,
$24 = 3^3 - 3 = (u_1 + u_2 + e)^3 - (u_1 + u_2 + e) = 8.3 = 8(u_1 + u_2 + e)$
$24 = (u_1 + u_2 + e)^3 - (u_1 + u_2 + e)$
$= (u_1 + u_2 + e)[(u_1 + u_2 + e)^2 - 1]$
$= (u_1 + u_2 + e)(1 + 2\, u_1u_2 + 5e)$
$= u_1 + 2u_2 + 5u_1e + u_2 + 2u_1 + 5u_2e + e + 2u_1u_2e + 5e$
$= 3u_1 + 3u_2 + 5u_1e + 5u_2e + 2u_1u_2e + 6e.$
Now, $3u_1 + 3u_2 + 5u_1e + 5u_2e + 2u_1u_2e + 6e = 8(u_1 + u_2 + e)$.
This implies that $5u_1 + 5u_2 - 10e - 6e + 8e - 2u_1u_2e = 0$. Multiply both sides by e, hence $5u_1e + 5u_2e - 8e - 2u_1u_2e = 0 \Rightarrow 10e - 8e = 2u_1u_2e \Rightarrow 2e = 2u_1u_2e$ or, $2u_1e = 2u_2e$.

Proof (2): Since $24 = 3^3 - 3 = (u_1 + u_2 + e)^3 - (u_1 + u_2 + e)$, from (1) $24 = 3u_1 + 3u_2 + 5u_1e + 5u_2e + 2u_1u_2e + 6e = 3(u_1 + u_2) + 16e + 2e = 3(3 - e) + 18e \Rightarrow 24 - 9 = 15e \Rightarrow 15 = 15e$.
Proof (3): Since $3 = u_1 + u_2 + e$, $3 - e = u_1 + u_2$ by squaring two both side we get $(3 - e)^2 = (u_1 + u_2)^2 \Rightarrow 9 - 6e + e = {u_1}^2 + 2u_1u_2 + {u_2}^2 \Rightarrow 9 - 5e = 2 + 2u_1u_2 \Rightarrow 7 - 5e = 2u_1u_2$ multiply both side by 3 we get $21 - 15e = 6u_1u_2$ from (2) we get $21 - 15 = 6u_1u_2 \Rightarrow 6 = 6u_1u_2$ multiply both side by u_1 we get $6u_1 = 6u_2$.
Proof (4): Since $9 - 3 = 3^2 - 3 = (u_1 + u_2 + e)^2 - (u_1 + u_2 + e)$
$\Rightarrow 6 = ({u_1}^2 + 2u_1(u_2 + e) + (u_2 + e)^2) - u_1 - u_2 - e$

$= 1 + 2u_1u_2 + 2u_1e + 2u_2e + e - u_1 - u_2 - e$
$= 2 + 2u_1u_2 + 4e - u_1 - u_2$
$\Rightarrow 4 = 2u_1u_2 + 4e - u_1 - u_2$ multiply both sides by 6
$\Rightarrow 24 = 12u_1u_2 + 24e - 6u_1 - 6u_2$ from (3) we get, $6u_1u_2 = 6$
$\Rightarrow 24 = 12 + 24e - 12u_1 \Rightarrow 12u_1 = 15e + 9e - 12$
$\Rightarrow 12u_1 = 3 + 9e$ multiply both sides by 5 where $15e = 15$
$\Rightarrow 60u_1 = 15 + 45e,\ 60u_1 = 60u_2 = 60e = 60.$ ■

Proposition 2.13: Let 4 be a 2-invo-clean element in a ring R. Then $240 = 0$ and the elements $\{30, 60, 90, 120, 150, 190, 210\}$ are nilpotent.
Proof: Let $4 = u_1 + u_2 + e \ldots (1)$,
then $e = 4 - u_1 - u_2$.
Since $e^2 = e \Rightarrow (4 - u_1 - u_2)^2 = 4 - u_1 - u_2$
$\Rightarrow 16 - 8u_1 - 8u_2 + {u_1}^2 + 2u_1u_2 + {u_2}^2 = 4 - u_1 - u_2$
$\Rightarrow 14 + 2u_1u_2 = 7u_1 + 7u_2, \ldots (2)$
by adding $7e$ to both side we get
$14 + 2u_1u_2 + 7e = 7u_1 + 7u_2 + 7e = 7(u_1 + u_2 + e) = 7(4)$
$\Rightarrow 2u_1u_2 + 7e = 28 - 14 = 14 \ldots (3)$,
multiply both sides by e we get
$2u_1u_2e = 7e \ldots (4)$,
by squaring two both sides, we get
$4e = 49e \Rightarrow 45e = 0 \ldots (5)$.
Now,
$60 = 4^3 - 4 = (u_1 + u_2 + e)^3 - (u_1 + u_2 + e)$
$= [(u_1 + u_2 + e)^2](u_1 + u_2 + e)^2 - (u_1 + u_2 + e)$
$= [{u_1}^2 + 2u_1u_2 + 2u_1e + {u_2}^2 + 2u_2e + e^2](u_1 + u_2 + e) - u_1 - u_2 - e$
$= 2u_1 + 2u_2 + 2e + 2u_1u_2e + u_1e + 2u_2 + 2u_1 + 2u_1u_2e + 2e + u_2e + 2e + 2u_1u_2e + 2u_1e + 2u_2e + e - u_1 - u_2 - e$
$= 3u_1 + 3u_2 + 6e + 6u_1u_2e + 3u_1e + 3u_2e$
$= 3(u_1 + u_2) + 6e + 6u_1u_2e + 3(u_1 + u_2)e$
$= 3(4 - e) + 6e + 6u_1u_2e + 3(4 - e)e$
$= 12 - 3e + 6e + 6u_1u_2e + 12e - 3e$
$\Rightarrow 48 = 6u_1u_2e + 12e = 3(2u_1u_2e) + 12e$, from Equation (4) we get $48 = 3(7e) + 12e = 33e$ multiply both side by e we get
$48e = 33e \Rightarrow 15e = 0 \ldots (6)$.
Now, $33e = 48$ by adding $12e$ to both sides we get $48 + 12e = 33e + 12e = 45e$ from (5) $48 = -12e$ by adding $-3e$ to both sides we get $48 - 3e = -12e - 3e = -15e = 0$ from (6) $48 = 3e$ multiply both sides by 5 we get $240 = 15e = 0$. Hence, $240 = 0$.
Now, $(30)^4 = (2.3.5)^4 = 2^4.3^4.5^4 = 16.3.5.3^3 5^3 = 240.3^3 5^3 = 0$,
$(60)^2 = (4.3.5)^2 = 16.3.5.3.5 = 240.3.5 = 0$,
$(90)^4 = (2.3.3.5)^4 = 16.3.5.3^7.5^3 = 240.3^7.5^3 = 0$,
$(120)^2 = (4.3.5.2)^2 = 16.3.5.2^2.3.5 = 240.2^2.3.5 = 0$,
$(150)^4 = (2.3.5.5)^4 = 16.3.5.3^3.5^7 = 240.3^3.5^7 = 0$,
$(180)^2 = (4.3.5.3)^2 = 16.3.5.3^3.5 = 240.3^3.5 = 0$,
$(210)^4 = (2.3.5.7)^4 = 16.3.5.3^3.5^3.7^4 = 240.3^3.5^3.7^4 = 0.$ ■

Proposition 2.14: If R_1, and R_2 are 2-invo-clean rings. Then $R_1 \times R_2$ is a 2-invo-clean ring.
Proof: Let $z \in R_1 \times R_2$, then there exists $p \in R_1, q \in R_2$ such that $z = (p, q)$ since R_1, and R_2 are 2-invo-clean rings then there exists $u_1, u_2 \in invo(R_1), u_3, u_4 \in$

$invo(R_2), e_1 \in Idmp(R_1),$ and $e_2 \in Idmp(R_2)$ such that $p = u_1 + u_2 + e_1, q = u_3 + u_4 + e_2$.
So, $z = (p,q) = (u_1 + u_2 + e_1, u_3 + u_4 + e_2) = (u_1, u_3) + (u_2, u_4) + (e_1, e_2)$ where $(u_1, u_3)^2 = (u_1, u_3).(u_1, u_3) = ({u_1}^2, {u_3}^2) = (1,1), (u_1, u_3) \in invo(R_1 \times R_2)$. Similarly, $(u_2, u_4) \in invo(R_1 \times R_2)$, $(e_1, e_2) \in Idmp(R_1 \times R_2)$. So, z is a 2-invo-clean element, and z is an arbitrary element. Hence, $R_1 \times R_2$ is a 2-invo-clean ring.■
In particular, [10] introduced the following concept: The ring R be strongly nil-clean when $r \in R$ is written as $r = e + n$, $e \in Idmp(R)$ and $n \in Nil(R)$, $en = ne$.

Theorem 2.15: Let R be a strongly 2-invo-clean ring, $2 \in Nil(R)$. Then
1) R is a strongly nil-clean ring.
2) For any $a \in R$, $a(a-1) \in Nil(R)$.
Proof: (1) Since R is a strongly 2-invo-clean, then for all $a \in R$, $a = u_1 + u_2 + e$, $u_1, u_2 \in invo(R)$, $e \in Idmp(R)$. Now, $(u_1 + u_2)^2 = {u_1}^2 + 2u_1u_2 + {u_2}^2 = 2(1 + u_1u_2)$, since $2 \in Nil(R)$, there is $n \in Z^+$ such that $2^n = 0$, $(u_1 + u_2)^{2n} = 0$, $u_1 + u_2 \in Nil(R)$, so $a = e + (u_1 + u_2) = e + w$, $w \in Nil(R)$. Therefore, R is a strongly nil-clean ring.
Proof (2): Since a is a strongly 2-invo-clean element, then $a = u_1 + u_2 + e$, $a(a-1) = (u_1 + u_2 + e)[(u_1 + u_2 + e) - 1] = 2(1 + u_1u_2 + u_1e + u_2e) - (u_1 + u_2)$,$[-(u_1 + u_2)]^2 = 0$. Hence, $a(a-1) \in Nil(R)$. ■

3. Application in Graph Theory.

Graph theory plays an important role in abstract algebra and in many other branches of mathematics. It can be viewed as a tool for representing relationships between elements, and where the elements are represented as vertices and the relationships between them as edges. It provides a visual framework for studying and understanding the algebraic structure of rings.

Definition 3.1: For a 2-invo-clean ring R, we define a graph denoted by $Cl_2(R)$ contains a vertex set $\mathcal{V}(Cl_2(R)) = \{(u_1, u_2, e): u_1, u_2 \in invo(R), e \in Idmp(R)\}$ and the graph has the edge set $\mathcal{F}(Cl_2(R)) = \{h_1h_2: h_1 = (u_1, u_2, e_1), h_2 = (u_3, u_4, e_2), u_1 + u_3 = 0 \; or \; u_2 + u_4 = 0$ or, $e_1.e_2 = 0, u_i \in invo(R), e_i \in Idmp(R), u_1 + u_3 + e_1, u_2 + u_4 + e_2 \in 2 - invo_c(R)\}$.
To clarify the definition, we take the following example:
let $\mathcal{V}(Cl_2(Z_3)) = \{[1,1,1], [1,2,1], [1,2,0], [1,1,0], [1,2,1], [2,2,0], [2,1,1], [2,1,0]\}$.
Then $\mathcal{F}(Cl_2(Z_3)) = \{[1,1,1][1,2,1], [1,1,1][1,2,0], [1,1,1][1,1,0], [1,1,1][1,2,1],$
$[1,1,1][2,2,0], [1,1,1][2,1,1], [1,1,1][2,1,0], [1,2,1][1,2,0],$
$[1,2,1][1,1,0], [1,2,1][1,2,1], [1,2,1][2,2,0], [1,2,1][2,1,1],$
$[1,1,1][2,1,0], [1,2,0][1,2,0], [1,2,0][1,1,0], [1,2,0][1,2,1],$
$[1,2,0][2,2,0], [1,2,0][2,1,1], [1,2,0][2,1,0], [1,1,0][1,1,0],$
$[1,1,0][1,2,1], [1,1,0][2,2,0], [1,1,0][2,1,1], [1,1,0][2,1,0],$
$[1,2,1][2,2,0], [1,2,1][2,1,1], [1,2,1][2,1,0], [2,2,0][2,2,0],$
$[2,2,0][2,1,1], [2,2,0][2,1,0], [2,1,1][2,1,0], [2,1,0][2,1,0]\}$.

The loops at the vertices [1,1,0], [1,2,0] will be neglected, because the graph we will take be a simple graph. The girth of $Cl_2(Z_3)$ is three. For similar works and more references, see [11-13].
We use a program in Python Language to obtain all connected graphs produced from Definition 3.1. Next, we will study some invariant properties like the Wiener index, average degree, average distance and some topological indices for resulting graphs in addition to some polynomials.

The following shows the graphs corresponding structures: 2-invo-clean rings Z_n, $n = 2,3,4,5,6,8$.

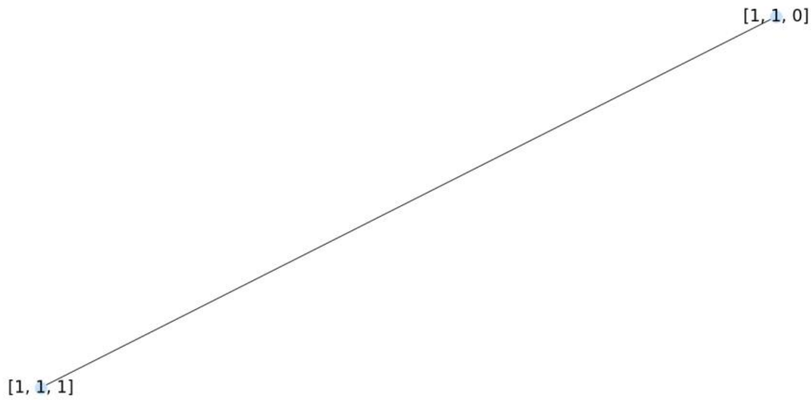

Figure 1: The graph $Cl_2(Z_2)$

$H(\mathrm{Cl}_2(Z_2); x) = 2 + x$
$W(\mathrm{Cl}_2(Z_2)) = 1$, $ad(\mathrm{Cl}_2(Z_2)) = 1$, $\mathcal{D}(\mathrm{Cl}_2(Z_2)) = 1$
$M_{vu}\,(\mathrm{Cl}_2(Z_2); x, y) = xy$
$PT(\mathrm{Cl}_2(Z_2)) = 1.0$, $ST(\mathrm{Cl}_2(Z_2)) = 0.7$,
$AGT(\mathrm{Cl}_2(Z_2)) = 1.0$, $ABCT(\mathrm{Cl}_2(Z_2)) = 0.0$

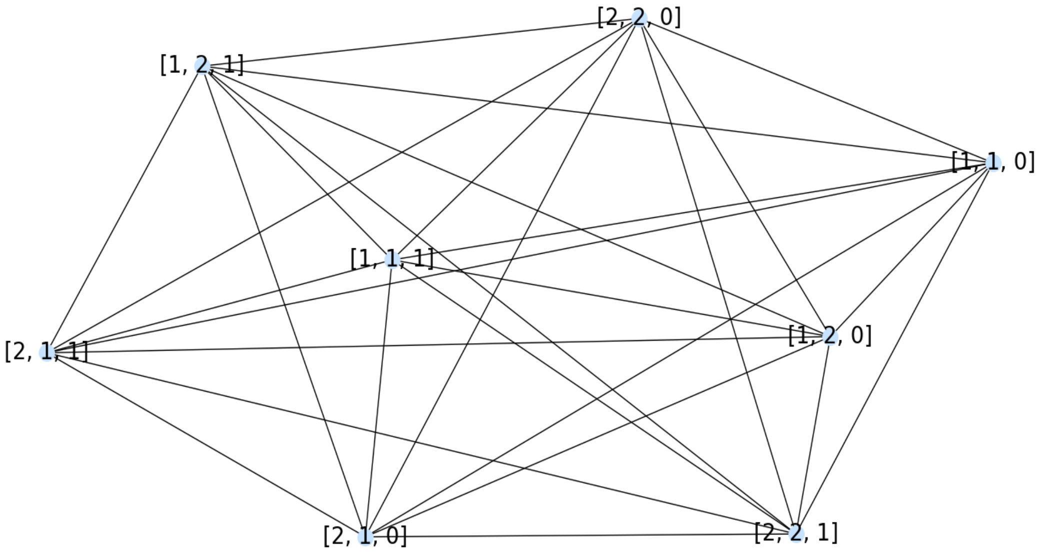

Figure 2: The graph of $Cl_2(Z_3)$

$H(Cl_2(Z_3); x) = 8 + 28x$
$W(Cl_2(Z_3)) = 28$, $ad(Cl_2(Z_3)) = 7$, $\mathcal{D}(Cl_2(Z_3)) = 1$
$M_{vu}\,(Cl_2(Z_3); x, y) = 28x^7y^7$
$PT(Cl_2(Z_3)) = 3.99$, $ST(Cl_2(Z_3)) = 7.48$,
$AGT(Cl_2(Z_3)) = 28.0$, $ABCT(Cl_2(Z_2)) = 13.85$

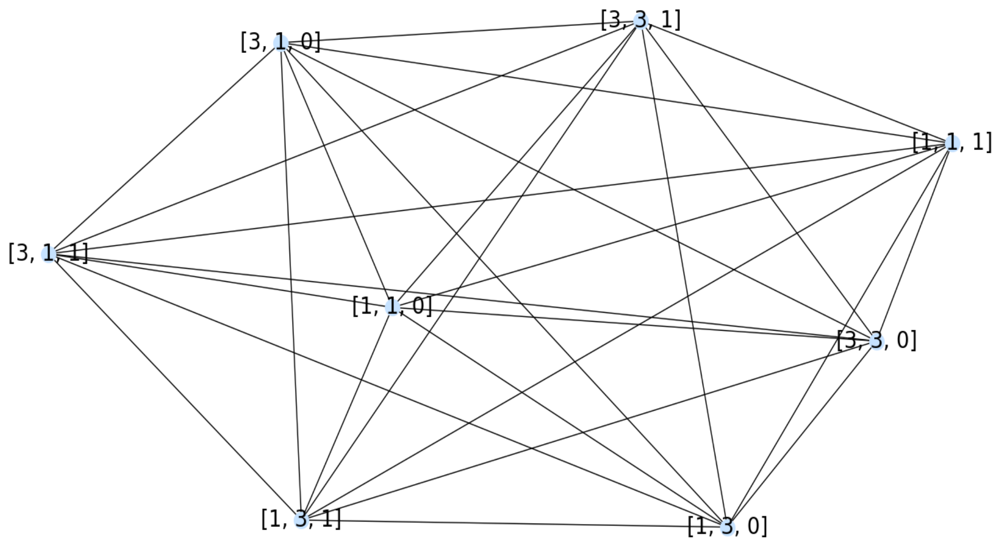

Figure 3: The graph $Cl_2(Z_4)$

$H(Cl_2(Z_4);x) = 8 + 28x$
$W(Cl_2(Z_4)) = 28$, $ad(Cl_2(Z_4)) = 7$, $\mathcal{D}(Cl_2(Z_4)) = 1$
$M_{vu}\,(Cl_2(Z_4);x,y) = 28x^7y^7$
$PT\big(Cl_2(Z_4)\big) = 3.99$, $ST\big(Cl_2(Z_4)\big) = 7.48$,
$AGT\big(Cl_2(Z_4)\big) = 28.0$, $ABCT\big(Cl_2(Z_4)\big) = 13.85$

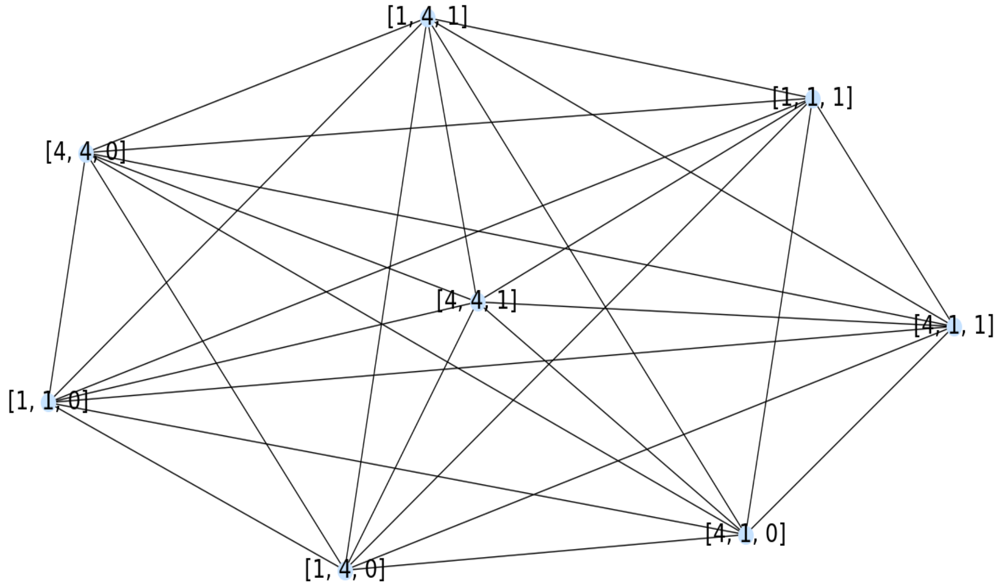

Figure 4: The graph $Cl_2(Z_5)$

$H(Cl_2(Z_5);x) = 8 + 28x$
$W(Cl_2(Z_5)) = 28$, $ad(Cl_2(Z_5)) = 7$, $\mathcal{D}(Cl_2(Z_5)) = 1$
$M_{vu}\,(Cl_2(Z_5);x,y) = 28x^7y^7$
$PT\big(Cl_2(Z_5)\big) = 3.99$, $ST\big(Cl_2(Z_5)\big) = 7.48$,
$AGT\big(Cl_2(Z_5)\big) = 28.0$, $ABCT\big(Cl_2(Z_5)\big) = 13.85$

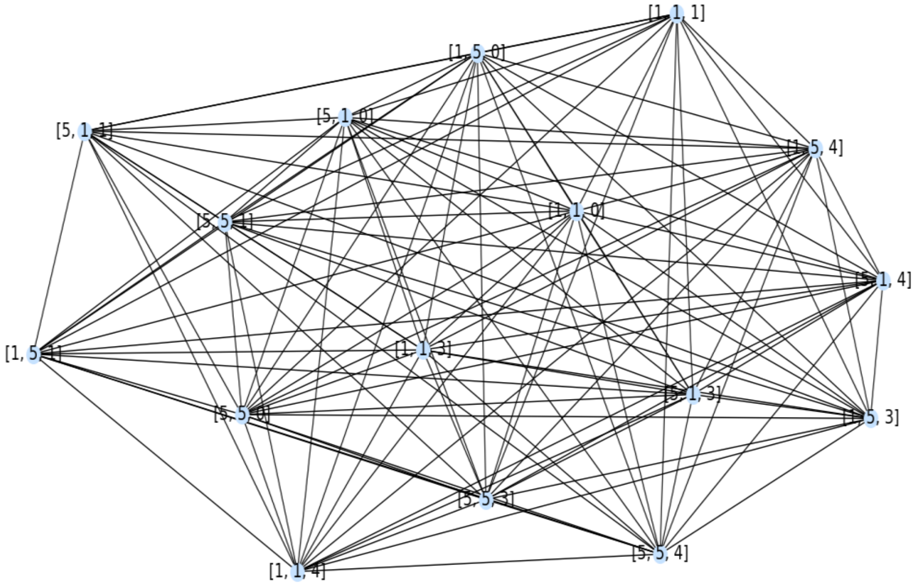

Figure 5 : The graph of $Cl_2(Z_6)$

$H(Cl_2(Z_6); x) = 16 + 112x + 8x^2$

$W(Cl_2(Z_6)) = 128$, $ad(Cl_2(Z_6)) = 14$, $\mathcal{D}(Cl_2(Z_6)) = 1.06$

$$M_{vu}(Cl_2(Z_6); x, y) = 6x^{13}y^{13} + 12x^{13}y^{14} + 6x^{13}y^{15} + 12x^{14}y^{13} + 28x^{14}y^{14} + 12x^{14}y^{15} + 10x^{15}y^{13} + 20x^{15}y^{14} + 6x^{15}y^{15}$$

$PT\big(Cl_2(Z_6)\big) = 7.99$, $ST\big(Cl_2(Z_6)\big) = 21.14$,

$AGT\big(Cl_2(Z_6)\big) = 112.07$, $ABCT\big(Cl_2(Z_6)\big) = 40.78$

Theorem 3.2: For any 2-invo-clean ring, the graph $Cl_2(R)$ is a connected, and the diameter smaller or equal to 2.

Proof: Let the vertex $(u, v, 0) \in \mathcal{V}\big(Cl_2(R)\big)$, then the vertex $(u, v, 0)$ is adjacent to every vertex in $Cl_2(R)) - \{(u, v, 0)\}$ because that, $Cl_2(R)$ it will be connected graph.

Now, to show that the graph $Cl_2(R)$ has a diameter smaller than or equal to 2. Let $(u_1, v_1, e_1), (u_2, v_2, e_2) \in \mathcal{V}\big(Cl_2(R)\big)$, The following instances are what we have:

Case 1: if $e_1 = 0$, then $d\big((u_1, v_1, e_1), (u_2, v_2, e_2)\big) = 1$, and if $e_2 = 0$, then $d\big((u_1, v_1, e_1), (u_2, v_2, e_2)\big) = 1$.

Case 2: if $u_1 + v_1 \neq 0$, $u_2 + v_2 \neq 0$ and $e_1.e_2 \neq 0$, $u_i, v_i \in invo(R)$, and $e_i \in Idmp(R)$, $i = 1, 2$ then $d\big((u_1, v_1, e_1), (u_2, v_2, e_2)\big) = 2$. ■

Theorem 3.3: For all $|Cl_2(R)| \geq 3$, then, the girth $g\big(Cl_2(R)\big)$ of a graph $Cl_2(R)$ is three .

Proof: Let *R* be an associative ring with identity; thus, the two vertices $(-1, -1, 0), (1,1,0) \in \mathcal{V}\big(Cl_2(R)\big)$ such that $(1,1,0)$ and $(-1, -1, 0)$ are adjacent. Since $|Cl_2(R)| \geq 3$ thus there exists (u_1, v_1, e) such that $(u_1, v_1, e) \neq (1,1,0), (-1, -1, 0)$ and (u_1, v_1, e) is adjacent to $(1,1,0)$ and $(-1, -1, 0)$ by Theorem 3.1, hence $g\big(Cl_2(R)\big) = 3$. ■

Below are several ring diagrams for the values $n = 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60$, representing the relationship between the order of rings and the topological index.

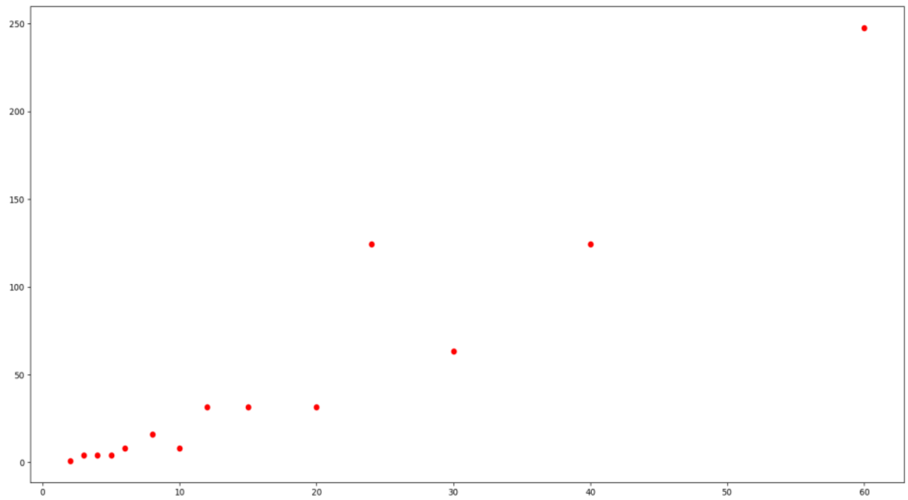

Figure 6: Product connectivity index of $Cl_2(Z_n)$.

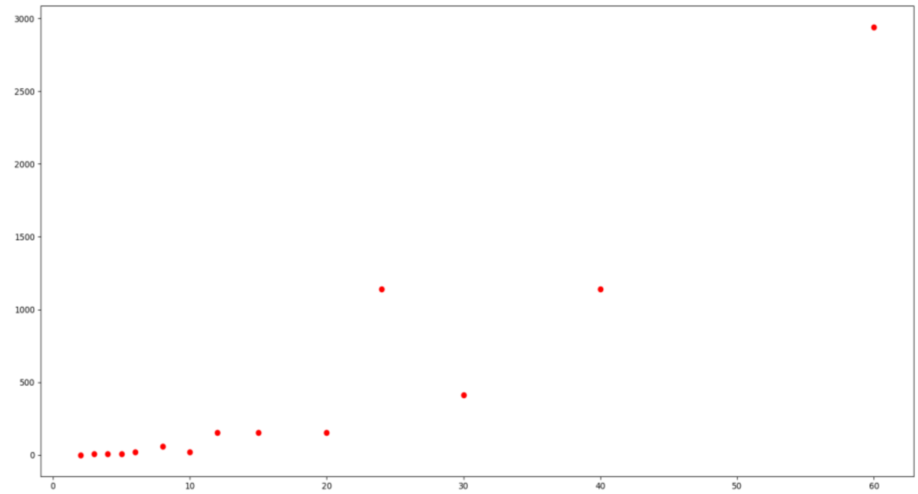

Figure 7: Sum connectivity index of $Cl_2(Z_n)$.

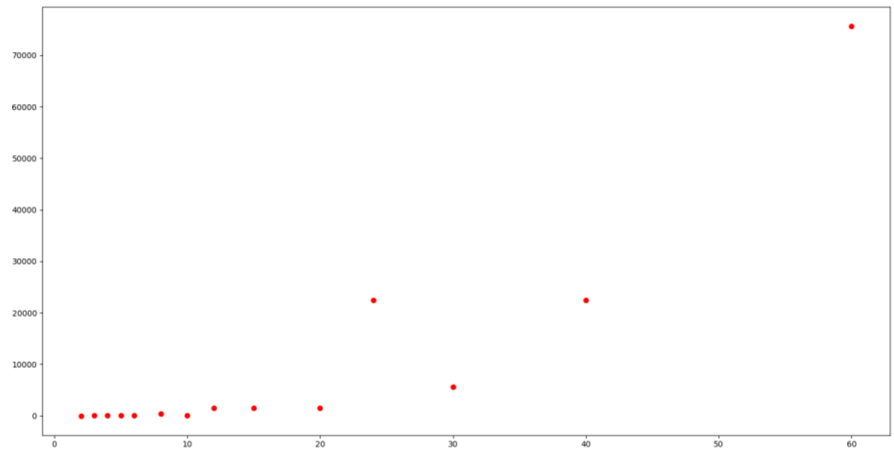

Figure 8 : Arithmetic-geometric index of $Cl_2(Z_n)$.

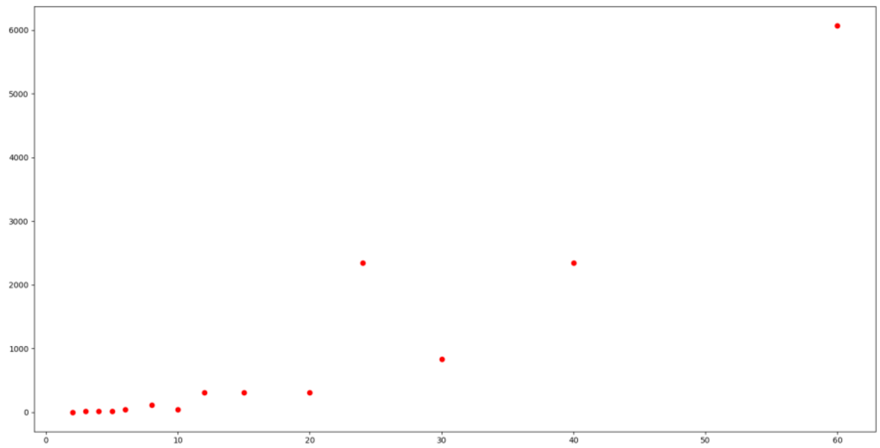

Figure 9: Atom bond connectivity index of $Cl_2(Z_n)$.

In Figures 14, 15, 16, and 17, we noticed that the order of the rings with the four topological indexes has the same behavior. Our expectation is that if a new topological index is added using the same approach as these indexes, it will exhibit the same behavior with 2-invo-clean rings.

4. Conclusions

The new rings that has been defined is actually an extension of the invo-t-clean and invo-clean rings. There are many theorems and results that have been followed by the new rings as an extension of the previous mention rings and there are properties that are not present in the previous rings. In this paper to observe the behavior of the 2-invo-clean rings with the topological indexes, which have the same behavior as different topological indexes. We expect that if we take another topological index that relies on them or the same methodology for the topological indexes, the result will also have the same behavior.

5. Acknowledgements

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