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T-Stable-extending Modules and Strongly T- stable Extending Modules

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Abstract

In this paper we introduce the notions of t-stable extending and strongly t-stable extending modules. We investigate properties and characterizations of each of these concepts. It is shown that a direct sum of t-stable extending modules is t-stable extending while with certain conditions a direct sum of strongly t-stable extending is strongly t-stable extending. Also, it is proved that under certain condition, a stable submodule of t-stable extending (strongly t-stable extending) inherits the property.

Keywords: extending modules, S-extending module, t-stable extending modules, and strongly t-stable extending modules.

المقاسات الموسعة المستقرة من النمطT والمقاسات الموسعة المستقرة بقوة من النمط T

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الخلاصه

في هذا البحث ، نقدم مفاهيم مقاسات التمديد المستقرة من النمط T والمستقرة بقوة من النمط T. نحن نتحرى خصائص ومميزات كل من هذه المفاهيم. من الواضح أن الجمع المباشر للمقاسات الموسعة المستقرة من النمط T – هو مقاسات موسعة من النمط T –بينما تحت شروط معينه ، يكون الجمع المباشر للمقاسات الموسعة المستقرةالقوية من النمط T مقاسات موسعة مستقرة قوية من النمط T كذلك يكون المقاس الجزئي الموسع المستقر من مقاس موسع مستقر من النمط T(مقاس موسع مستقر قوي من النمط T) بتوارث الخاصية.

Introduction

Let R be a ring with unity and M be a right R-module. A submodule N of M is called essential in M ($N \leq_{ess} M$) if $N \cap K = (0), K \leq M$ implies $K = (0)$. "A submodule N of M is called closed in M if it has no proper essential extension in M, that means if $N \leq_{ess} W$, where $W \leq M$, then $N = W$ [1], [2] ". It is known that for any submodule N of M, there exists a submodule H of M, such that $N \leq_{ess} H$, hence H is a closed submodule of M, H is called a closure of N [3]. Asgari [4] introduced the notion of t-essential submodule, where a submodule N of M is called t-essential (denoted by $N \leq_{tes} M$) if whenever $W \leq M$, $N \cap W \leq Z_2(M)$ implies $W \leq Z_2(M)$, where $Z_2(M)$ is the second

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singular submodule defined by $Z\left(\frac{M}{Z(M)}\right) = \frac{Z_2(M)}{Z(M)}$ [1], where $Z(M) = \{x \in M: xI = (0)$ for some essential ideal of $R\}$. Equivalently, $Z(M) = \{x \in M: ann(x) \leq_{ess} R\}$ and $ann(x) = \{r \in R: xr = 0\}$. M is called singular (nonsingular) if $Z(M) = M(Z(M) = 0)$. Note that $Z_2(M) = \{x \in M: xI = (0)$ for some t-essential ideal I of $R\}$. M is called Z_2-torsion if $Z_2(M) = M$. Asgari introduced the concept of t-closed submodule where a submodule N is called t-closed ($\leq_{tc} M$) if N has no proper t-essential extension in M [4]. It is clear that every t-closed submodule is closed, but the converse is not true. However, under the class of nonsingular, the two concepts are equivalent. Asgari [5] stated that for any submodule N of M, there exists a t-closed submodule H of M such that $N \leq_{tes} H$. H is called a t-closure of N . A module M is called extending if for every submodule N of M there exists a direct summand $W(W \leq^{\oplus} M)$ such that $N \leq_{ess} W$ [6]. Equivalently, M is an extending module if every closed submodule is a direct summand. As a generalization of extending modules, Asgari [4] introduced the concept of t-extending module, where a module M is t-extending if every t-closed submodule is a direct summand. Equivalently, M is t-extending if every submodule of M is t-essential in a direct summand. The notion of a strongly extending module is introduced in another study [7], which is a subclass of the class of extending module, where an R-module M is called strongly extending if each submodule of M is essential in a fully invariant direct summand of M, and a submodule N of M is called fully invariant if for each $f \in End(M)$, $f(N) \leq N$ [8]. A submodule N of an R-module M is called stable if for each R-homomorphism $f: N \to M, f(N) \leq N$[9]. It is clear that every stable submodule is fully invariant but not conversely**.** An R-module M is fully stable if every submodule of M is stable [9]. An R-module M is called strongly t-extending if every submodule is t-essential in a stable direct summand. Equivalently, M is strongly t-extending if every t-closed submodule is a fully invariant direct summand [10]. Saad [7] introduced the stable extending (S-extending) modules as a generalization of FI-extending modules. An R-module M is called stable extending (S-extending) if every stable submodule of M is essential in a direct summand of M. A ring R is left (right) S-extending if R is S-extending left (right) R-module and M is called FI-extending if every fully invariant submodule of M is essential in a direct summand of M[11]

In this paper, we introduce the concepts of t-stable extending and strongly t-stable extending modules. The class of t-stable extending modules contains the class of stable extending, and the class of strongly t-stable contains the class of t-stable extending and it is contained in the class of strongly t-extending.

In section two we study t-stable extending modules and their relationships with other related modules. Among other results in this section, we prove that an R-module M is a t-stable-extending R-module if and only if for each stable submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $A \leq M_1$ and $A + M_2 \leq_{tes} M$. An R-module M is t-stable extending if and only if for each stable submodule K of M, there exist $e = e^2 \in End(E(M))$ such that $K \leq_{tes} e(E(M))$ and $e(M) \leq M$ where $E(M)$ is the injective hull of M. Let M be a stable injective relative to a stable submodule X. If M is t-stable extending, then so is X.

In section three, we study strongly t-stable extending modules. Many properties are given.

2. T-Stable-extending Modules

In this section we introduce the concept of t-stable extending modules which is a generalization of S-extending modules.

First we give the following definitions.

Definition 2.1: An R-module M is called t-stable extending if every stable submodule of M is t-essential in a direct summand. A ring R is called right t-stable extending if R is a right t-stable extending R-module.

Recall that an R-module is t-uniform if every submodule of M is t-essential in M [12]. As a generalization of t-uniform module, we present the following concept.

Definition 2.2: An R-module is called stable-t- uniform if every stable submodule of M is t-essential in M.

Remarks and Examples 2.3:

(1) It is clear that every S-extending module (or t-extending module) is t-stable extending, for example:

(i)For arbitrary Z-module M, $E(M)\oplus Z_2\oplus Z_8$ is t-extending [4], so it is t-stable extending. Also $Z_2\oplus Q$ as Z-module is S-extending, so it is t-stable extending.

Recall that an R-module M is called t-continuous if M satisfies the following: M is t-extending, and every submodule of M which contains $Z_2(M)$ and isomorphic to direct summand of M is itself a direct summand **[3].** Hence, every t-continuous module is t-stable extending. Hence, we can give the following examples:

(I)By **[6, Example 2.6(2)],** Let R be a Z_2-torsion ring (e.g $R=\frac{Z}{P^2Z}$, for a prime number P) and set $T=\begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$. T^2 t-continuous T-module. It follows that T^2 is a t-stable extending module. However, T^2 is not stable extending. Hence T^2 is not stable extending.

(II) Let R be a ring and M be an R-module and $I\leq_{ess} R$. The R-module $E(M)\oplus\frac{R}{I}$ is t-continuous [6, Example 2.6(1)], so it is t-stable extending. In particular if $M=Z_p$ as Z-module. Then $Z_{P^\infty}\oplus\frac{Z}{<4>}\simeq Z_{P^\infty}\oplus Z_4$ is t-stable

(2) Let M be a nonsingular R-module. Then M is S-extending if and only if M is t-stable extending.

Proof: since M is non-singular, then the two concepts essential and t-essential coincide [5]. Hence the two concepts, S-extending and t-stable extending, are equivalent.

(3) If M is a singular module then M is t-stable extending.

Proof: since M is a singular module then Z_2(M)=M and for every submodule N of M,N+ Z_2 (M)= N+M=M$\leq_{ess}$ M, hence N$\leq_{tes}$M by[5,Prop1.1]. But M is a direct summand of M, so every stable submodule of M is t-essential in a direct summand. Thus M is t-stable extending

(4) Every FI-t-extending is t-stable-extending where M is FI-t-extending if every fully invariant is t-essential in a direct summand.

Proof: Let N be a stable submodule of M. Then N is fully invariant, hence N is t-essential in a direct summand.

(5) The converse of (4) holds if M is FI-quasi-injective, where an R-module M is called FI-quasi-injective if for each fully invariant submodule N of M, each R-homomorphism *f: N*⟼*M* can be extended to an R-endomorphism g*: M*⟼*M* [7].

Proof: Let N be a fully invariant submodule of M. By [7, Proposition 3.1.19] N is stable. Hence by t-stable extending property of M, N is t-essential in direct summand. Thus M is a FI-t-extending.

(6) t-stable extending module need not be extending, for example the Z-module $Z_8\oplus Z_2$ is not extending but it is S-extending by [7, Remarks and Examples 3.1.3(3)] hence it is t-stable extending.

(7) Every stable t-uniform (hence every t-uniform) is t-stable extending.

Proof: Let N be a stable submodule of M. Hence $N\leq_{tes} M$. But $M\leq^{\oplus} M$, so N is t-essential in a direct summand.

Recall that an R-module M is called an S-indecomposable if (0) , M are the only stable direct summand. M is S-extending and S-indecomposable if M is S-uniform. ″An R-module M is called stable uniform (shortly, S-uniform) if every stable submodule of M is essential in M ″ [7]. However we have:

Proposition 2.4: If M is t- stable extending and indecomposable, then M is stable t- uniform.

Proof: Let N be a stable submodule in M. Then $N\leq_{tes} W$ for some $W\leq^{\oplus} M$. Since M is indecomposable, $W=M$. Thus $N\leq_{tes} M$ and so M is a t-stable uniform.

Note that a stable t- uniform module does not imply indecomposable, for example Z_6 as Z-module is stable t- uniform, but Z_6 is not indecomposable. Also, Z_6 is not S-indecomposable.

Proposition 2.5: Let M be an R-module. If M is t-stable extending, then every stable t-closed submodule is a direct summand and the converse holds if every t-closure of stable submodule is stable.

Proof: Let N be a stable t-closed submodule. Since M is t-stable extending, $N\leq_{tes} W$ for some $W\leq^{\oplus} M$. Hence $N=W\leq^{\oplus} M$, since N is a t-closed. Now if N is a stable submodule of M, then $N\leq_{tes} W$, where W is a t-closure of N [5,Lemma 2.3]. By hypothesis, W is stable, and so W is stable t-closed, which implies $W\leq^{\oplus} M$. Thus N is t-essential in a direct summand and M is t-stable extending.

Proposition 2.6: Let M be an R-module which satisfies that the t-closure of any submodule is stable. Then M is t-stable extending if and only if M t-extending**.**

Proof: ⇒ Let N be a t-closed of M. Hence N is a t-closure of N and so by hypothesis, N is stable. But M is t-stable extending, so there exists $W \leq^{\oplus} M$ such that $N \leq_{tes} W$. Thus $N = W$ because N is t-closed and so M is t-extending.

⇐ If M is t-extending, then by Remarks and Examples 2.3(1), M is t-stable extending.

Corollary 2.7: Let M be a fully stable R-module. Then the following statements are equivalent:

(1) M is a t-stable extending module;

(2) M is a t-extending module ;

(3) M is a strongly t-extending module.

Proof: Since M is a fully stable R-module, and the t-closure of any submodule of M is stable . Then (1) ⇔ (2) follows by Proposition 2.6.

(1)⇒(3) Let $N \leq M$. Since M is fully stable, then N is stable. Hence N is t-essential in a direct summand W. But W is stable in M. Then N is t-essential in a stable direct summand and so M is strongly t-extending.

(3)⇒(2) obvious.

Proposition 2.8: Let M be an R-module that satisfies that the t-closure of any submodule is stable. Then the following statements are equivalent:

(1) M is a t-stable extending module;

(2) Every stable t-closed submodule of M is a direct summand;

(3) Every stable submodule is t-essential in stable direct summand.

Proof: (1)⇒(2) Let N be a stable t-closed submodule. Condition (1) implies N is t-essential in a direct summand W. Hence $N = W \leq^{\oplus} M$ since N is a t-closed.

(2)⇒(3) Let N be a stable submodule in M. Then N has a t-closure W; such that $N \leq_{tes} W$ and W is a t-closed. But W is stable by hypothesis , so that W is t-closed stable. Then by condition (2) $W \leq^{\oplus} M$ and hence N is t-essential in a stable direct summand.

(3)⇒(1) clear.

The following are characterizations of the t-stable extending modules.

Theorem 2.9: An R-module M is t-stable-extending if and only if for each stable submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $A \leq M_1$ and $A + M_2 \leq_{tes} M$.

Proof: ⇒ Suppose M is t-stable-extending. Let A be a stable submodule of M. Then $A \leq_{tes} M_1 \leq^{\oplus} M$, hence $M_1 \oplus M_2 = M$ for some $M_2 \leq M$. It follows that $A \oplus M_2 \leq_{tes} M_1 \oplus M_2 = M$(since $A \leq_{tes} M_1$ and $M_2 \leq_{tes} M_2$ [5, Corollary1.3].

⇐ Let A be a stable submodule of M. By hypothesis, there is a decomposition $M = M_1 \oplus M_2$ with $A \leq M_1$ and $A + M_2 \leq_{tes} M = M_1 \oplus M_2$. It follows that $A \leq_{tes} M_1$ by [5, Corollary 1.3]. Thus $A \leq_{tes} M_1 \leq^{\oplus} M$. Therefore M is t-stable-extending.

The following is another characterization of t-stable extending modules.

Theorem 2.10: An R-module M is t-stable extending if and only if for each stable submodule K of M, there exists $e = e^2 \in End(E(M))$ such that $K \leq_{tes} e(E(M))$ and $e(M) \leq M$ where $E(M)$ is the injective hull of M.

Proof:⇒ Assume M is t-stable extending. Let K be a stable submodule of M. Then there exists $D \leq^{\oplus} M$ of M such that $K \leq_{tes} D$ and so there is $H \leq M$ such that $= D \oplus H$. Hence $E(M) = E(D) \oplus E(H)$. Let $e: E(M) \mapsto E(D)$ be the projection endomorphism from $E(M)$ onto $E(D)$. Clearly $e^2 = e(e$ is idempotent). Thus we have $e(M) \leq (D \oplus H)$. Also, $K \leq_{tes} D \leq_{ess} E(D)$ implies $K \leq_{tes} E(D) = e(E(M))$.

⇐ Let K be a stable submodule of M. By hypothesis, There exists $e \in End\big(E(M)\big)$, $e^2 = e$ such that $K \leq_{tes} e(E(M))$ and $e(M) \leq M$.Since $M \leq_{tes} M$, then $K \cap M \leq_{tes} e\big(E(M)\big) \cap M = e(M)$. It is easy to see that $e\big(E(M)\big) \cap M = e(M)$. Also, since $K \cap M = K$, hence $K \leq_{tes} e(M)$. But $e(M) \leq^{\oplus} M$[7, Lemma 1.1.22], so K is t-essential in stable direct summand. Thus M is stable extending.

Lemma 2.11: Let $M = \oplus_{i \in I} M_i$. Let N be a stable submodule of M. Then $N = \oplus_{i \in I}(N \cap M_i)$ where $N \cap M_i$ is stable in M_i, $\forall\, i \in I$.

Proof: Let W be a stable submodule. Then $W = \oplus_{i\in I}(W \cap M_i)$ by [9 , Proposition 4.5] we claim that $N \cap M_i$ is stable in M_i, for each $i \in I$. To prove this, let $g: W \cap M_i \mapsto M_i$ be any R-homomorphism. Then $g(W \cap M_i) \subseteq M_i$. Consider the following $W = \oplus_{i\in I}(W \cap M_i \overset{\rho}{\to} W \cap M_i \overset{g}{\to} M_i \overset{i}{\to} M = \oplus_{i\in I} M_i$, where ρ is the natural projection and i is the inclusion mapping. Then $(i \circ g \circ \rho)(W) \subseteq W$ (since W is stable in M). But $(i \circ g \circ \rho)(W) = i \circ g(W \cap M_i) = i(g(W \cap M_i) = g(W \cap M_i)$. Thus $(W \cap M_i)(W) \subseteq W$. From above $g(W \cap M_i) \subseteq M_i$, so we get $g(W \cap M_i) \subseteq W \cap M_i$ and $W \cap M_i$ is a stable submodule of M_i, for each $i \in I$.

Theorem 2.12: A direct sum of t-stable extending modules is t-stable extending.

Proof: Suppose that $M = \oplus_{i\in I} M_i$, M_i is t-stable extending for each $i \in I$. Let W be a stable submodule of M. Then $W = \oplus_{i\in I}(W \cap M_i)$ and $W \cap M_i$ is stable in M_i for each $i \in I$ by Lemma 2.11 and so by the t-stable extending property of M_i, $W \cap M_i$ is t-essential in a direct summand N_i of M_i for each $i \in I$. Then $\oplus_{i\in I}(W \cap M_i) \leq_{tes} \oplus_{i\in I} N_i$ by [5,Coroallary 1.3]. Put $N = \oplus_{i\in I} N_i$, so $N \leq^{\oplus} M$. Thus $N \leq_{tes} N \leq^{\oplus} M$ and □ is t-stable extending.

Note that any direct sum of extending is S-extending [7, Corollary 3.2.2], hence by Remarks and Examples 2.4(2), it is t-stable extending.

By applying Theorem 2.12, each of $Z_p \oplus Z, Z_p \oplus Q$(for each prime number P) $Z \oplus Z, Z_2 \oplus Z_8, Z \oplus Z, Z \oplus Z \oplus Z$ ... as Z-module is t-stable extending. Not that $Z_2 \oplus Z_8$ and $Z \oplus Z \oplus Z$ ... are not extending. Note that by [7, Corollary 3.2.4] every finitely generated Z-module is S-extending, hence it is t-stable extending.

Proposition 2.13: Let M be an R-module which satisfies that the t-closure of any submodule is stable. If M is t-stable extending, then every direct summand is t-stable extending.

Proof: Let $N \leq^{\oplus} M$. Since M is t-stable extending, then M is t-extending by Proposition 2.6. Hence N is t-extending by [4, Proposition 2.14(1)]. It follows that N is FI-t-extending and hence by Remarks and Examples 2.3(3), N is t-stable extending.

Corollary 2.14: Let M be a fully stable R-module. If M is t-stable extending, then every direct summand is t-stable extending.

Recall that an R-module M has the summand intersection property (SIP) if the intersection of two direct summands of M is a direct summand [13]. Since S-extending and t-stable extending are equivalent in the class of nonsingular modules, thus we have every direct sumand of t-stable extending module M(where M is nonsingular with SIP) is t-stable extending module. Also, we have by [2, Corollary 3.2.7, Corollary 3.2.8 and Corollary 3.2.9] the following:

1- Let Mbe a nonsingular SS-module (that is every direct summand is stable). If M t-stable extending, then every direct summand is t-stable extending.

2- Every direct summand right ideal of a nonsingular t-stable extending commutative ring is t-stable extending.

3- Every direct summand of nonsingular cyclic Z-module is t-stable extending.

An R-module M is called stable-injective relative to X (simply, S-X-injective) if for each stable submodule A of X, each R-homomorphism $f: A \longmapsto M$ can be extended to an R-homomorphism $g: X \longmapsto M$. " [7, Definition 3.2.10].

By using the procedure of the proof of Theorem 3.2.14 [7], we have the following Lemma.

Lemma 2.15: Let M be a stable injective module relative to a stable submodule X of M. If $A \subseteq X$ such that A is a stable in X, then A is stable in M.

Proof: Let $f \in Hom(A, M)$. Since M is stable injective relative to X, there exists an R-homomorphism $g: X \longmapsto M$ such that $g \circ i = f$ where i is the inclusion mapping from A into X. It follows that $g(X) \subseteq X$, since X is stable in M. So $g \circ i(A) = g(A) \subseteq X$; that is $g|_A : A \longmapsto X$. But A is stable in X, so that $g|_A(A) \subseteq A$. Thus $f(A) \subseteq A$ and A is stable in M.

Proposition 2.16: Let M be a stable injective relative to a stable submodule X. If M t-stable extending, then so is X.

Proof: To prove X is t-stable. Let A be a stable submodule of X. By Lemma 2.15, A is stable in M. Since M is t-stable extending, there exists $D \leq^{\oplus} M$ such that $A \leq_{tes} D$ it follows that $M = D \oplus D'$ for some $D' \subseteq M$ and so $A = X \cap D \leq_{tes} X \cap D \leq^{\oplus} M$ by (5, Corollary 1.3]

3. Strongly t-stable extending modules

In this section, we extend the notion of t-stable extending modules into strongly t-stable extending modules. We study these classes of modules and their relations with some related concepts.

Definition 3.1: An R-module M is called strongly t-stable extending if each stable submodule N of M. N is t-essential in a stable direct summand.

Remarks and Examples 3.2:

(1) It is clear that every strongly t-stable extending is t-stable extending

(2) Every strongly t-extending (hence every Z_2-torsion) module is strongly t-stable extending. In particular, each of Z-module $M = Z_n \oplus Z$ where n is a positive integer is strongly t-extending (see [10, Example 3.3]. Thus M is strongly t-stable extending.

(3) The converse of (2) is not true as the following example shows: Let M be the Z-module $Z \oplus Z$. Let N be a stable submodule of M. Then $N = (N \cap Z) \oplus (N \cap Z)$, where $N \cap Z$ is stable in Z by Lemma 2.11. Since the only stable submodules of Z are Z, (0), then $N = Z \oplus Z$ or $N = (0) \oplus (0)$ and hence $N \leq_{tes} N \leq^{\oplus} M$. Thus M is a strongly t-stable extending module. On the other hand, $N = Z \oplus (0)$ is t-closed(closed) and N is not a fully invariant direct summand, since there exists $f: M \mapsto M$, such that $f(x, y) = (y, x)$ for each $(x, y) \in M$ and so $f(N) = f\big(Z \oplus (0)\big) = (0) \oplus Z \nleq N$.

(4) Recall that an R-module M is called weak duo if every direct summand is fully invariant [14]. Let M be a week duo. Then M is strongly t-stable extending if and only if M is a t-stable extending module.

Proof: $\Rightarrow$ It follows by (1)

$\Leftarrow$ Let N be a stable submodule of M. Then $N \leq_{tes} W \leq^{\oplus} M$. Since M is weak duo, W is a fully invariant in M and then by [7, Lemma 2.1.6] W is stable. Thus M is strongly t-stable extending.

(5) Let M be a fully stable module. Then the following are equivalent:

(1) M is t-stable extending;

(2) M is t-extending;

(3) M is strongly t-stable extending;

(4) M is strongly t-extending;

(6) Every stable t-uniform module is strongly t-stable extending.

(7) If M is S-indecomposable and M is strongly t-stable extending, then M is a stable t-uniform.

Proof: Let N be a stable submodule of M. Since M is strongly t-stable extending, $N \leq_{tes} W \leq^{\oplus} M$, W is a fully invariant in M. Then by[7,Lemma 2.1.6], $W$is stable in M, but $N$is S-indecomposable, so $W = M$. Thus $N \leq_{tes} M$ and M is a stable t-uniform.

(8) If M is S-uniform, then M is strongly t-stable extending and$M$ is S-indecomposable.

(9) Let M be an indecomposable module. Then M is strongly t-stable extending if and only if M is t-stable extending.

(10) If M is a FI-t-extending, then M is strongly t-stable extending. The converse holds if M is FI-quasi injective.

Proof: Let N be a stable submodule of M. Then N is fully invariant, hence by [11, Theorem 2.2 (1) $\Leftrightarrow$ (7)] N is t-essential in a fully invariant direct summand, say W. By [7, Lemma 2.1.6] W is stable. Thus M is strongly t-stable extending.

Proposition 3.3: Let M be an R-module which satisfies that the t-closure of any submodule is stable. Then the following statements are equivalent:

(1) M is strongly t-stable extending;

(2) M is t-stable extending;

(3) M is t-extending;

(4) Every stable t-closed is a direct summand;

(5) M is strongly t-extending.

Proof: (1) $\Rightarrow$ (2) Let N be a stable submodule of N. Then by definition of strongly t-stable extending, N is a t-esential in a fully invariant direct summand. Thus M is t-stable extending.

(3) $\Rightarrow$ (4) Since M is t-extending , every t-closed is a direct summand, so it is clear that every stable t- closed is a direct summand.

(2) $\Leftrightarrow$ (4) It follows by Proposition 2.8.

(2) $\Leftrightarrow$ (3) It follows by Proposition 2.6.

(4) ⇒ (1) Let N be a stable submodule of M. Then there exists a t-closure of N say W such that $N \leq_{tes} W$. By hypothesis, W is stable t-closed of M, hence $W \leq^{\oplus} M$. Thus M is strongly t-stable extending.
(5) ⇒ (1) It follows by Remarks and Examples 3.2(2).
(1) ⇒ (5) Let N be a t-closed of M. Hence N is a t-closure of N and so by hypthesis N is stable. Since M is strongly t-stable extending, $N \leq_{tes} W$ for some stable direct summand W. It follows that $N = W$, since N is t-closed. Thus N is a stable direct summand and M is strongly t-extending.
Recall that an Rmodule M is a multiplication module if for each $N \leq M$, there exists an ideal I of R such that $N = MI$ [15].
Proposition 3.4: Let M be a multiplication t-extending. Then$M$ is strongly t-stable extending.
Proof: Let N be a stable submodule of M. Since M is t-stable extending, then there exists $H \leq^{\oplus} M$ such that $N \leq tes \ H \leq^{\oplus} M$. But M is a multiplication module implies H is a fully invariant submodule of M and so by [7, Lemma2.1.6], H is stable. Thus M is t-essential in stable direct summand of M. Therefore, M is strongly t-stable extending.
Corollary 3.5: Every cyclic t-stable extending module over a commutative ring is strongly t-stable extending.
Corollary 3.6: Every commutative t-stable extending ring is strongly t-stable extending.
The following is a characterization of strongly t-stable extending modules.
Theorem 3.7: Let M be an R-module. M is strongly t-stable extending if for each stable submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $A \leq M_1$ and M_1 is a stable submodule of$M$ and $A + M_2 \leq_{tes} M$.
Proof:⇒ Let A be a stable submodule of M. Since M is strongly t-stable extending, $A \leq_{tes} M_1 \leq^{\oplus} M$ and M_1 is stable in M. Hence $M = M_1 \oplus M_2$ for some $M_2 \leq M$. Since $A \leq_{tes} M_1$, $M_2 \leq_{tes} M_2$, then $A + M_2 \leq_{tes} M_1 \oplus M_2 = M$, by [5, Corollary 1.3].
⇐ Let A be a stable submodule of M. By hypothesis, there is a decomposition $M = M_1 \oplus M_2$ such that $A \leq M_1$, M_1 is stable in M and $A + M_2 \leq_{tes} M$. Since $A + M_2 = A \oplus M_2 \leq_{tes} M = M_1 \oplus M_2$, then $A \leq_{tes} M_1$. But M_1 is a stable direct summand of M. Thus M is strongly t-stable extending.
Theorem 3.8: Let $M = M_1 \oplus M_2$, where M_1 and M_2 are R-module , such that M is an abelian module $(annM_{1_R} \oplus annM_{2_R} = R)$. If M_1 and M_2 are strongly t-stable extending, then $M = M_1 \oplus M_2$ is strongly t-stable extending.
Proof: : Let N be a stable submodule of M. By Lemma 2.11, $N = (N \cap M_1) \oplus (N \cap M_2)$ where $N \cap M_1$ is stable in M_1, $N \cap M_2$ is stable in M_2. Put $N_1 = (N \cap M_1), N_2 = (N \cap M_2)$. Since M_1 and M_2 are strongly t-stable extending, there exist $W_1 \leq^{\oplus} M_1$, $W_2 \leq^{\oplus} M_2$ and W_i is stable in M_i for i = 1,2 and $N_i \leq_{tes} W_i$. It follows that $N_1 \oplus N_2 \leq_{tes} W_1 \oplus W_2$ by [5, Corollary 1.3]. Since $W_1 \leq^{\oplus} M_1$, $W_2 \leq^{\oplus} M_2$, then $W_1 \oplus W_2 \leq^{\oplus} M$. On other hand M is abelian (or $(annM_{1_R} \oplus annM_{2_R} = R)$implies $Hom(M_1, M_2) = 0$, $Hom(M_2, M_1) = 0$,by[14,Theorem4.6].Hence $End(M) \simeq \begin{pmatrix} End(M_1) & Hom(M_2, M_1) \\ Hom(M_1, M_2) & End(M_2) \end{pmatrix} \simeq \begin{pmatrix} End(M_1) & 0 \\ 0 & End(M_2) \end{pmatrix}$. Hence for each $f \in End(M), f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$, $f_1 \in End(M_1), f_2 \in End(M_2)$ and $f(W_1 \oplus W_2) = f(W_1) \oplus f(W_2)$. But W_1 and W_2 are stable in M_1, M_2 respectively and so that $f(W_1) \subseteq W_1, f(W_2) \subseteq W_2$. Thus $f(W_1 \oplus W_2) \subseteq W_1 \oplus W_2$, hence $W_1 \oplus W_2$ is a fully invariant in M, $W_1 \oplus W_2 \leq^{\oplus} M$, then [2,Lemma 2.1.6] $W_1 \oplus W_2$ is stable in M.

Now we ask the following: Is the property of being strongly t-stable extending inherit to a submodule?
First we give the following
Definition 3.9: An R-module M is said to be stable-injective if M is stable-injective to N(M is S-N-injective), where N is any R-module.
Theorem 3.10: Let M be a stable-injective R-module. If M is strongly t-stable extending, then every stable submodule of M is strongly t-stable extending.
Proof: Let X be a stable submodule of M. To prove X is strongly t-stable extending, let A be a stable submodule of X. Since M is stable-njective, then M stable-injective relative to X and hence by Lemma 2.15, A stable submodule of M. Now M is strongly t-stable extending and A is stable in M imply there

exists a stable direct summand D such that $A \leq_{tes} D \leq^{\oplus} M$. Thus $M = D \oplus D'$ for some $D' \leq M$. Since X is stable in , $X = (X \cap D) \oplus (X \cap D')$ where $X \cap D$ is stable of D, $X \cap D'$ is stable of D' by Lemma 2.11. Now $A \leq_{tes} D$ implies $A = X \cap A \leq_{tes} X \cap D$ by [3,Corllary 1.3]. But$(X \cap D) \leq^{\oplus} X$, so that $A \leq_{tes} X \cap A \leq^{\oplus} X$. We claim that $X \cap D$ is stable in X. Since D is stable of M and $X \cap D$ is stable in D, then $X \cap D$ is stable of M by Lemma 2.15. But $X \cap D$ is stable in M and $X \cap D \subseteq X$ imply $X \cap D$ is stable in X.

Proposition 3.11: Let M be an R-module which satisfies that the t-closure of any submodule is stable. If M is strongly t-stable extending, then every direct summand is strongly t-stable extending.
Proof: Let $W \leq^{\oplus} M$. Since M satisfies that the t-closure of any submodule is stable, then by (Proposition 3.3) M is strongly t-extending and so by [8, Theorem 3.5] W is strongly t-extending. Thus by Remarks and Examples 3.2(2), W is strongly t-stable extending.
Corollary 3.12: Let M be a fully stable R-module. If M is strongly t-stable extending, then every direct summand is strongly t-stable extending.

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